PHY 712 Electrodynamics
10-10:50 AM MWF Online

Discussion for Lecture 24:

Complete reading of Chap. 9 & 10

A. Superposition of radiation

B. Scattered radiation

				Exam	
21	Mon: 03/22/2021	Chap. 8	EM waves in wave guides		
22	Wed: 03/24/2021	Chap. 9	Radiation from localized oscillating sources	#15	03/26/2021
23	Fri: 03/26/2021	Chap. 9	Radiation from oscillating sources	#16	03/29/2021
24	Mon: 03/29/2021	Chap. 9 & 10	Radiation and scattering	#17	03/31/2021
25	Wed: 03/31/2021	Chap. 11	Special Theory of Relativity		
26	Fri: 04/02/2021	Chap. 11	Special Theory of Relativity		

PHY 712 -- Assignment #17

March 29, 2021

Finish reading Chapters 9 and 10 in **Jackson** .

1. Work problem 9.16(a) in **Jackson**. Note that you can use an approach similar to that discussed in Section 9.4 of the textbook, replacing the "center-fed" antenna with the given antenna configuration.

Ph.D. DEFENSE

TUESDAY • MARCH 30, 2021

"Dynamics of Charge Carrier Traps in Organic Semiconductors"

Hamna Iqbal

Dr. Oana Jurchescu, Advisor

Department of Physics

Wake Forest University

Oral Defense

9:00 am – 11:00 am EST

Organic semiconductors (OSCs) are becoming an integral part of our lives as active components of various optoelectronic devices given their low-cost processing, light weight, chemical versatility by molecular design, and compatibility with flexible substrates. These systems undergo considerable electronic and structural transformations during device fabrication and operation, which can profoundly impact their performance and stability. Characterization techniques that can elucidate the mechanisms of the time-dependent transformations occurring in these materials and devices are needed to guide the design and processing to yield high-performance and stable devices. In this dissertation, a highly efficient method is introduced to elucidate the microscopic processes occurring within

Your questions –
From Gao -- What factors decide whether the electric dipole radiates or not? I found the famula is the same in electrostatics and radiation, and I am [wondering how this is justified].

Comments about HW 15 & 16 -- If you are stuck on the last part of the question, using Jackson's clever factorization, don't worry about that. However, for the record, note that $k=\omega_0/c$.

Electromagnetic waves from time harmonic sources – review:

$$\text{For scalar potential (Lorentz gauge, } k \equiv \frac{\omega}{c}\text{)}$$

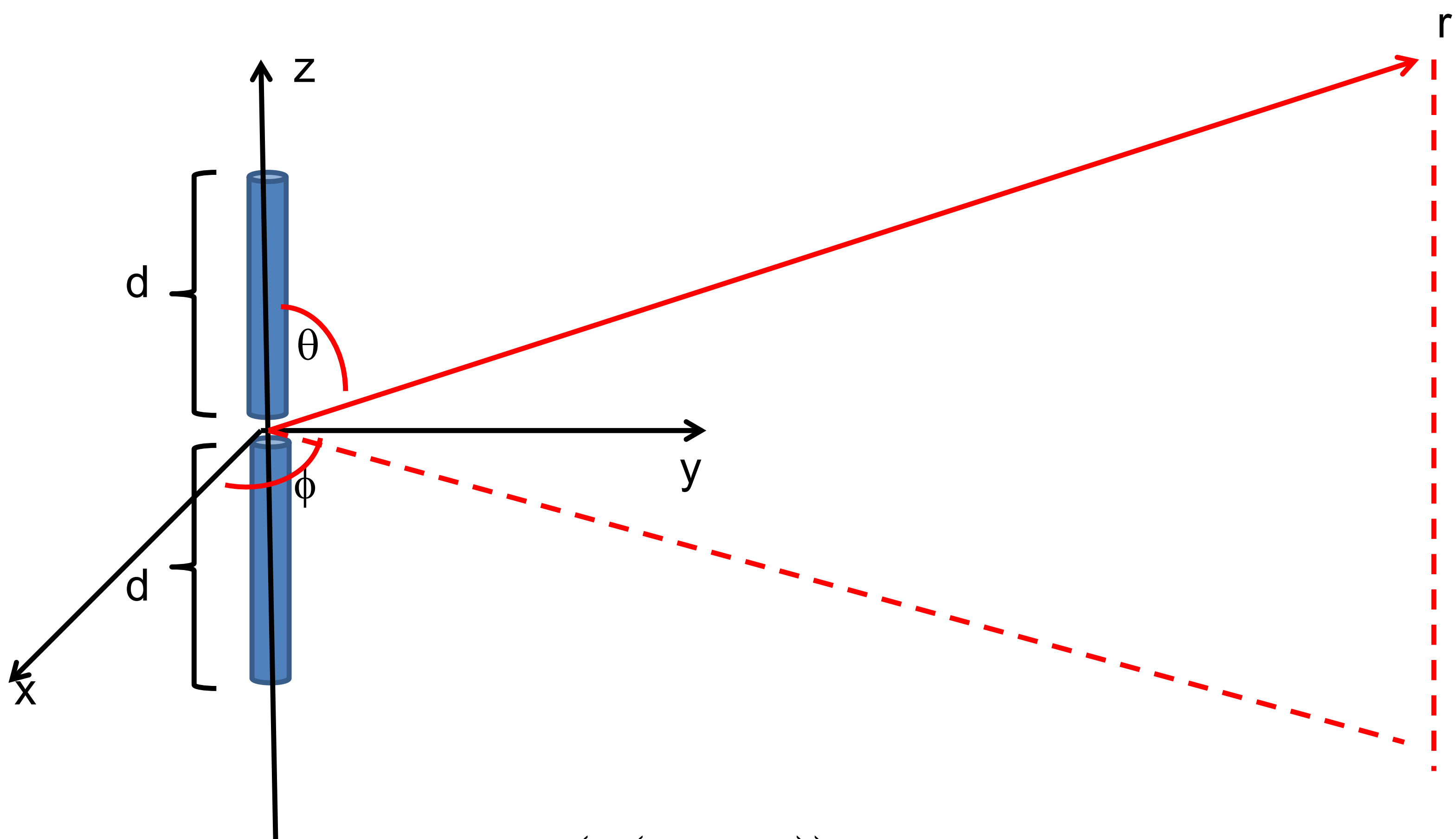
$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\rho}(\mathbf{r}',\omega)$$

$$\text{For vector potential (Lorentz gauge, } k \equiv \frac{\omega}{c}\text{)}$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \tilde{\mathbf{A}}_0(\mathbf{r},\omega) + \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

Consider antenna source (center-fed)

Note – these notes differ from previous formulation d/2 ←→ d



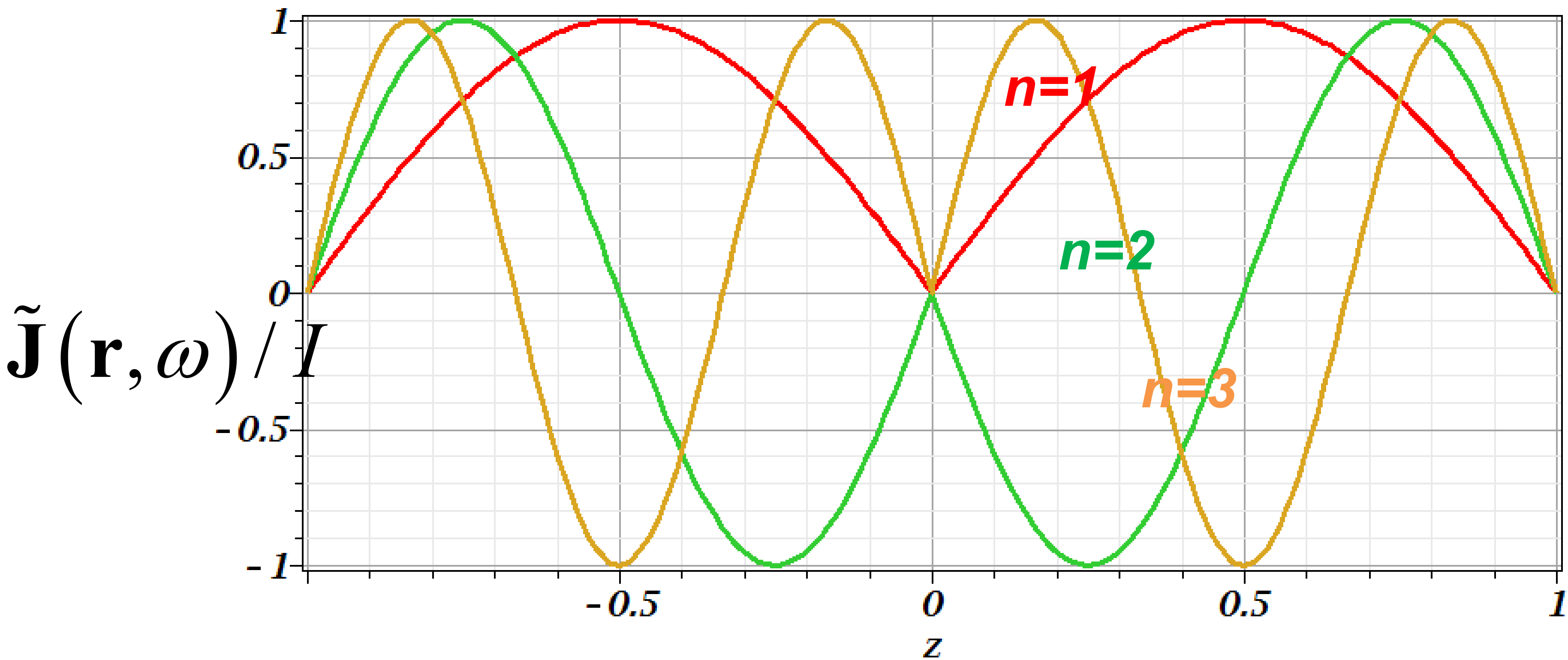
$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} I \sin\left(k\left(d - |z|\right)\right)\delta(x)\delta(y) \quad \text{for } -d \le z \le d$$

