PHY 712 Electrodynamics
10-10:50 AM MWF Online

Notes for Lecture 25:

Start reading Chap. 11

- A. Equations in cgs (Gaussian) units
- B. Special theory of relativity
- C. Lorentz transformation relations

In this lecture we will jump to Chapter 11 of Jackson and the special theory of relativity.

PHYSICS COLLOQUIUM

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THURSDAY
APRIL 1, 2021
ZOOM link

"From Fukushima to the Future: Lessons Learned and New Developments"

The day before a huge tsunami hit the coast of Japan on March 11, 2011, nuclear power appeared to be poised for a "renaissance" in much of the world. However, the tsunami resulted in a major accident at the Fukushima nuclear power plant, causing the world to hit the pause button on nuclear power development. In the 9 years since that accident, the industry has focused on understanding the underlying causes of the accident and modifying current nuclear plants and operations based on the lessons learned. Now, new nuclear power plants are being built in several countries, and more are being planned. This talk will address the accident and its aftermath, including major changes that have been made at existing plants, as well as the status of nuclear power today in different countries, and how advanced nuclear reactor concepts might affect the future of nuclear power.

Dr. Gail Marcus
Independent Consultant
Nuclear Power Technology and Policy
Washington, DC

4:00 pm

21	Mon: 03/22/2021	Chap. 8	EM waves in wave guides		
22	Wed: 03/24/2021	Chap. 9	Radiation from localized oscillating sources	#15	03/26/2021
23	Fri: 03/26/2021	Chap. 9	Radiation from oscillating sources	#16	03/29/2021
24	Mon: 03/29/2021	Chap. 9 & 10	Radiation and scattering	#17	03/31/2021
25	Wed: 03/31/2021	Chap. 11	Special Theory of Relativity	#18	04/02/2021
26	Fri: 04/02/2021	Chap. 11	Special Theory of Relativity		

PHY 712 -- Assignment #18

March 31, 2021

Begin reading Chapter 11 in **Jackson** .

1. Derive the relationships between the component of the electric and magnetic field components E_1, E_2, E_3, B_1, B_2, and B_3 as measured in the stationary frame of reference and the components E'_1, E'_2, E'_3, B'_1, B'_2, and B'_3 measured in the moving frame of reference. Note that the reverse relationships are given in Eq. 11.148.

Units - SI vs Gaussian

Below is a table comparing SI and Gaussian unit systems. The fundamental units for each system are so labeled and are used to define the derived units.

Variable	SI		Gaussian		SI/Gaussian
	Unit	Relation	Unit	Relation	
length	m	fundamental	cm	fundamental	100
mass	kg	fundamental	gm	fundamental	1000
time	s	fundamental	s	fundamental	1
force	N	$kg \cdot m^2/s$	$dyne$	$gm \cdot cm^2/s$	10^5
current	A	fundamental	$statampere$	$statcoulomb/s$	$\frac{1}{10c}$
charge	C	$A \cdot s$	$statcoulomb$	$\sqrt{dyne \cdot cm^2}$	$\frac{1}{10c}$

Now – jumping to Chapter 11 of Jackson. Fortunately/unfortunately Jackson decided to use cgs Gaussian units starting in Chapter 11. Here is a table of comparison.

Basic equations of electrodynamics

CGS (Gaussian)	SI
$\nabla \cdot \mathbf{D} = 4\pi\rho$	$\nabla \cdot \mathbf{D} = \rho$
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$
$\mathbf{F} = q(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})$	$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$
$u = \frac{1}{8\pi}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$	$u = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$
$\mathbf{S} = \frac{c}{4\pi}(\mathbf{E} \times \mathbf{H})$	$\mathbf{S} = (\mathbf{E} \times \mathbf{H})$

More tables of comparison of the two unit schemes.

More relationships

CGS (Gaussian)

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = \epsilon\mathbf{E}$$

$$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} = \frac{1}{\mu}\mathbf{B}$$

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

MKS (SI)

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon\mathbf{E}$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} = \frac{1}{\mu}\mathbf{B}$$

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

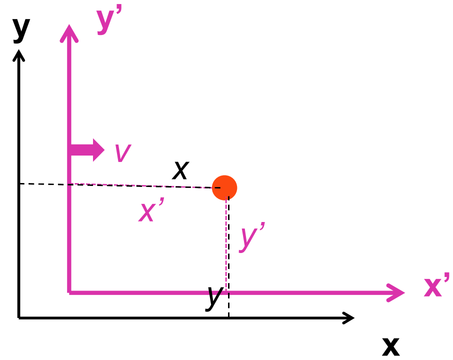
$$\epsilon \quad \Leftrightarrow \quad \epsilon / \epsilon_0$$

$$\mu \quad \Leftrightarrow \quad \mu / \mu_0$$

More relationships.

