

**PHY 712 Electrodynamics
10-10:50 AM MWF Online**

Notes for Lecture 24:

Complete reading of Chap. 9 & 10

A. Superposition of radiation

B. Scattered radiation

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In this lecture, we will continue to focus on radiation from sources with pure harmonic time dependence with frequency ω , considering effects of superposition of multiple such sources (leading to interference) and also considering (re)radiation due to scattering of electromagnetic waves.

				LECTURE	
21	Mon: 03/22/2021	Chap. 8	EM waves in wave guides		
22	Wed: 03/24/2021	Chap. 9	Radiation from localized oscillating sources	#15	03/26/2021
23	Fri: 03/26/2021	Chap. 9	Radiation from oscillating sources	#16	03/29/2021
24	Mon: 03/29/2021	Chap. 9 & 10	Radiation and scattering	#17	03/31/2021
25	Wed: 03/31/2021	Chap. 11	Special Theory of Relativity		
26	Fri: 04/02/2021	Chap. 11	Special Theory of Relativity		

PHY 712 -- Assignment #17

March 29, 2021

Finish reading Chapters 9 and 10 in **Jackson**.

1. Work problem 9.16(a) in **Jackson**. Note that you can use an approach similar to that discussed in Section 9.4 of the textbook, replacing the "center-fed" antenna with the given antenna configuration.

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The assigned homework deals with radiation from an antenna with a slightly different configuration than covered in the textbook and in the lecture notes.

Ph.D. DEFENSE

TUESDAY
•
MARCH 30, 2021

“Dynamics of Charge Carrier Traps in Organic Semiconductors”

Organic semiconductors (OSCs) are becoming an integral part of our lives as active components of various optoelectronic devices given their low-cost processing, light weight, chemical versatility by molecular design, and compatibility with flexible substrates. These systems undergo considerable electronic and structural transformations during device fabrication and operation, which can profoundly impact their performance and stability. Characterization techniques that can elucidate the mechanisms of the time-dependent transformations occurring in these materials and devices are needed to guide the design and processing to yield high-performance and stable devices. In this dissertation, a highly efficient method is introduced to elucidate the microscopic processes occurring within

Hamna Iqbal

Dr. Oana Jurchescu, Advisor
Department of Physics
Wake Forest University

Oral Defense
9:00 am – 11:00 am EST

Electromagnetic waves from time harmonic sources – review:

For scalar potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\Phi}(\mathbf{r}, \omega) = \tilde{\Phi}_0(\mathbf{r}, \omega) + \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\rho}(\mathbf{r}', \omega)$$

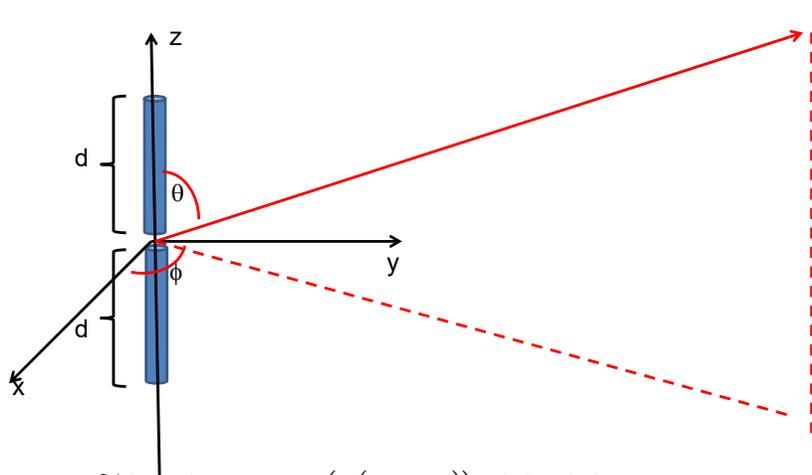
For vector potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \tilde{\mathbf{A}}_0(\mathbf{r}, \omega) + \frac{\mu_0}{4\pi} \int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

Review of equations that we have been using for the time Fourier transforms of the scalar and vector potentials due to their corresponding Fourier transforms of the charge and current densities.

Consider antenna source (center-fed)

Note – these notes differ from previous formulation $d/2 \leftrightarrow d$



$$\tilde{\mathbf{J}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} I \sin(k(d - |z|)) \delta(x) \delta(y) \quad \text{for } -d \leq z \leq d$$

$$k \equiv \frac{\omega}{c}$$

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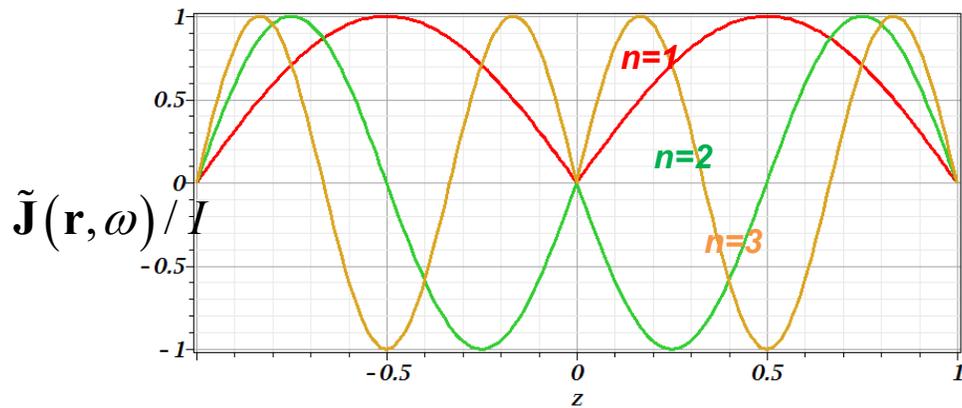
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Specifically, consider an antenna. For convenience, we are using a slightly different notation from the previous lecture as noted at the top of the slide.