$$k \equiv \frac{\omega}{c}$$

Consider antenna source -- continued

$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} I \sin\left(k\left(d - |z|\right)\right)\delta(x)\delta(y) \qquad \text{for} \quad -d \le z \le d$$

$$\text{for} \quad k \equiv \frac{\omega}{c} = \frac{n\pi}{d}; \qquad n = 1, 2, 3....$$



Consider antenna source -- continued

$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} I \sin\left(k\left(d-|z|\right)\right)\delta(x)\delta(y) \quad \text{for} \quad -d \le z \le d$$

$$k \equiv \frac{\omega}{c}$$

Vector potential from source:

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

For $r >> d$ $\qquad \tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r' e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r} I \int_{-d}^{d} dz' \; e^{-ikz'\cos\theta} \quad \sin\left(k\left(d-|z'|\right)\right)$$

Consider antenna source -- continued

$$\widetilde{\mathbf{A}}(\mathbf{r},\omega) \approx \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r} I \int_{-d}^{d} dz\, e^{-ikz\cos\theta} \;\; \sin\left(k\left(d-|z|\right)\right)$$

$$= \;\hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{e^{ikr}}{kr} 2I\left[\frac{\cos(kd\cos\theta)-\cos(kd)}{\sin^2\theta}\right]$$

In the radiation zone:

$$\widetilde{\mathbf{B}}(\mathbf{r},\omega) = \nabla\times\widetilde{\mathbf{A}}(\mathbf{r},\omega) \approx ik\hat{\mathbf{r}}\times\widetilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$\widetilde{\mathbf{E}}(\mathbf{r},\omega) \approx -ikc\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\widetilde{\mathbf{A}}(\mathbf{r},\omega)\right)$$

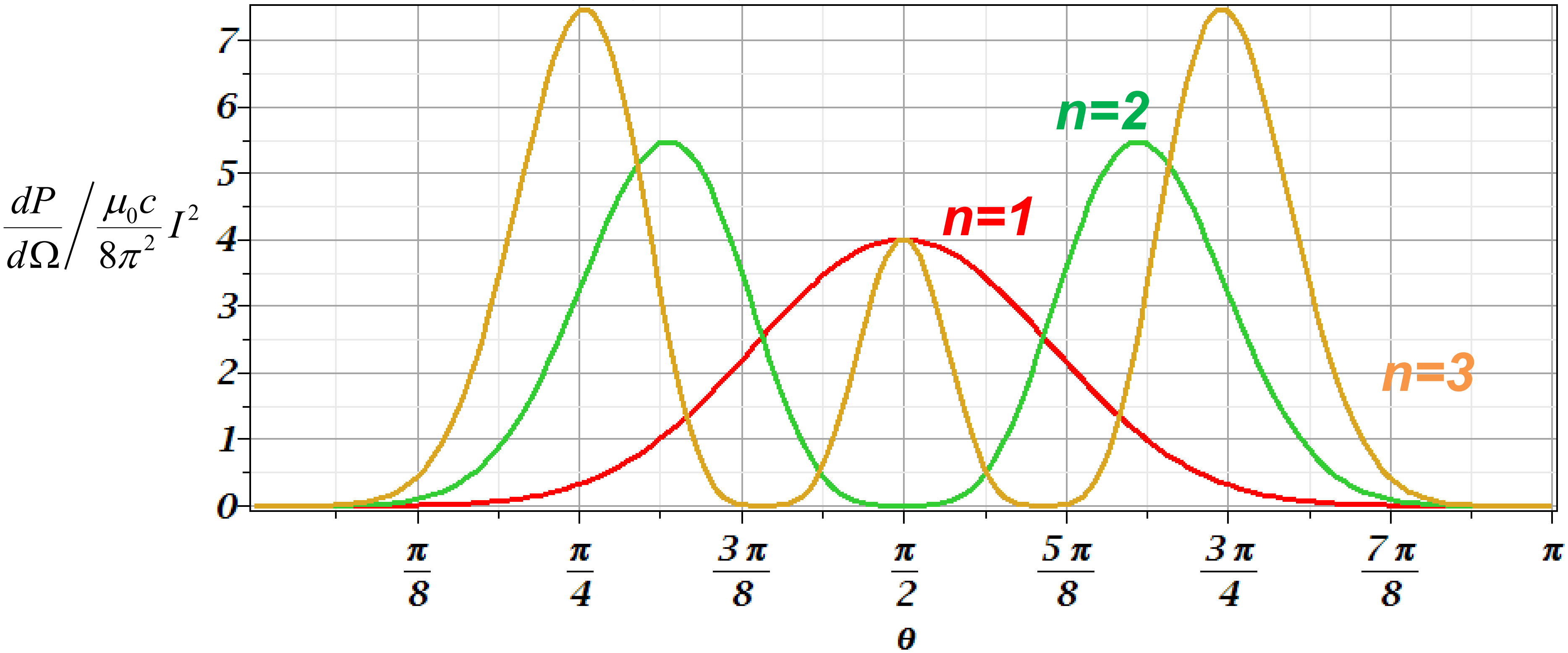
$$\frac{dP}{d\Omega} = \frac{1}{2\mu_0} r^2\hat{\mathbf{r}}\cdot\Re\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)\times\widetilde{\mathbf{B}}^*(\mathbf{r},\omega)\right) = \frac{k^2 c}{2\mu_0} r^2\left(\left|\widetilde{\mathbf{A}}(\mathbf{r},\omega)\right|^2 - \left|\hat{\mathbf{r}}\cdot\widetilde{\mathbf{A}}(\mathbf{r},\omega)\right|^2\right)$$

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2\left[\frac{\cos(kd\cos\theta)-\cos(kd)}{\sin\theta}\right]^2$$