Notions of special relativity

- The basic laws of physics are the same in all frames of reference (at rest or moving at constant velocity).
- The speed of light in vacuum *c* is the same in all frames of reference.



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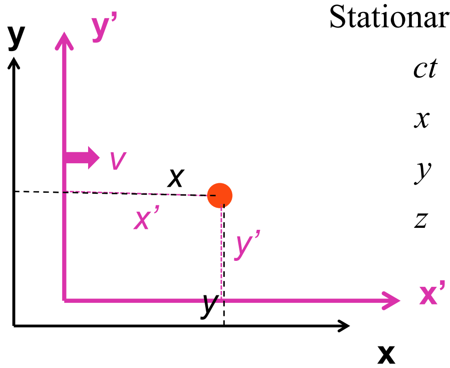
Now – jumping into the story of special relativity. The black frame corresponds to a (stationary) frame. The purple coordinate system is moving relative to it along the x axis at a speed of v. The red dot is measured differently in the two frames of reference.

Lorentz transformations

Convenient notation :

$$\beta \equiv \frac{v}{c}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$



Stationary frame		Moving frame
ct	$=$	$\gamma(ct'+\beta x')$
x	$=$	$\gamma(x'+\beta ct')$
y	$=$	y'
z	$=$	z'

The notation with beta and gamma is defined. The question is how the four parameters, ct, x,y,z measured in the stationary reference frame are related to the corresponding four variables ct’,x’y’,z’ measured in the moving frame and vice versa? The consensus is that the Lorentz transformation is the correct correspondence.

Lorentz transformations -- continued

$\beta \equiv \frac{v}{c} \qquad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\mathcal{L} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathcal{L}^{-1} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \mathcal{L} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} \qquad \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Notice:

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2$$

The Lorentz transformation expressed in matrix form. Also note that the four variables have an invariant.

Examples of other 4-vectors applicable to the Lorentz transformation:

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\beta \equiv \frac{v}{c} \qquad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

$$\mathcal{L} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathcal{L}^{-1} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix} = \mathcal{L} \begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} \qquad \begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix}$$

Note: $\omega t - \mathbf{k}\cdot\mathbf{r} = \omega' t' - \mathbf{k}'\cdot\mathbf{r}'$

In free space:

$$\left(\frac{\omega}{c}\right)^2 - k^2 = \left(\frac{\omega'}{c}\right)^2 - k'^2 = 0$$

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \mathcal{L} \begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} \qquad \begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix}$$

Note: $E^2 - p^2c^2 = E'^2 - p'^2 c^2$

Other 4 vectors that obey the Lorentz transformation --

The Doppler Effect

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\mathcal{L} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathcal{L}^{-1} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \omega / c \\ k_x \\ k_y \\ k_z \end{pmatrix} = \mathcal{L} \begin{pmatrix} \omega' / c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} \qquad \begin{pmatrix} \omega' / c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} \omega / c \\ k_x \\ k_y \\ k_z \end{pmatrix} \qquad \text{Note: } \omega t - \mathbf{k}\cdot\mathbf{r} = \omega' t' - \mathbf{k}'\cdot\mathbf{r}'$$

$$\omega'/c = \gamma(\omega/c - \beta k_x) \qquad k'_x = \gamma(k_x - \beta\omega/c)$$

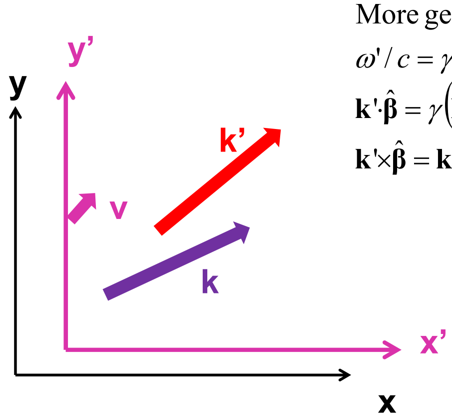
$$k'_y = k_y \qquad k'_z = k_z$$

Some details about the frequency/wavevector 4-vector and the Doppler effect for electromagnetic waves.