Consider antenna source -- continued

$$\tilde{\mathbf{J}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} I \sin(k(d - |z|)) \delta(x) \delta(y) \quad \text{for } -d \leq z \leq d$$

$$\text{for } k \equiv \frac{\omega}{c} = \frac{n\pi}{d}; \quad n = 1, 2, 3, \dots$$



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The plot indicates how the current varies along the z axis of the antenna for the center-fed configuration.

Consider antenna source -- continued

$$\tilde{\mathbf{J}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} I \sin(k(d - |z|)) \delta(x) \delta(y) \quad \text{for } -d \leq z \leq d$$

$$k \equiv \frac{\omega}{c}$$

Vector potential from source:

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \frac{\mu_0}{4\pi} \int d^3 r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

$$\text{For } r \gg d \quad \tilde{\mathbf{A}}(\mathbf{r}, \omega) \approx \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3 r' e^{-ik\hat{\mathbf{r}} \cdot \mathbf{r}'} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) \approx \hat{\mathbf{z}} \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} I \int_{-d}^d dz' e^{-ikz' \cos \theta} \sin(k(d - |z'|))$$

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Evaluation of the vector potential far from the antenna.

Consider antenna source -- continued

$$\begin{aligned}\tilde{\mathbf{A}}(\mathbf{r}, \omega) &\approx \hat{\mathbf{z}} \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} I \int_{-d}^d dz e^{-ikz \cos \theta} \sin(k(d - |z|)) \\ &= \hat{\mathbf{z}} \frac{\mu_0}{4\pi} \frac{e^{ikr}}{kr} 2I \left[\frac{\cos(kd \cos \theta) - \cos(kd)}{\sin^2 \theta} \right]\end{aligned}$$

In the radiation zone :

$$\tilde{\mathbf{B}}(\mathbf{r}, \omega) = \nabla \times \tilde{\mathbf{A}}(\mathbf{r}, \omega) \approx ik\hat{\mathbf{r}} \times \tilde{\mathbf{A}}(\mathbf{r}, \omega)$$

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) \approx -ikc\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \tilde{\mathbf{A}}(\mathbf{r}, \omega))$$

$$\frac{dP}{d\Omega} = \frac{1}{2\mu_0} r^2 \hat{\mathbf{r}} \cdot \Re(\tilde{\mathbf{E}}(\mathbf{r}, \omega) \times \tilde{\mathbf{B}}^*(\mathbf{r}, \omega)) = \frac{k^2 c}{2\mu_0} r^2 \left(|\tilde{\mathbf{A}}(\mathbf{r}, \omega)|^2 - |\hat{\mathbf{r}} \cdot \tilde{\mathbf{A}}(\mathbf{r}, \omega)|^2 \right)$$

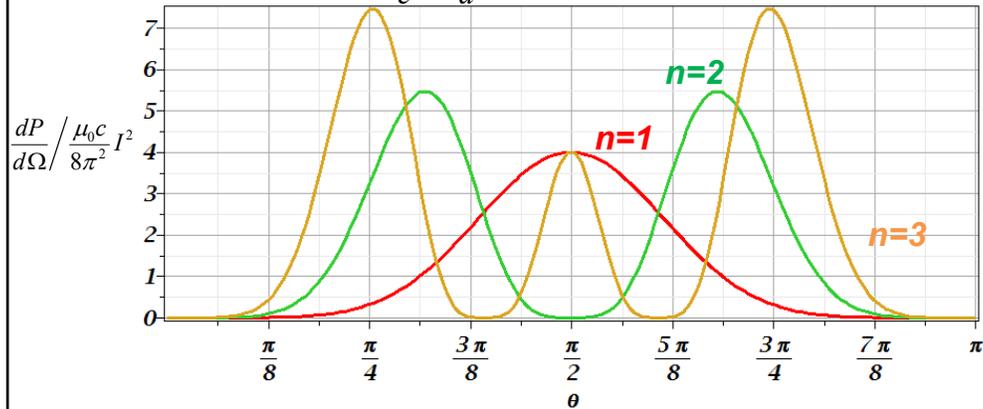
$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[\frac{\cos(kd \cos \theta) - \cos(kd)}{\sin \theta} \right]^2$$

Some details for evaluating the power per unit solid angle.

Consider antenna source -- continued

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[\frac{\cos(kd \cos \theta) - \cos(kd)}{\sin \theta} \right]^2$$

$$\text{for } k \equiv \frac{\omega}{c} = \frac{n\pi}{d}; \quad n = 1, 2, 3, \dots$$



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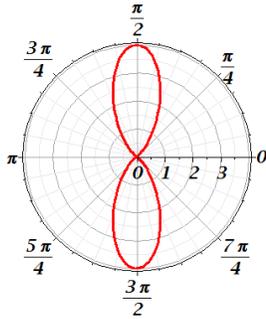
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Plot of the power distribution as a function of angle for this case.