Consider antenna source -- continued

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta} \right]^2$$

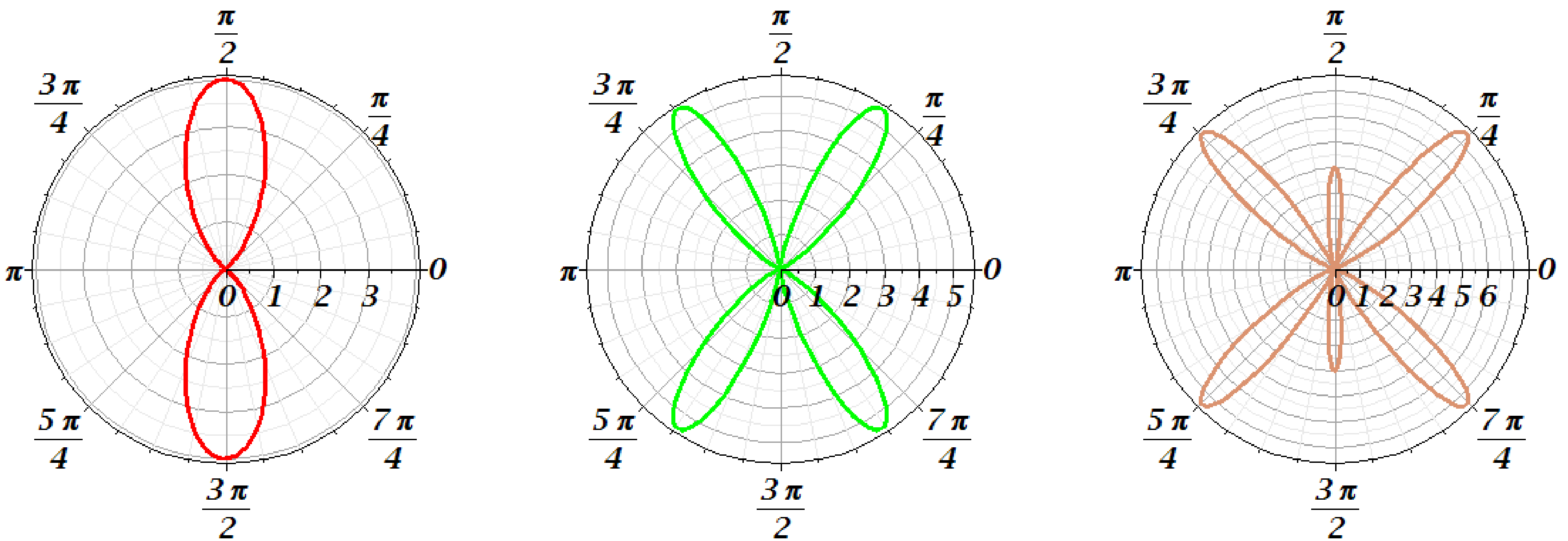
$$\text{for } k \equiv \frac{\omega}{c} = \frac{n\pi}{d}; \qquad n = 1, 2, 3....$$



Consider antenna source -- continued

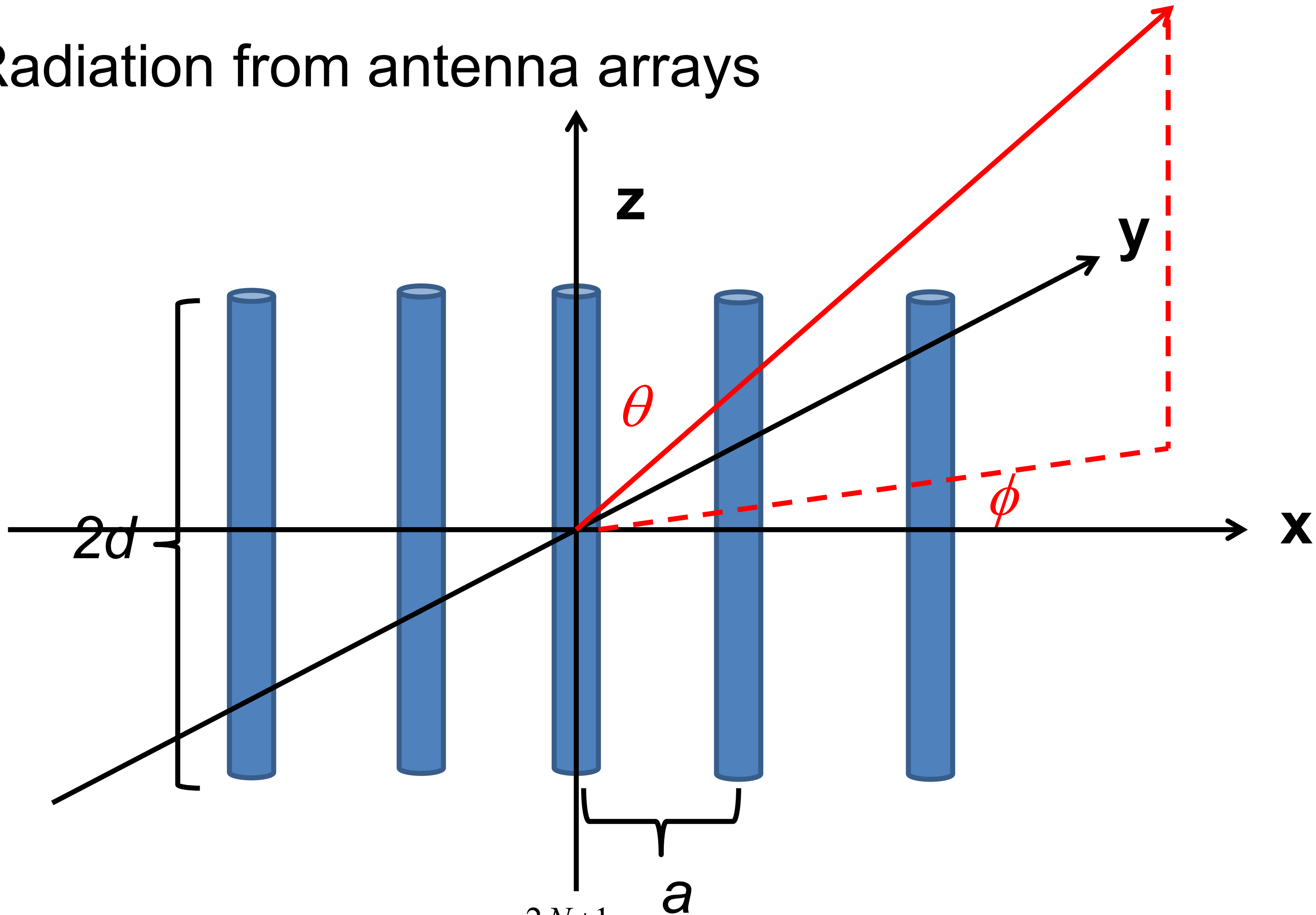
$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta} \right]^2$$

For $kd = n\pi$:



n=1 ***n=2*** ***n=3***

Radiation from antenna arrays



$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} I \sin\left(k\left(d-|z|\right)\right) \sum_{j=1}^{2N+1} \delta\left(x-(N+1-j)a\right)\delta(y) \quad \text{for } -d \le z \le d$$

$$k \equiv \frac{\omega}{c} = \frac{n\pi}{d}; \qquad n = 1,2,3\ldots.$$

Radiation from antenna arrays -- continued

Vector potential from array source :

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)=\frac{\mu_0}{4\pi}\int d^3r'\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)\approx\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r' e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{J}}(\mathbf{r},\omega)=\hat{\mathbf{z}}I\sin\big(k(d-|z|)\big)\sum_{j=1}^{2N+1}\delta\big(x-(N+1-j)a\big)\delta(y)\quad \text{for } -d\le z\le d$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)\approx\hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\sum_{j=-N}^{N}e^{-ikaj\sin\theta\cos\phi}\right)I\int_{-d}^{d}dz\, e^{-ikz\cos\theta}\ \sin\big(k(d-|z|)\big)$$

$$\sum_{j=-N}^{N}e^{-ikaj\sin\theta\cos\phi}=\frac{\sin\left(\frac{1}{2}ka(2N+1)\sin\theta\cos\phi\right)}{\sin\left(\frac{1}{2}ka\sin\theta\cos\phi\right)}$$

Radiation from antenna arrays -- continued

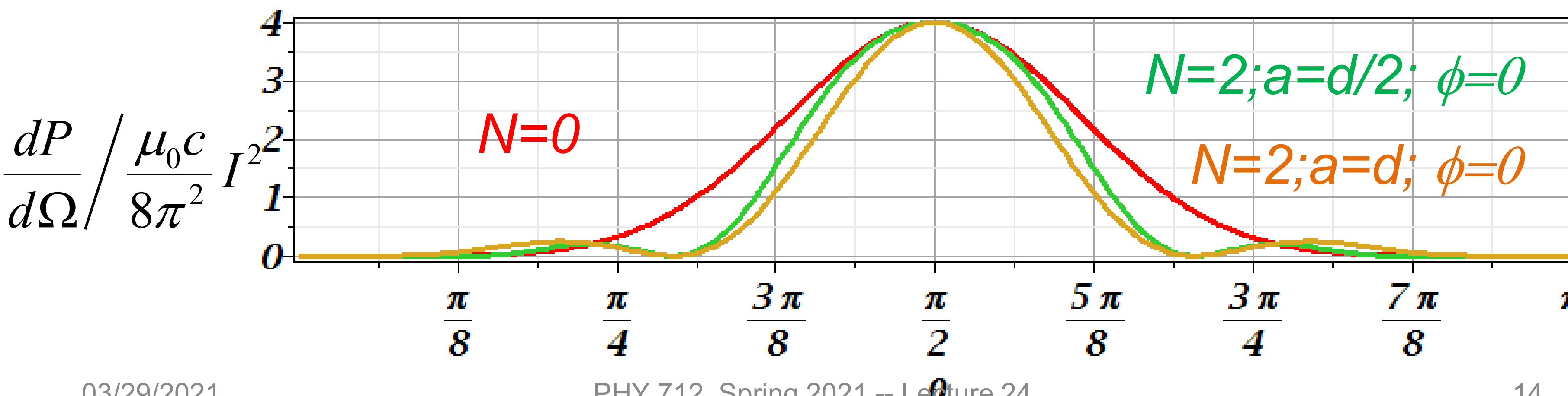
In the radiation zone:

$$\widetilde{\mathbf{B}}(\mathbf{r},\omega)=\nabla\times\widetilde{\mathbf{A}}(\mathbf{r},\omega)\approx ik\hat{\mathbf{r}}\times\widetilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$\widetilde{\mathbf{E}}(\mathbf{r},\omega)\approx -ikc\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\widetilde{\mathbf{A}}(\mathbf{r},\omega)\right)$$