The Doppler Effect -- continued

$$\omega'/c = \gamma(\omega/c - \beta k_x) \qquad k'_x = \gamma(k_x - \beta\omega/c)$$

$$k'_y = k_y \qquad k'_z = k_z$$



More generally:

$$\omega'/c = \gamma(\omega/c - \boldsymbol{\beta}\cdot\mathbf{k}) \equiv \gamma(\omega/c - \beta k\cos\theta)$$

$$\mathbf{k}'\cdot\hat{\boldsymbol{\beta}} = \gamma(\mathbf{k}\cdot\hat{\boldsymbol{\beta}} - \beta\omega/c) \equiv k'\cos\theta' = \gamma(k\cos\theta - \beta\omega/c)$$

$$\mathbf{k}'\times\hat{\boldsymbol{\beta}} = \mathbf{k}\times\hat{\boldsymbol{\beta}}$$

For $\theta = 0$: $(k = \omega/c)$

$$\omega' = \omega\gamma(1-\beta) \qquad \Rightarrow \omega' = \omega\sqrt{\frac{1-\beta}{1+\beta}}$$

For $\theta \neq 0$: $(k = \omega/c)$

$$\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - \beta)}$$

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Doppler effect continued.

Electromagnetic Doppler Effect (θ=0)

$$\omega' = \omega\sqrt{\frac{1-\beta}{1+\beta}} \qquad \beta \approx \frac{v_{\text{source}} - v_{\text{detector}}}{c}$$

More precisely: $\beta = \dfrac{v_{\text{source}} - v_{\text{detector}}}{c\left(1 - \dfrac{v_{\text{source}}v_{\text{detector}}}{c^2}\right)}$

(Thanks to E. Carlson)

Sound Doppler Effect (θ=0)

$$\omega' = \omega\left(\frac{1 \pm v_{\text{detector}}/c_s}{1 \mp v_{\text{source}}/c_s}\right)$$

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Discussion about the Doppler effect for electromagnetic waves and sound waves.

Lorentz transformation of the velocity

Stationary frame		Moving frame
ct	$=$	$\gamma\left(ct'+\beta x'\right)$
x	$=$	$\gamma\left(x'+\beta ct'\right)$
y	$=$	y'
z	$=$	z'

For an infinitesimal increment:

Stationary frame		Moving frame
cdt	$=$	$\gamma\left(cdt'+\beta dx'\right)$
dx	$=$	$\gamma\left(dx'+\beta cdt'\right)$
dy	$=$	dy'
dz	$=$	dz'

Now consider the measurement of velocity in the two different reference frames.

Lorentz transformation of the velocity -- continued

Stationary frame		Moving frame
cdt	$=$	$\gamma(cdt'+\beta dx')$
dx	$=$	$\gamma(dx'+\beta cdt')$
dy	$=$	dy'
dz	$=$	dz'

Define: $u_x \equiv \frac{dx}{dt} \quad u_y \equiv \frac{dy}{dt} \quad u_z \equiv \frac{dz}{dt}$

$$u'_x \equiv \frac{dx'}{dt'} \quad u'_y \equiv \frac{dy'}{dt'} \quad u'_z \equiv \frac{dz'}{dt'}$$

$$\frac{dx}{dt} = \frac{\gamma(dx'+\beta cdt')}{\gamma(dt'+\beta dx'/c)} = \frac{u'_x + v}{1+vu'_x/c^2} = u_x$$

$$\frac{dy}{dt} = \frac{dy'}{\gamma(dt'+\beta dx'/c)} = \frac{u'_y}{\gamma(1+vu'_x/c^2)} = u_y$$

Evaluating the infinitesimals to determine the velocity relationships.

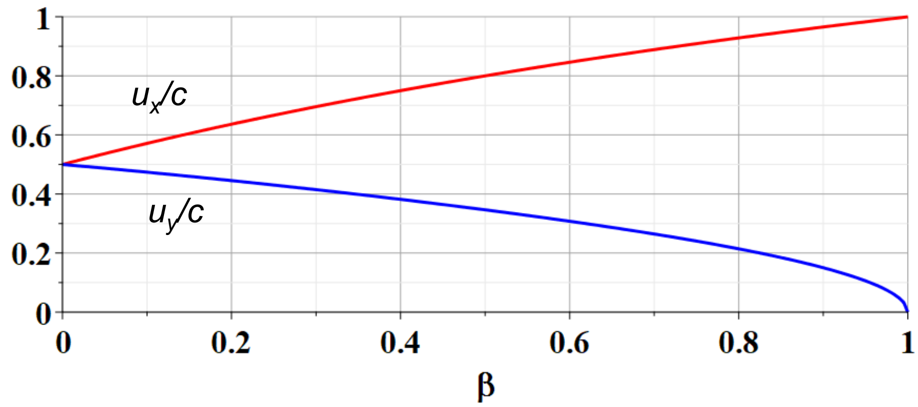
Summary of velocity relationships

$$u_x = \frac{u'_x + v}{1 + vu'_x / c^2}$$

$$u_y = \frac{u'_y}{\gamma\left(1 + vu'_x / c^2\right)} \equiv \frac{u'_y}{\gamma_v\left(1 + vu'_x / c^2\right)}$$

$$u_z = \frac{u'_z}{\gamma\left(1 + vu'_x / c^2\right)} \equiv \frac{u'_z}{\gamma_v\left(1 + vu'_x / c^2\right)}$$