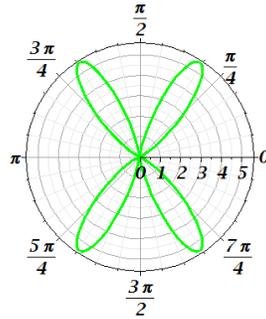
Consider antenna source -- continued

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[\frac{\cos(kd \cos \theta) - \cos(kd)}{\sin \theta} \right]^2$$

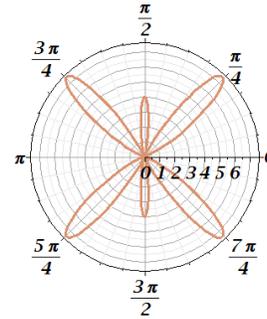
For $kd = n\pi$:



$n=1$



$n=2$



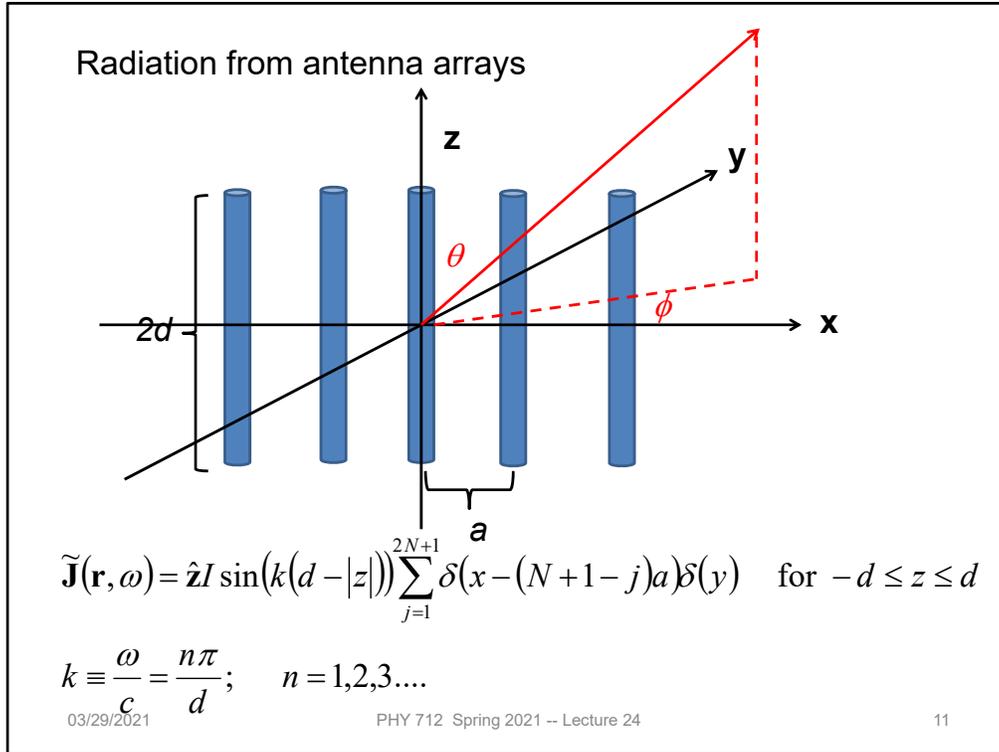
$n=3$

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Polar plots of the power distribution.



Now consider the case of several antennas, in this case each antenna is oriented along the z-axis and $2N+1$ of them are arranged in a line along the x-axis.

Radiation from antenna arrays -- continued

Vector potential from array source :

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \frac{\mu_0}{4\pi} \int d^3 r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\mathbf{J}}(\mathbf{r}', \omega) \approx \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3 r' e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

$$\tilde{\mathbf{J}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} I \sin(k(d-|z|)) \sum_{j=1}^{2N+1} \delta(x - (N+1-j)a) \delta(y) \quad \text{for } -d \leq z \leq d$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) \approx \hat{\mathbf{z}} \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \left(\sum_{j=-N}^N e^{-ikaj \sin \theta \cos \phi} \right) I \int_{-d}^d dz e^{-ikz \cos \theta} \sin(k(d-|z|))$$

$$\sum_{j=-N}^N e^{-ikaj \sin \theta \cos \phi} = \frac{\sin(\frac{1}{2} ka(2N+1) \sin \theta \cos \phi)}{\sin(\frac{1}{2} ka \sin \theta \cos \phi)}$$

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Analyzing the same equations as before, keeping the leading terms for the limit that $kr \rightarrow \text{infinity}$. Here we see that the x-axis dependence involves evaluating a geometric series which can be done analytically as shown.

Radiation from antenna arrays -- continued

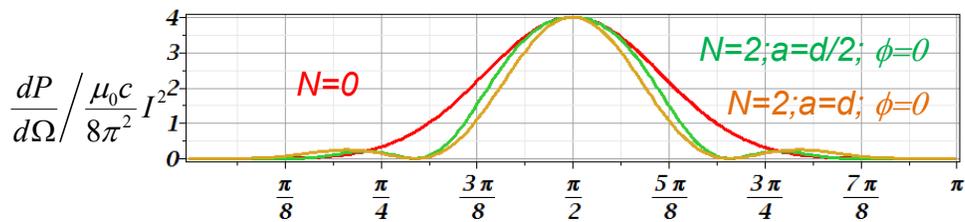
In the radiation zone :

$$\tilde{\mathbf{B}}(\mathbf{r}, \omega) = \nabla \times \tilde{\mathbf{A}}(\mathbf{r}, \omega) \approx ik\hat{\mathbf{r}} \times \tilde{\mathbf{A}}(\mathbf{r}, \omega)$$