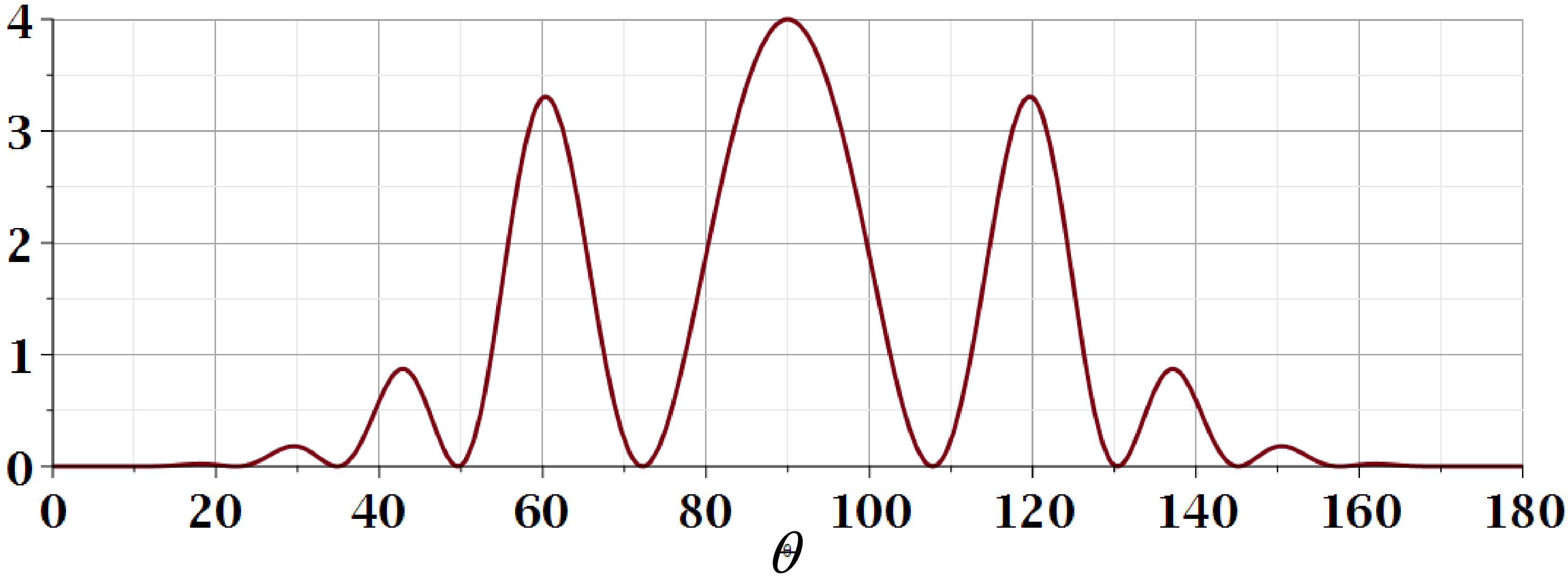
$$\frac{dP}{d\Omega}=\frac{1}{2\mu_0}r^2\hat{\mathbf{r}}\cdot\Re\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)\times\widetilde{\mathbf{B}}^*(\mathbf{r},\omega)\right)=\frac{k^2cr^2}{2\mu_0}\left(\left|\widetilde{\mathbf{A}}(\mathbf{r},\omega)\right|^2-\left|\hat{\mathbf{r}}\cdot\widetilde{\mathbf{A}}(\mathbf{r},\omega)\right|^2\right)$$

$$\frac{dP}{d\Omega}=\frac{\mu_0 c}{8\pi^2}I^2\left[\frac{\cos(kd\cos\theta)-\cos(kd)}{\sin\theta}\right]^2\left[\frac{\sin\left(\frac{1}{2}ka(2N+1)\sin\theta\cos\phi\right)}{\sin\left(\frac{1}{2}ka\sin\theta\cos\phi\right)}\right]^2$$



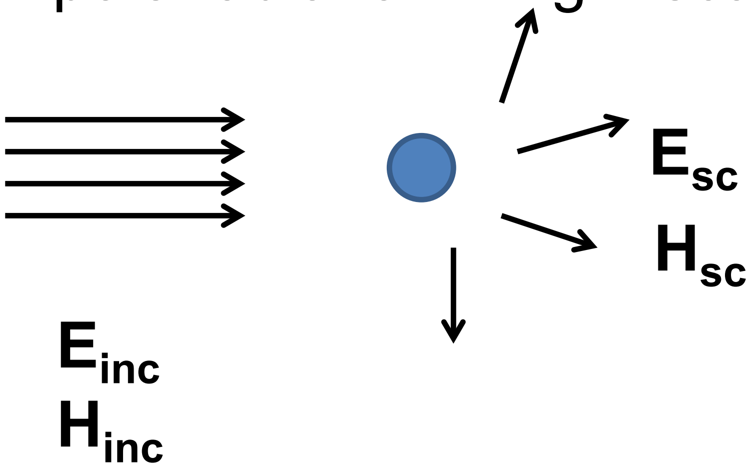
$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta} \right]^2 \left[\frac{\sin\left(\frac{1}{2}ka(2N+1)\sin\theta\cos\varphi\right)}{\sin\left(\frac{1}{2}ka\sin\theta\cos\varphi\right)} \right]^2$$

Example for $\phi = 0, N = 10, \quad kd = \pi = 2ka$



Additional amplitude patterns can be obtained by controlling relative phases of antennas.

Dipole radiation in light scattering by small (dielectric) particles

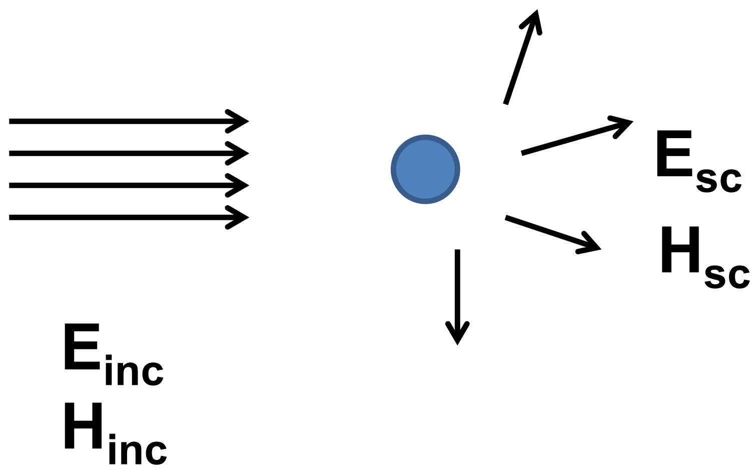


$$\mathbf{E}_{\mathrm{inc}} = \hat{\boldsymbol{\varepsilon}}_0 E_0 e^{ik\hat{\mathbf{k}}_0 \cdot \mathbf{r}} \qquad \mathbf{H}_{\mathrm{inc}} = \frac{1}{\mu_0 c}\hat{\mathbf{k}}_0 \times \mathbf{E}_{\mathrm{inc}}$$

In electric dipole approximation :

$$\mathbf{E}_{\mathrm{sc}} = \frac{1}{4\pi\varepsilon_0} k^2 \frac{e^{ikr}}{r}\left(\left(\hat{\mathbf{r}} \times \mathbf{p}\right) \times \hat{\mathbf{r}}\right) \qquad \mathbf{H}_{\mathrm{sc}} = \frac{1}{\mu_0 c}\hat{\mathbf{r}} \times \mathbf{E}_{\mathrm{sc}}$$

Dipole radiation in light scattering by small (dielectric) particles



$$\mathbf{E}_{\text{inc}} = \hat{\boldsymbol{\varepsilon}}_0 E_0 e^{ik\hat{\mathbf{k}}_0\cdot\mathbf{r}}$$

$$\mathbf{H}_{\text{inc}} = \frac{1}{\mu_0 c}\hat{\mathbf{k}}_0 \times \mathbf{E}_{\text{inc}}$$

In electric dipole approximation:

$$\mathbf{E}_{\text{sc}} = \frac{1}{4\pi\varepsilon_0} k^2 \frac{e^{ikr}}{r}\left((\hat{\mathbf{r}}\times\mathbf{p})\times\hat{\mathbf{r}}\right)$$

$$\mathbf{H}_{\text{sc}} = \frac{1}{\mu_0 c}\hat{\mathbf{r}}\times\mathbf{E}_{\text{sc}}$$

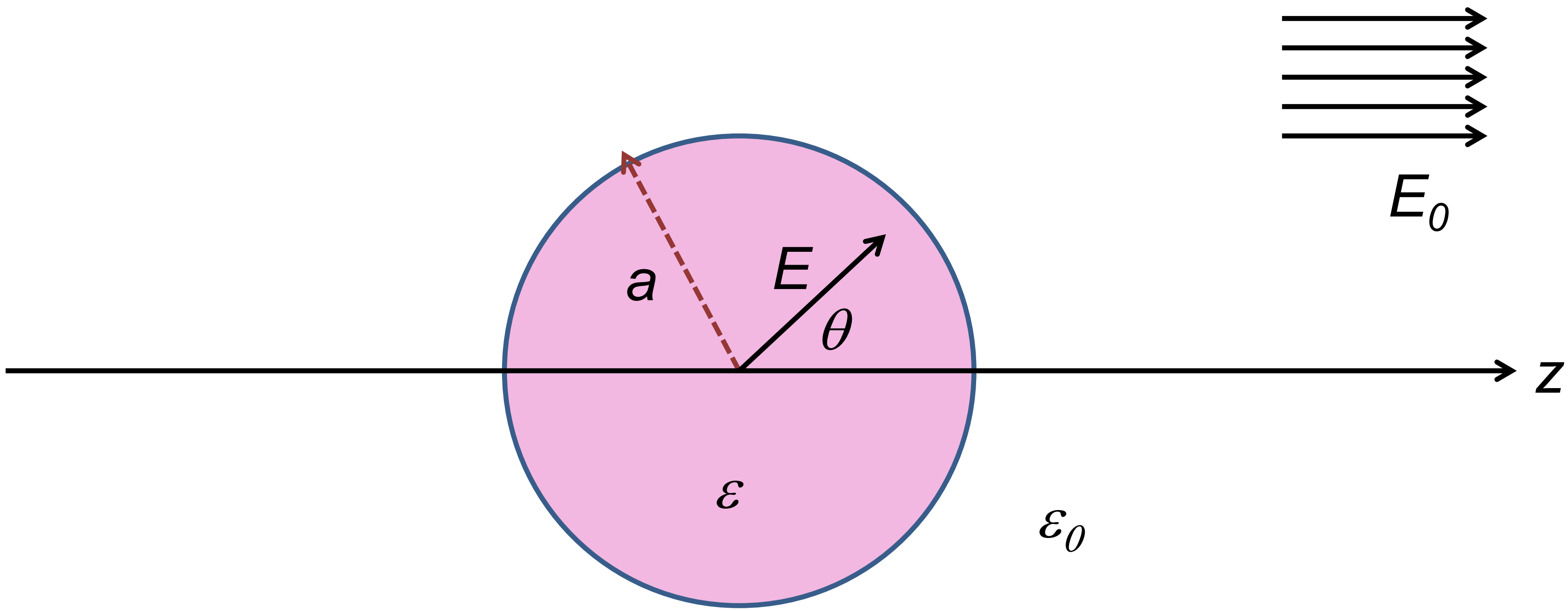
Scattering cross section :

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\boldsymbol{\varepsilon}};\hat{\mathbf{k}}_0,\hat{\boldsymbol{\varepsilon}}_0\right) = \frac{r^2\hat{\mathbf{r}}\cdot\left\langle \mathbf{S}_{sc}\right\rangle_{avg}}{\hat{\mathbf{k}}_0\cdot\left\langle \mathbf{S}_{inc}\right\rangle_{avg}}$$

$$= \frac{r^2\left|\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{E}_{sc}\right|^2}{\left|\hat{\boldsymbol{\varepsilon}}_0\cdot\mathbf{E}_{inc}\right|^2} = \frac{k^4}{\left(4\pi\varepsilon_0 E_0\right)^2}\left|\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{p}\right|^2$$

Recall previous analysis for electrostatic case:

Boundary value problems in the presence of dielectrics – example:



$$\text{At } r = a: \quad \varepsilon \frac{\partial \Phi_<(\mathbf{r})}{\partial r} = \varepsilon_0 \frac{\partial \Phi_>(\mathbf{r})}{\partial r}$$

$$\frac{\partial \Phi_<(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_>(\mathbf{r})}{\partial \theta}$$

Boundary value problems in the presence of dielectrics – example -- continued:

$$\Phi_<(\mathbf{r}) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$\Phi_>(\mathbf{r}) = \sum_{l=0}^{\infty} \left(B_l r^l + \frac{C_l}{r^{l+1}} \right) P_l(\cos\theta)$$

At $r = a$: $\varepsilon \dfrac{\partial \Phi_<(\mathbf{r})}{\partial r} = \varepsilon_0 \dfrac{\partial \Phi_>(\mathbf{r})}{\partial r}$

$$\frac{\partial \Phi_<(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_>(\mathbf{r})}{\partial \theta}$$

For $r \to \infty$ $\Phi_>(\mathbf{r}) = -E_0 r \cos\theta$

Solution -- only $l = 1$ contributes

$$B_1 = -E_0$$

$$A_1 = -\left(\frac{3}{2 + \varepsilon/\varepsilon_0} \right) E_0 \qquad C_1 = \left(\frac{\varepsilon/\varepsilon_0 - 1}{2 + \varepsilon/\varepsilon_0} \right) a^3 E_0$$

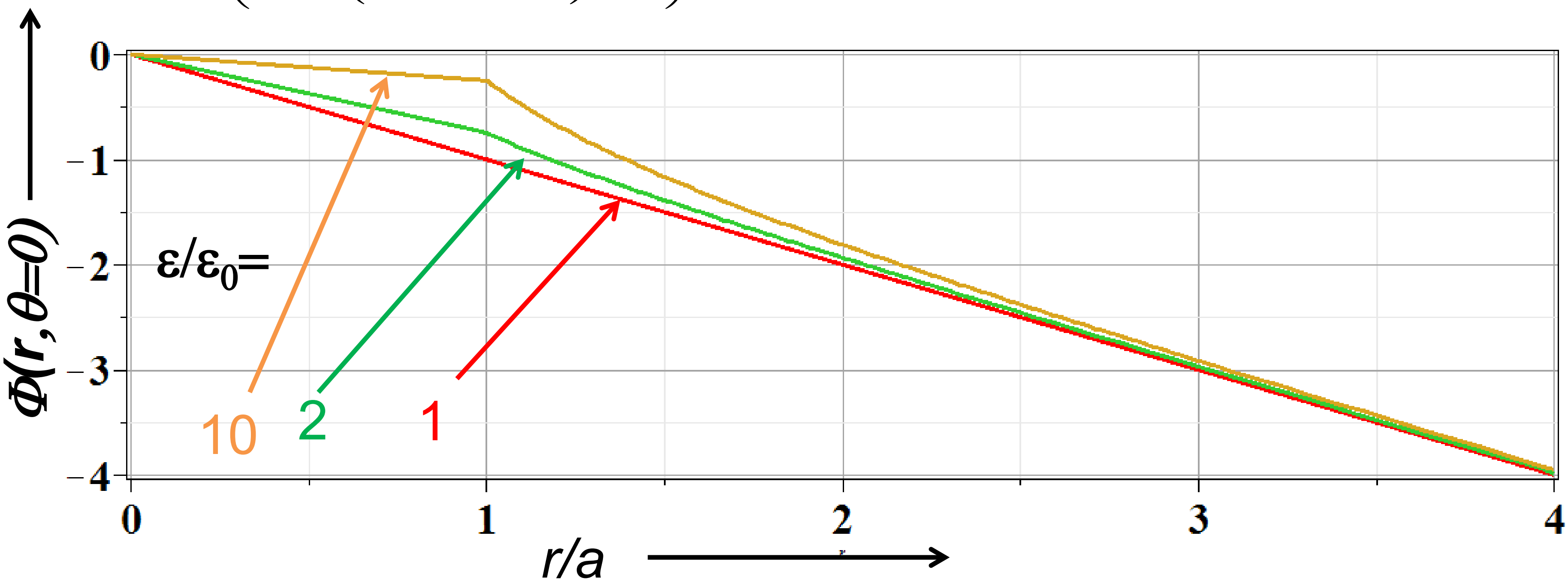
Boundary value problems in the presence of dielectrics – example -- continued:

$$\Phi_<(\mathbf{r}) = -\left(\frac{3}{2+\varepsilon/\varepsilon_0}\right)E_0\, r\cos\theta$$

$$\Phi_>(\mathbf{r}) = -\left(r - \left(\frac{\varepsilon/\varepsilon_0 - 1}{2+\varepsilon/\varepsilon_0}\right)\frac{a^3}{r^2}\right)E_0\cos\theta$$