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) \approx -ikc\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \tilde{\mathbf{A}}(\mathbf{r}, \omega))$$

$$\frac{dP}{d\Omega} = \frac{1}{2\mu_0} r^2 \hat{\mathbf{r}} \cdot \Re(\tilde{\mathbf{E}}(\mathbf{r}, \omega) \times \tilde{\mathbf{B}}^*(\mathbf{r}, \omega)) = \frac{k^2 cr^2}{2\mu_0} \left(|\tilde{\mathbf{A}}(\mathbf{r}, \omega)|^2 - |\hat{\mathbf{r}} \cdot \tilde{\mathbf{A}}(\mathbf{r}, \omega)|^2 \right)$$

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[\frac{\cos(kd \cos \theta) - \cos(kd)}{\sin \theta} \right]^2 \left[\frac{\sin(\frac{1}{2} ka(2N+1) \sin \theta \cos \phi)}{\sin(\frac{1}{2} ka \sin \theta \cos \phi)} \right]^2$$



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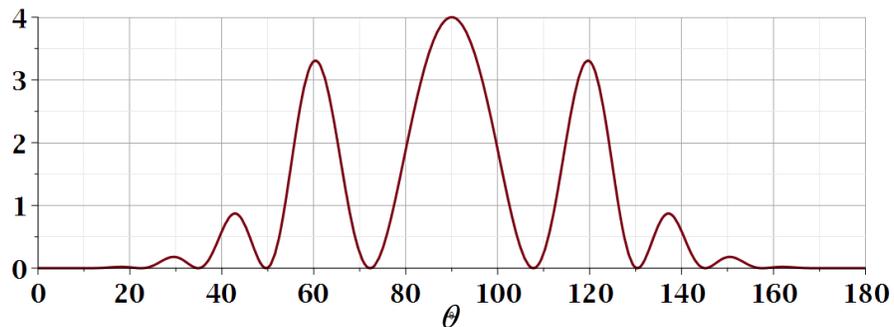
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Carrying out the integrations and simplifying the expressions, we get the results. The plots here refer to $\phi=0$, which corresponds to the observation of the radiation along the x-axis.

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[\frac{\cos(kd \cos \theta) - \cos(kd)}{\sin \theta} \right]^2 \left[\frac{\sin\left(\frac{1}{2}ka(2N+1)\sin \theta \cos \phi\right)}{\sin\left(\frac{1}{2}ka \sin \theta \cos \phi\right)} \right]^2$$

Example for $\phi = 0, N = 10, kd = \pi = 2ka$



Additional amplitude patterns can be obtained by controlling relative phases of antennas.

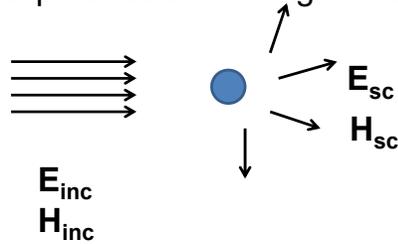
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Plot of the power for another case. Obviously, there is a lot of variety with antenna arrays which are used extensively for communications and other technologies.

Dipole radiation in light scattering by small (dielectric) particles



$$\mathbf{E}_{\text{inc}} = \hat{\mathbf{e}}_0 E_0 e^{ik\hat{\mathbf{k}}_0 \cdot \mathbf{r}} \quad \mathbf{H}_{\text{inc}} = \frac{1}{\mu_0 c} \hat{\mathbf{k}}_0 \times \mathbf{E}_{\text{inc}}$$

In electric dipole approximation :

$$\mathbf{E}_{\text{sc}} = \frac{1}{4\pi\epsilon_0} k^2 \frac{e^{ikr}}{r} ((\hat{\mathbf{r}} \times \mathbf{p}) \times \hat{\mathbf{r}}) \quad \mathbf{H}_{\text{sc}} = \frac{1}{\mu_0 c} \hat{\mathbf{r}} \times \mathbf{E}_{\text{sc}}$$

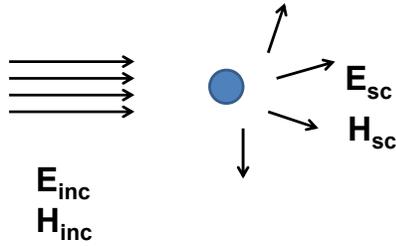
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Now consider a different radiation source – that is re-radiation from matter interacting with light (such as sunlight). Here we will simplify the analysis and assume that the matter is in the form of uniform sphere. This topic is covered in Chapter 10 of Jackson.

Dipole radiation in light scattering by small (dielectric) particles



$$\mathbf{E}_{\text{inc}} = \hat{\boldsymbol{\epsilon}}_0 E_0 e^{ik\hat{\mathbf{k}}_0 \cdot \mathbf{r}}$$

$$\mathbf{H}_{\text{inc}} = \frac{1}{\mu_0 c} \hat{\mathbf{k}}_0 \times \mathbf{E}_{\text{inc}}$$

In electric dipole approximation:

$$\mathbf{E}_{\text{sc}} = \frac{1}{4\pi\epsilon_0} k^2 \frac{e^{ikr}}{r} ((\hat{\mathbf{r}} \times \mathbf{p}) \times \hat{\mathbf{r}})$$

$$\mathbf{H}_{\text{sc}} = \frac{1}{\mu_0 c} \hat{\mathbf{r}} \times \mathbf{E}_{\text{sc}}$$

Scattering cross section :

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{r}}, \hat{\boldsymbol{\epsilon}}; \hat{\mathbf{k}}_0, \hat{\boldsymbol{\epsilon}}_0) = \frac{r^2 \hat{\mathbf{r}} \cdot \langle \mathbf{S}_{\text{sc}} \rangle_{\text{avg}}}{\hat{\mathbf{k}}_0 \cdot \langle \mathbf{S}_{\text{inc}} \rangle_{\text{avg}}}$$

$$= \frac{r^2 |\hat{\boldsymbol{\epsilon}} \cdot \mathbf{E}_{\text{sc}}|^2}{|\hat{\boldsymbol{\epsilon}}_0 \cdot \mathbf{E}_{\text{inc}}|^2} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2} |\hat{\boldsymbol{\epsilon}} \cdot \mathbf{p}|^2$$

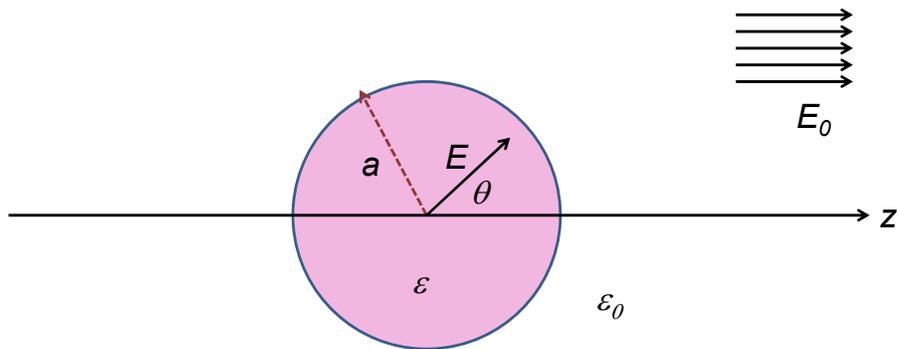
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We will assume that the incident light is in the form of an ideal plane wave, and analyze the re-radiated light as a spherical wave far from the particle itself. The unit vectors $\hat{\boldsymbol{\epsilon}}_0$ and $\hat{\boldsymbol{\epsilon}}$ reference the incident polarization of the light and the scattered polarization direction of the light, respectively. The cross section is defined as the scattered power per unit incident power.

Recall previous analysis for electrostatic case:
 Boundary value problems in the presence of dielectrics
 – example:



$$\text{At } r = a: \quad \epsilon \frac{\partial \Phi_{<}(\mathbf{r})}{\partial r} = \epsilon_0 \frac{\partial \Phi_{>}(\mathbf{r})}{\partial r}$$

$$\frac{\partial \Phi_{<}(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_{>}(\mathbf{r})}{\partial \theta}$$

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Analyzing the source of re-radiation, we need to recall how a spherical dielectric of radius a interacts with a constant electric field. We can use the results we obtained in Chapter 4 when we considered the situation as an electrostatic boundary value problem. Here the z direction is the direction of the incident electric field, not the wave vector direction.

Boundary value problems in the presence of dielectrics
 – example -- continued:

$$\Phi_{<}(\mathbf{r}) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta)$$

$$\Phi_{>}(\mathbf{r}) = \sum_{l=0}^{\infty} \left(B_l r^l + \frac{C_l}{r^{l+1}} \right) P_l(\cos \theta)$$

At $r = a$: $\varepsilon \frac{\partial \Phi_{<}(\mathbf{r})}{\partial r} = \varepsilon_0 \frac{\partial \Phi_{>}(\mathbf{r})}{\partial r}$

$$\frac{\partial \Phi_{<}(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_{>}(\mathbf{r})}{\partial \theta}$$

For $r \rightarrow \infty$ $\Phi_{>}(\mathbf{r}) = -E_0 r \cos \theta$

Solution -- only $l = 1$ contributes

$$B_1 = -E_0$$

$$A_1 = -\left(\frac{3}{2 + \varepsilon / \varepsilon_0} \right) E_0 \quad C_1 = \left(\frac{\varepsilon / \varepsilon_0 - 1}{2 + \varepsilon / \varepsilon_0} \right) a^3 E_0$$

These are the results from the electrostatic case discussed previously.

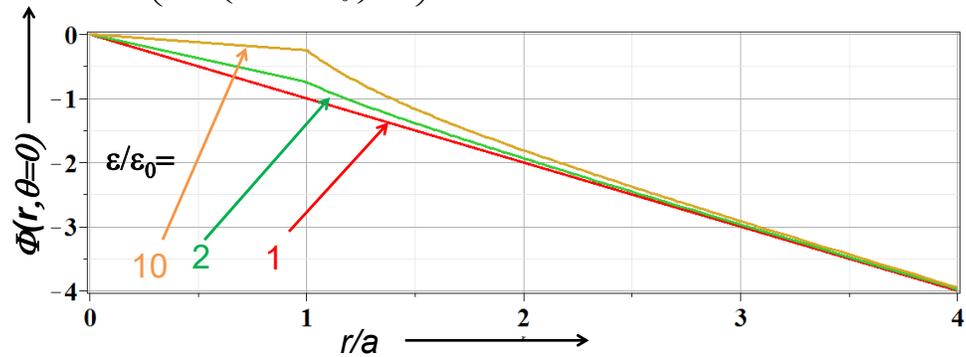
Boundary value problems in the presence of dielectrics
 – example -- continued:

$$\Phi_{<}(\mathbf{r}) = -\left(\frac{3}{2 + \epsilon/\epsilon_0}\right) E_0 r \cos \theta$$

Induced dipole moment:

$$\Phi_{>}(\mathbf{r}) = -\left(r - \left(\frac{\epsilon/\epsilon_0 - 1}{2 + \epsilon/\epsilon_0}\right) \frac{a^3}{r^2}\right) E_0 \cos \theta$$

$$\mathbf{p} = 4\pi a^3 \epsilon_0 \left(\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2}\right) \mathbf{E}_0$$



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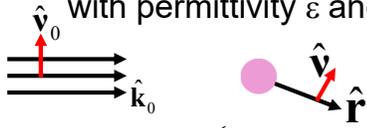
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Continued results obtained previously for the electrostatic problem.