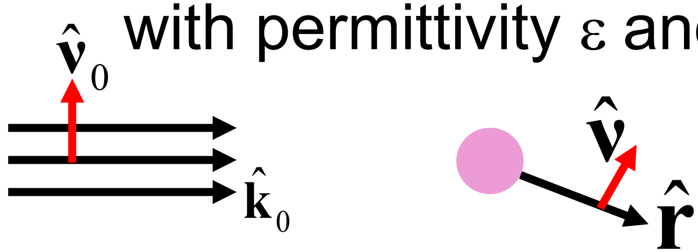
Induced dipole moment:

$$\mathbf{p} = 4\pi a^3\varepsilon_0\left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right)\mathbf{E}_0$$



Estimation of scattering dipole moment:

Suppose the scattering particle is a dielectric sphere with permittivity ε and radius *a*:



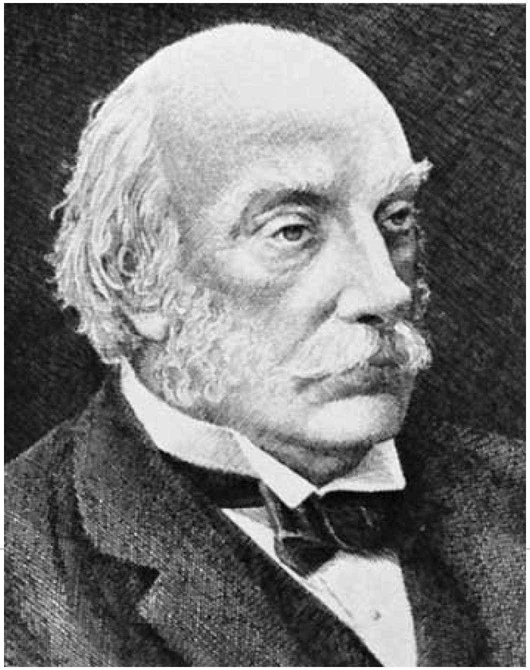
$$\mathbf{p} = 4\pi a^3 \varepsilon_0 \left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right) \mathbf{E}_{inc} \qquad \mathbf{E}_{\text{inc}} = \hat{\mathbf{v}}_0 E_0 e^{ik\hat{\mathbf{k}}_\mathbf{0}\cdot\mathbf{r}}$$

Scattering cross section:

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}}, \hat{\mathbf{v}}; \hat{\mathbf{k}}_\mathbf{0}, \hat{\mathbf{v}}_0\right) = \frac{r^2 \left|\hat{\mathbf{v}} \cdot \mathbf{E}_{sc}\right|^2}{\left|\hat{\mathbf{v}}_0 \cdot \mathbf{E}_{inc}\right|^2} = \frac{k^4}{\left(4\pi\varepsilon_0 E_0\right)^2} \left|\hat{\mathbf{v}} \cdot \mathbf{p}\right|^2$$

$$= k^4 a^6 \left| \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right|^2 \left|\hat{\mathbf{v}} \cdot \hat{\mathbf{v}}_0\right|^2$$

https://www.britannica.com/biography/John-William-Strutt-3rd-Baron-Rayleigh

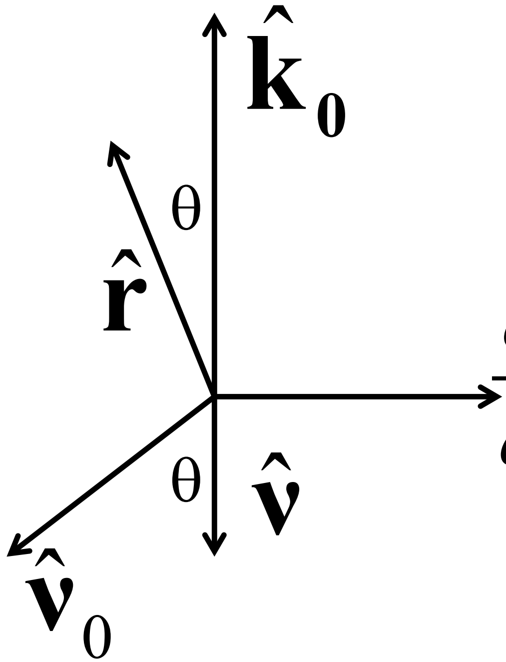


WRITTEN BY: R. Bruce Lindsay
See Article History

Alternative Titles: John William Strutt, 3rd Baron Rayleigh of Terling Place

Lord Rayleigh, in full **John William Strutt, 3rd Baron Rayleigh of Terling Place**, (born November 12, 1842, Langford Grove, Maldon, Essex, England—died June 30, 1919, Terling Place, Witham, Essex), English physical scientist who made fundamental discoveries in the fields of acoustics and optics that are basic to the theory of wave propagation in fluids. He received the Nobel Prize for Physics in 1904 for his successful isolation of argon, an inert atmospheric gas.

Scattering by dielectric sphere with permittivity ε and radius *a*:

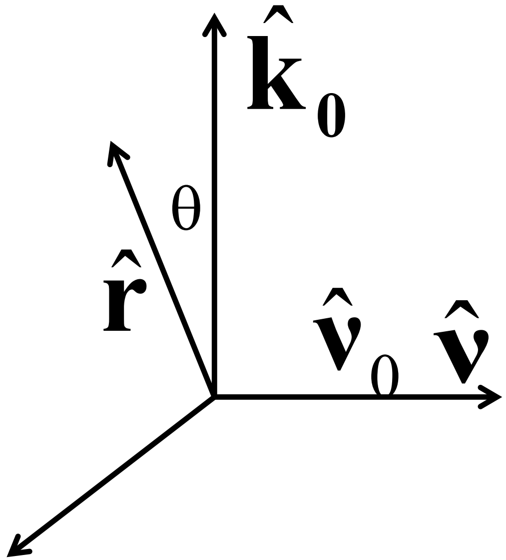


For $\mathbf{E}_{\mathbf{inc}}$ polarized in scattering plane:

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\mathbf{v}};\hat{\mathbf{k}}_0,\hat{\mathbf{v}}_0\right) == k^4 a^6 \left|\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right|^2 \left|\hat{\mathbf{v}}\cdot\hat{\mathbf{v}}_0\right|^2$$

$$= k^4 a^6 \left|\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right|^2 \cos^2\theta$$

Scattering by dielectric sphere with permittivity ε and radius *a*:



For $\mathbf{E}_{inc}$ polarized perpendicular to scattering plane:

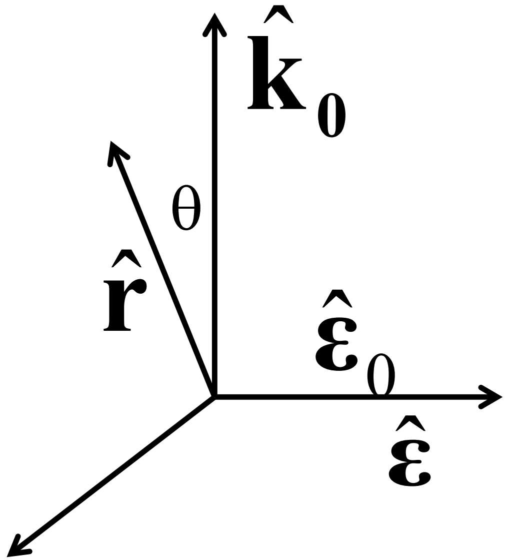
$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\mathbf{v}};\hat{\mathbf{k}}_0,\hat{\mathbf{v}}_0\right)=k^4a^6\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2\left|\hat{\mathbf{v}}\cdot\hat{\mathbf{v}}_0\right|^2$$

$$=k^4a^6\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2$$

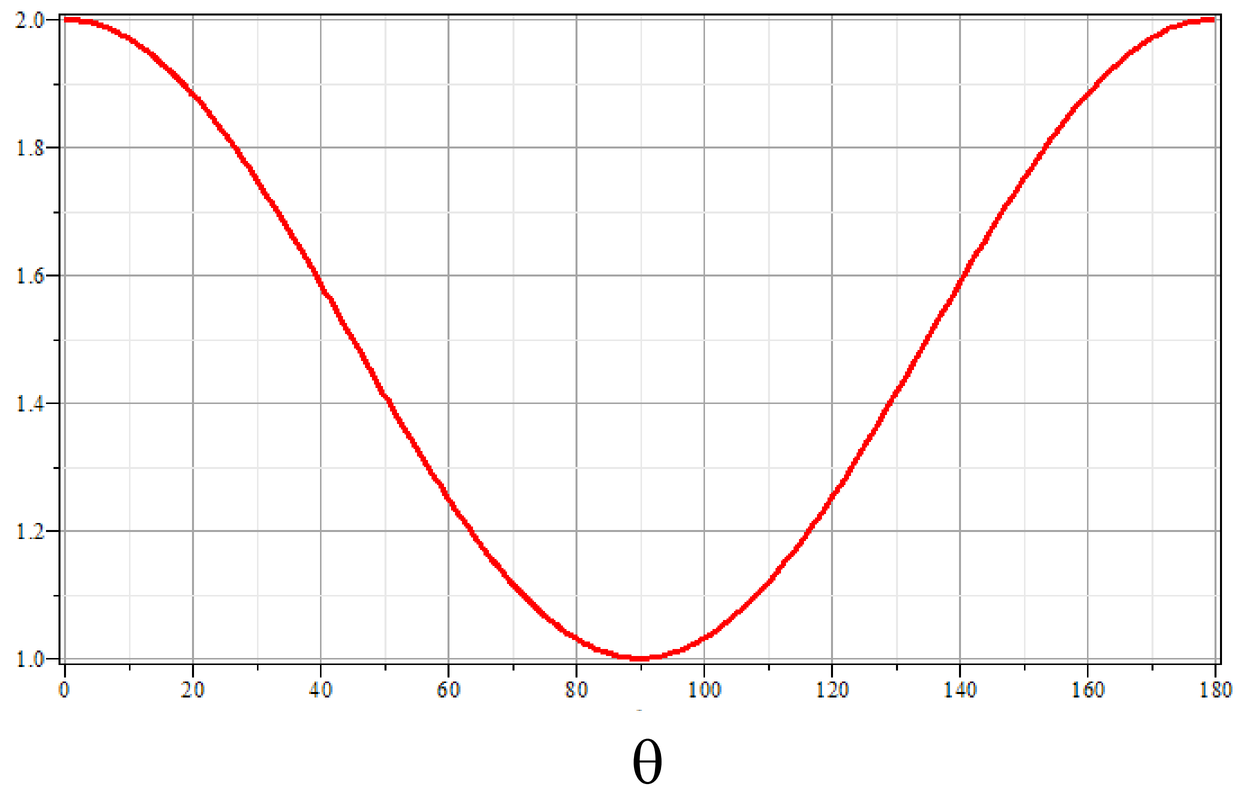
Assuming both incident polarizations are equally likely, average cross section is given by:

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\mathbf{v}};\hat{\mathbf{k}}_0,\hat{\mathbf{v}}_0\right)=\frac{k^4a^6}{2}\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2\left(\cos^2\theta+1\right)$$

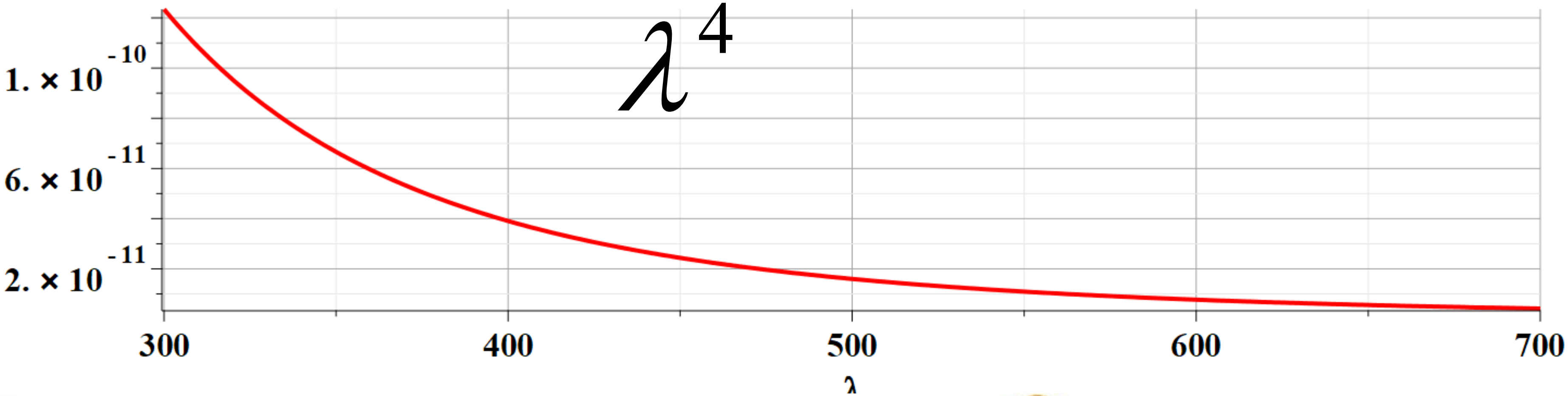
Scattering by dielectric sphere with permittivity ε and radius *a*:

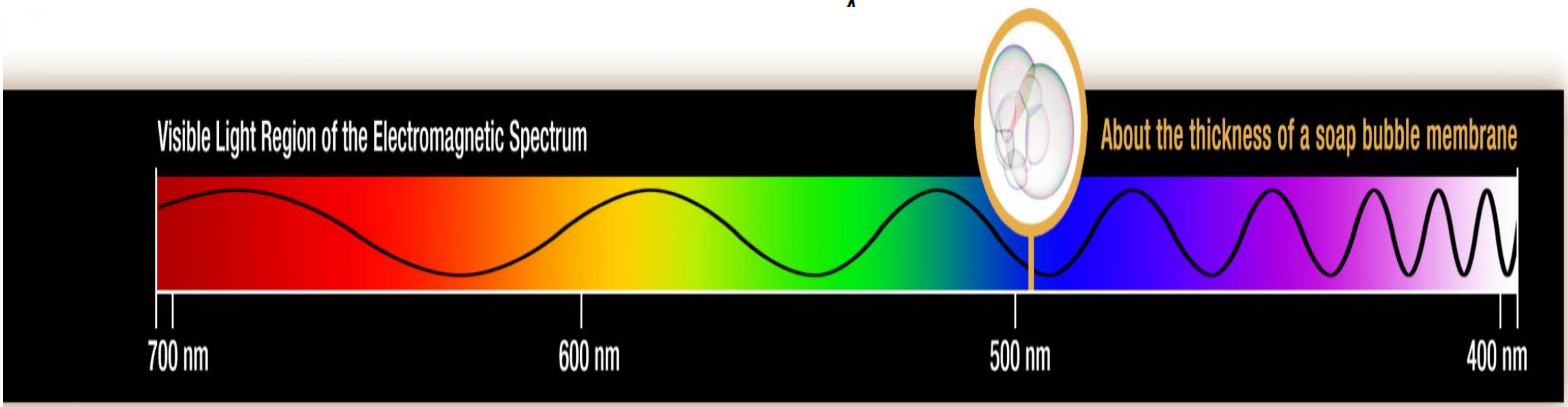


$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\boldsymbol{\varepsilon}};\hat{\mathbf{k}}_0,\hat{\boldsymbol{\varepsilon}}_0\right) = \frac{k^4a^6}{2}\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2\left(\cos^2\theta+1\right)$$



$$\frac{1}{\lambda^4}$$





Brief introduction to multipole expansion of electromagnetic fields (Chap. 9.7)

Sourceless Maxwell's equations

in terms of $\mathbf{E}$ and $\mathbf{H}$ fields with time dependence $e^{-i\omega t}$:

$$\nabla \times \mathbf{E} = ikZ_0\mathbf{H} \qquad \nabla \times \mathbf{H} = -ik\mathbf{E} / Z_0$$