Estimation of scattering dipole moment:

Suppose the scattering particle is a dielectric sphere with permittivity ϵ and radius a :



$$\mathbf{p} = 4\pi a^3 \epsilon_0 \left(\frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right) \mathbf{E}_{inc} \quad \mathbf{E}_{inc} = \hat{\mathbf{v}}_0 E_0 e^{ik\hat{\mathbf{k}}_0 \cdot \mathbf{r}}$$

Scattering cross section:

$$\begin{aligned} \frac{d\sigma}{d\Omega}(\hat{\mathbf{r}}, \hat{\mathbf{v}}; \hat{\mathbf{k}}_0, \hat{\mathbf{v}}_0) &= \frac{r^2 |\hat{\mathbf{v}} \cdot \mathbf{E}_{sc}|^2}{|\hat{\mathbf{v}}_0 \cdot \mathbf{E}_{inc}|^2} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2} |\hat{\mathbf{v}} \cdot \mathbf{p}|^2 \\ &= k^4 a^6 \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2 |\hat{\mathbf{v}} \cdot \hat{\mathbf{v}}_0|^2 \end{aligned}$$

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Jumping back to the scattering problem, assuming that the same mathematics can be translated to this case -- Here we have used bold epsilon to reference the polarization directions. These directions are always perpendicular to the light propagation directions. The not bold epsilons indicate the permittivity functions which are functions of the harmonic frequency of the light involved. The final result was derived by Lord Raleigh.

<https://www.britannica.com/biography/John-William-Strutt-3rd-Baron-Rayleigh>



WRITTEN BY: [R. Bruce Lindsay](#)
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Alternative Titles: John William Strutt, 3rd Baron Rayleigh of Terling Place

Lord Rayleigh, in full **John William Strutt, 3rd Baron Rayleigh of Terling Place**, (born November 12, 1842, Langford Grove, [Maldon, Essex](#), England—died June 30, 1919, Terling Place, Witham, Essex), English physical scientist who made fundamental discoveries in the fields of [acoustics](#) and [optics](#) that are basic to the theory of [wave propagation](#) in fluids. He received the [Nobel Prize](#) for Physics in 1904 for his successful isolation of argon, an inert atmospheric gas.

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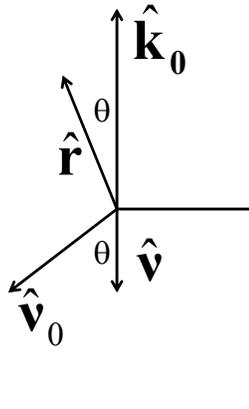
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Some information about Lord Rayleigh on the web.

Scattering by dielectric sphere with permittivity ϵ and radius a :

For \mathbf{E}_{inc} polarized in scattering plane:



$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{r}}, \hat{\mathbf{v}}; \hat{\mathbf{k}}_0, \hat{\mathbf{v}}_0) = k^4 a^6 \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2 |\hat{\mathbf{v}} \cdot \hat{\mathbf{v}}_0|^2$$

$$= k^4 a^6 \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2 \cos^2 \theta$$

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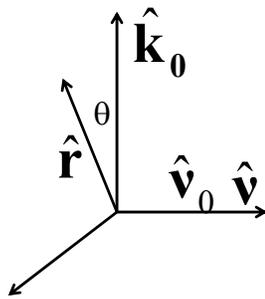
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In this analysis, we consider the case where the incident wavevector (along the vertical axis) and the polarization direction (epsilon₀) are in the same plane as the observed scattered light (direction of $\hat{\mathbf{r}}$). In this case, the dot product of the incident and scattered polarizations give a factor of $\cos(\theta)$ as show.

Scattering by dielectric sphere with permittivity ϵ and radius a :

For \mathbf{E}_{inc} polarized perpendicular to scattering plane:



$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{r}}, \hat{\mathbf{v}}; \hat{\mathbf{k}}_0, \hat{\mathbf{v}}_0) = k^4 a^6 \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2 |\hat{\mathbf{v}} \cdot \hat{\mathbf{v}}_0|^2$$

$$= k^4 a^6 \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2$$

Assuming both incident polarizations are equally likely, average cross section is given by:

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{r}}, \hat{\mathbf{v}}; \hat{\mathbf{k}}_0, \hat{\mathbf{v}}_0) = \frac{k^4 a^6}{2} \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2 (\cos^2 \theta + 1)$$

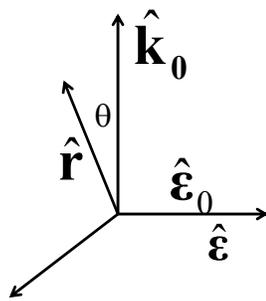
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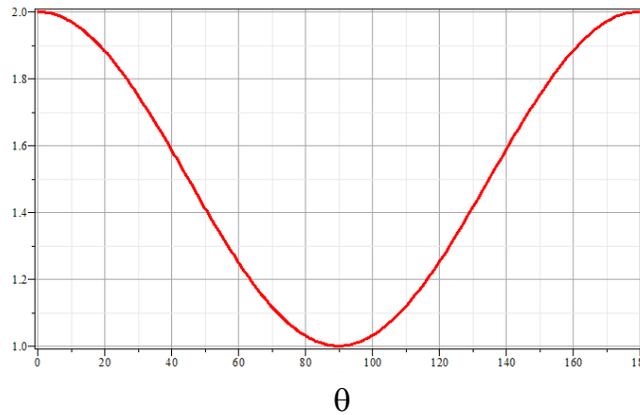
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In this case, the incident wavevector (along the vertical axis) and the observed scattered light (direction of $\hat{\mathbf{r}}$) are as before and again define the scattering plane. However, the polarization direction of incident light (ϵ_0) and the polarization direction of the scattered light (ϵ) are both perpendicular to the scattering plane and thus are parallel to each other, given 1 for their dot product. The last result indicates the cross section of the total scattered light assuming both polarizations are equally likely.

Scattering by dielectric sphere with permittivity ϵ and radius a :



$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{r}}, \hat{\boldsymbol{\epsilon}}; \hat{\mathbf{k}}_0, \hat{\boldsymbol{\epsilon}}_0) = \frac{k^4 a^6}{2} \left| \frac{\epsilon / \epsilon_0 - 1}{\epsilon / \epsilon_0 + 2} \right|^2 (\cos^2 \theta + 1)$$

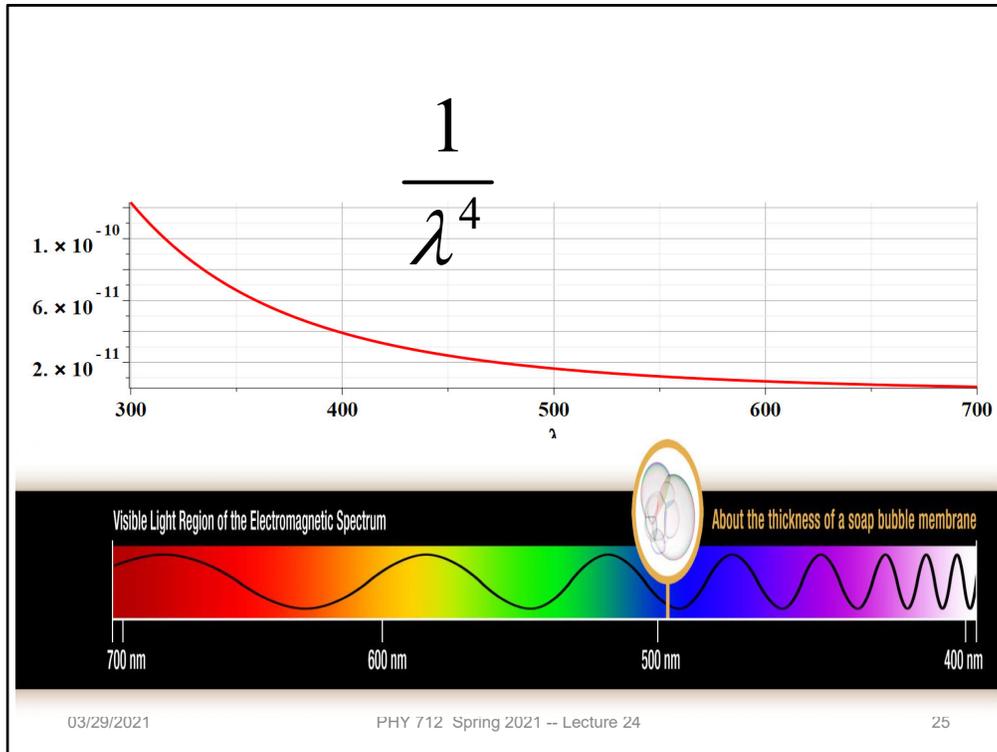


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The plot shows the angular dependence of the scattered light as a function of the angle theta.



In addition to the angular dependence of the scattered light, Raleigh scattering depends of the wavevector as k^4 which has the corresponding wavelength dependence indicated on this slide. The figure from the web shows the variation of wavelength for visible light. The analysis of Raleigh scattering thus tells us why the sky at mid day is blue and why it tends to be red at sun rise and sunset.

Brief introduction to multipole expansion of electromagnetic fields (Chap. 9.7)

Sourceless Maxwell's equations

in terms of \mathbf{E} and \mathbf{H} fields with time dependence $e^{-i\omega t}$:

$$\nabla \times \mathbf{E} = ikZ_0 \mathbf{H} \quad \nabla \times \mathbf{H} = -ik\mathbf{E} / Z_0$$

$$\nabla \cdot \mathbf{E} = 0 \quad \nabla \cdot \mathbf{H} = 0$$

where $k \equiv \omega / c$ and $Z_0 \equiv \sqrt{\mu_0 / \epsilon_0}$

Decoupled equations:

$$(\nabla^2 + k^2)\mathbf{E} = 0 \quad (\nabla^2 + k^2)\mathbf{H} = 0$$

$$\mathbf{H} = -\frac{i}{kZ_0} \nabla \times \mathbf{E} \quad \mathbf{E} = \frac{iZ_0}{k} \nabla \times \mathbf{H}$$

In the next few slides, we go over material presented in Section 9.7 of your textbook. I have personally never used this formalism, but recognize it as a powerful tool for analyzing fields from localized sources in terms of the fields themselves rather than using scalar and vector and scalar potentials. Please review this material as time permits.