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{H} = 0$$

where $k \equiv \omega / c$ and $Z_0 \equiv \sqrt{\mu_0 / \epsilon_0}$

Decoupled equations:

$$\left(\nabla^2 + k^2\right)\mathbf{E} = 0 \qquad \left(\nabla^2 + k^2\right)\mathbf{H} = 0$$

$$\mathbf{H} = -\frac{i}{kZ_0}\nabla \times \mathbf{E} \qquad \mathbf{E} = \frac{iZ_0}{k}\nabla \times \mathbf{H}$$

Multipole expansion of electromagnetic fields -- continued

Note that:

$$\left(\nabla^2 + k^2\right)\left(\mathbf{r}\cdot\mathbf{E}\right) = 0 \qquad\qquad \left(\nabla^2 + k^2\right)\left(\mathbf{r}\cdot\mathbf{H}\right) = 0$$

Convenient operators for angular momentum analysis

Define: $\mathbf{L} \equiv \frac{1}{i}\left(\mathbf{r}\times\nabla\right)$

Note that $\mathbf{r} \cdot \mathbf{L} = 0$

$$\nabla^2 = \frac{1}{r}\frac{\partial^2 r}{\partial r^2} - \frac{L^2}{r^2}$$

Eigenfunctions:

$$L^2 Y_{lm}(\theta,\phi) = -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]Y_{lm}(\theta,\phi) = l(l+1)Y_{lm}(\theta,\phi)$$

Multipole expansion of electromagnetic fields -- continued

Magnetic multipole field:

$$\mathbf{r} \cdot \mathbf{H}_{lm}^{M} \equiv \frac{l(l+1)}{k} g_l(kr) Y_{lm}(\theta,\phi)$$

$$\mathbf{r} \cdot \mathbf{E}_{lm}^{M} = 0$$

$$\mathbf{L} \cdot \mathbf{E}_{lm}^{M} = l(l+1) Z_0 g_l(kr) Y_{lm}(\theta,\phi)$$

spherical Bessel function

Electric multipole field:

$$\mathbf{r} \cdot \mathbf{E}_{lm}^{E} \equiv -Z_0 \frac{l(l+1)}{k} f_l(kr) Y_{lm}(\theta,\phi)$$

$$\mathbf{r} \cdot \mathbf{H}_{lm}^{E} = 0$$

$$\mathbf{L} \cdot \mathbf{H}_{lm}^{E} = l(l+1) f_l(kr) Y_{lm}(\theta,\phi)$$

spherical Bessel function

Multipole expansion of electromagnetic fields -- continued

Vector spherical harmonics: (for $l > 0$)

$$\mathbf{X}_{lm}(\theta,\phi) = \frac{1}{\sqrt{l(l+1)}}\mathbf{L}Y_{lm}(\theta,\phi)$$

Orthogonality conditions:

$$\int d\Omega\ \mathbf{X}_{l'm'}{}^{*}(\theta,\phi)\cdot\mathbf{X}_{lm}(\theta,\phi) = \delta_{ll'}\delta_{mm'}$$

$$\int d\Omega\ \mathbf{X}_{l'm'}{}^{*}(\theta,\phi)\cdot\left(\mathbf{r}\times\mathbf{X}_{lm}(\theta,\phi)\right) = 0$$

General expansion of fields:

$$\mathbf{H} = \sum_{lm}\left[a_{lm}^{E} f_l(kr)\mathbf{X}_{lm}(\theta,\phi) - \frac{i}{k}a_{lm}^{M}\nabla\times\left(g_l(kr)\mathbf{X}_{lm}(\theta,\phi)\right)\right]$$

$$\mathbf{E} = \sum_{lm}\left[\frac{i}{k}a_{lm}^{E}\nabla\times\left(f_l(kr)\mathbf{X}_{lm}(\theta,\phi)\right) + a_{lm}^{M} g_l(kr)\mathbf{X}_{lm}(\theta,\phi)\right]$$

Multipole expansion of electromagnetic fields -- continued

Time averaged power distribution of radiation far from source:

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}\left|\sum_{lm}(-i)^{l+1}\left[a^E_{lm}\mathbf{X}_{lm}(\theta,\phi)\times\hat{\mathbf{r}} + a^M_{lm}\mathbf{X}_{lm}(\theta,\phi)\right]\right|^2$$

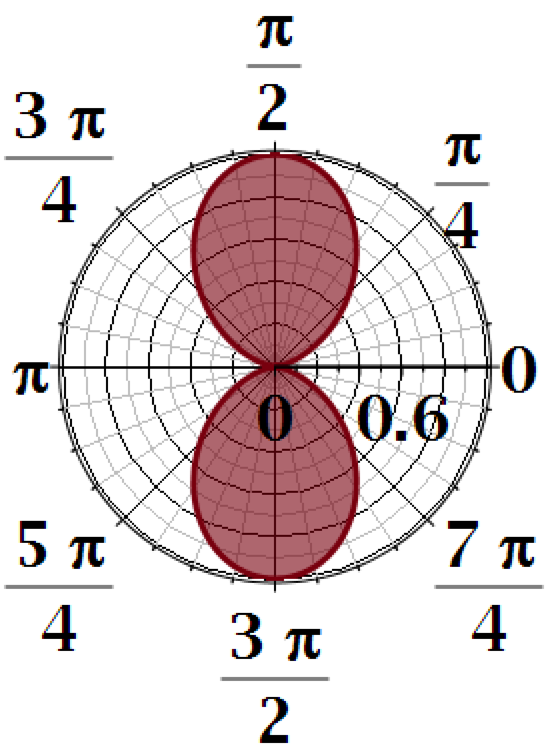
For a pure multipole radiation with either a^E_{lm} or a^M_{lm} :

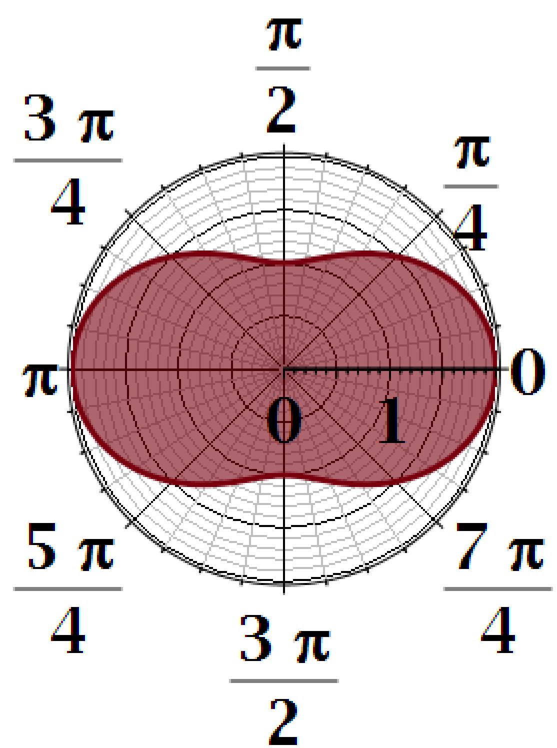
$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}\left|a_{lm}\right|^2\left|\mathbf{X}_{lm}(\theta,\phi)\right|^2$$

$$\left|\mathbf{X}_{lm}(\theta,\phi)\right|^2 = \frac{1}{2l(l+1)}\left(2m^2\left|Y_{lm}\right|^2 + (l+m)(l-m+1)\left|Y_{l(m-1)}\right|^2 + (l-m)(l+m+1)\left|Y_{l(m+1)}\right|^2\right)$$

For example: $l = 1$

$$\left|\mathbf{X}_{10}(\theta,\phi)\right|^2 = \frac{3}{8\pi}\sin^2\theta \qquad \left|\mathbf{X}_{11}(\theta,\phi)\right|^2 = \left|\mathbf{X}_{1-1}(\theta,\phi)\right|^2 = \frac{3}{16\pi}\left(1+\cos^2\theta \quad\right)$$





For example: $l = 2$

$$|\mathbf{X}_{20}(\theta,\phi)|^2 = \frac{15}{8\pi}\sin^2\theta\,\cos^2\theta \qquad |\mathbf{X}_{21}(\theta,\phi)|^2 = \frac{5}{16\pi}\left(1-3\cos^2\theta+4\cos^4\theta\right) \qquad |\mathbf{X}_{22}(\theta,\phi)|^2 = \frac{5}{16\pi}\left(1-\cos^4\theta\right)$$