Multipole expansion of electromagnetic fields -- continued

Note that:

$$(\nabla^2 + k^2)(\mathbf{r} \cdot \mathbf{E}) = 0 \quad (\nabla^2 + k^2)(\mathbf{r} \cdot \mathbf{H}) = 0$$

Convenient operators for angular momentum analysis

$$\text{Define: } \mathbf{L} \equiv \frac{1}{i}(\mathbf{r} \times \nabla)$$

Note that $\mathbf{r} \cdot \mathbf{L} = 0$

$$\nabla^2 = \frac{1}{r} \frac{\partial^2 r}{\partial r^2} - \frac{L^2}{r^2}$$

Eigenfunctions:

$$L^2 Y_{lm}(\theta, \phi) = - \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] Y_{lm}(\theta, \phi) = l(l+1) Y_{lm}(\theta, \phi)$$

Multipole expansion of electromagnetic fields -- continued

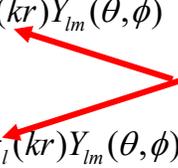
Magnetic multipole field:

$$\mathbf{r} \cdot \mathbf{H}_{lm}^M \equiv \frac{l(l+1)}{k} g_l(kr) Y_{lm}(\theta, \phi)$$

$$\mathbf{r} \cdot \mathbf{E}_{lm}^M = 0$$

$$\mathbf{L} \cdot \mathbf{E}_{lm}^M = l(l+1) Z_0 g_l(kr) Y_{lm}(\theta, \phi)$$

spherical Bessel function



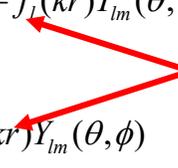
Electric multipole field:

$$\mathbf{r} \cdot \mathbf{E}_{lm}^E \equiv -Z_0 \frac{l(l+1)}{k} f_l(kr) Y_{lm}(\theta, \phi)$$

$$\mathbf{r} \cdot \mathbf{H}_{lm}^E = 0$$

$$\mathbf{L} \cdot \mathbf{H}_{lm}^E = l(l+1) f_l(kr) Y_{lm}(\theta, \phi)$$

spherical Bessel function



Multipole expansion of electromagnetic fields -- continued

Vector spherical harmonics: (for $l > 0$)

$$\mathbf{X}_{lm}(\theta, \phi) = \frac{1}{\sqrt{l(l+1)}} \mathbf{L} Y_{lm}(\theta, \phi)$$

Orthogonality conditions:

$$\int d\Omega \mathbf{X}_{l'm'}^*(\theta, \phi) \cdot \mathbf{X}_{lm}(\theta, \phi) = \delta_{ll'} \delta_{mm'}$$

$$\int d\Omega \mathbf{X}_{l'm'}^*(\theta, \phi) \cdot (\mathbf{r} \times \mathbf{X}_{lm}(\theta, \phi)) = 0$$

General expansion of fields:

$$\mathbf{H} = \sum_{lm} \left[a_{lm}^E f_l(kr) \mathbf{X}_{lm}(\theta, \phi) - \frac{i}{k} a_{lm}^M \nabla \times (g_l(kr) \mathbf{X}_{lm}(\theta, \phi)) \right]$$

$$\mathbf{E} = \sum_{lm} \left[\frac{i}{k} a_{lm}^E \nabla \times (f_l(kr) \mathbf{X}_{lm}(\theta, \phi)) + a_{lm}^M g_l(kr) \mathbf{X}_{lm}(\theta, \phi) \right]$$

Multipole expansion of electromagnetic fields -- continued

Time averaged power distribution of radiation far from source:

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2} \left| \sum_{lm} (-i)^{l+1} \left[a_{lm}^E \mathbf{X}_{lm}(\theta, \phi) \times \hat{\mathbf{r}} + a_{lm}^M \mathbf{X}_{lm}(\theta, \phi) \right] \right|^2$$

For a pure multipole radiation with either a_{lm}^E or a_{lm}^M :

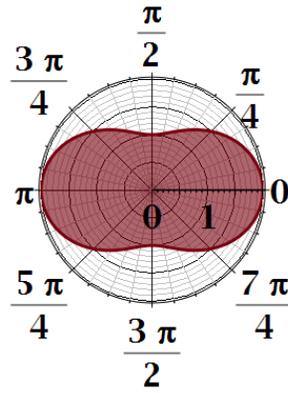
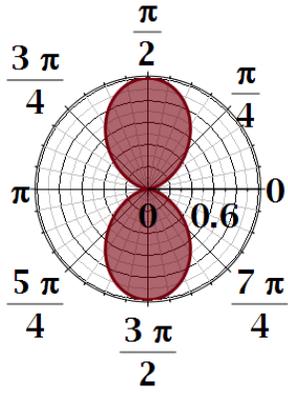
$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2} |a_{lm}|^2 |\mathbf{X}_{lm}(\theta, \phi)|^2$$

$$|\mathbf{X}_{lm}(\theta, \phi)|^2 = \frac{1}{2l(l+1)} \left(2m^2 |Y_m|^2 + (l+m)(l-m+1) |Y_{l(m-1)}|^2 + (l-m)(l+m+1) |Y_{l(m+1)}|^2 \right)$$

For example: $l = 1$

$$|X_{10}(\theta, \phi)|^2 = \frac{3}{8\pi} \sin^2 \theta$$

$$|X_{11}(\theta, \phi)|^2 = |X_{1-1}(\theta, \phi)|^2 = \frac{3}{16\pi} (1 + \cos^2 \theta)$$



For example: $l = 2$

$$|X_{20}(\theta, \phi)|^2 = \frac{15}{8\pi} \sin^2 \theta \cos^2 \theta \quad |X_{21}(\theta, \phi)|^2 = \frac{5}{16\pi} (1 - 3\cos^2 \theta + 4\cos^4 \theta) \quad |X_{22}(\theta, \phi)|^2 = \frac{5}{16\pi} (1 - \cos^4 \theta)$$