Example of velocity variation with β:
($u'_x/c = u'_y/c = 0.5$)



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Numerical evaluation of the velocity relationship.

Extention to tranformation of acceleration

$$\mathbf{a}_{\parallel} = \frac{\left(1-\frac{v^2}{c^2}\right)^{3/2}}{\left(1+\frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^3}\mathbf{a}'_{\parallel}$$

$$\mathbf{a}_{\perp} = \frac{\left(1-\frac{v^2}{c^2}\right)}{\left(1+\frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^3}\left(\mathbf{a}'_{\perp}+\frac{\mathbf{v}}{c^2}\times\left(\mathbf{a}'\times\mathbf{u}'\right)\right)$$

It is possible to take the derivatives of the velocities to get the accelerations. The proof of these results are left for you to fill in.

Comment –
The acceleration equations are obtained by taking the infinitesimal derivative of the velocity relationships and simplifying the expressions. (See Jackson Problem 11.5.)

Velocity transformations continued:

Consider: $u_x = \frac{u'_x + v}{1 + vu'_x / c^2}$ $\quad u_y = \frac{u'_y}{\gamma_v \left(1 + vu'_x / c^2\right)}$ $\quad u_z = \frac{u'_z}{\gamma_v \left(1 + vu'_x / c^2\right)}$.

Note that $\gamma_u \equiv \frac{1}{\sqrt{1-(u/c)^2}} = \frac{1 + vu'_x / c^2}{\sqrt{1-(u'/c)^2}\sqrt{1-(v/c)^2}} = \gamma_v \gamma_{u'} \left(1 + vu'_x / c^2\right)$

$\Rightarrow \gamma_u c = \gamma_v \left(\gamma_{u'} c + \beta_v \gamma_{u'} u'_x\right)$

$\Rightarrow \gamma_u u_x = \gamma_v \left(\gamma_{u'} u'_x + \gamma_{u'} v\right) = \gamma_v \left(\gamma_{u'} u'_x + \beta_v \gamma_{u'} c\right)$

$\Rightarrow \gamma_u u_y = \gamma_{u'} u'_y \qquad \gamma_u u_z = \gamma_{u'} u'_z$

Velocity 4-vector: $\begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix} = \mathcal{L}_v \begin{pmatrix} \gamma_{u'} c \\ \gamma_{u'} u'_x \\ \gamma_{u'} u'_y \\ \gamma_{u'} u'_z \end{pmatrix}$

It is apparent that the velocity 4 vector itself does not obey the Lorentz transformation. These identities show that we can construct a related 4 vector that does obey the Lorentz transformation.

Some details:

$$\gamma_u = \gamma_v \gamma_{u'} \left(1 + vu'_x / c^2\right) \quad \Rightarrow \left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{u'^2}{c^2}\right) = \left(1 - \frac{u^2}{c^2}\right)\left(1 + \frac{u_x v}{c^2}\right)^2$$

where $\quad u_x = \dfrac{u'_x + v}{1 + vu'_x / c^2} \quad u_y = \dfrac{u'_y}{\gamma_v \left(1 + vu'_x / c^2\right)} \quad u_z = \dfrac{u'_z}{\gamma_v \left(1 + vu'_x / c^2\right)}.$

$$\left(\frac{u_x^2}{c^2} + \frac{u_y^2}{c^2} + \frac{u_z^2}{c^2}\right)\left(1 + \frac{u_x v}{c^2}\right)^2 = \left(\frac{u'_x}{c} + \frac{v}{c}\right)^2 + \left(\frac{u'^2_y}{c^2} + \frac{u'^2_z}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)$$

$$\frac{u^2}{c^2}\left(1 + \frac{u_x v}{c^2}\right)^2 = \frac{u'^2}{c^2}\left(1 - \frac{v^2}{c^2}\right) + \left(1 + \frac{u_x v}{c^2}\right)^2 - \left(1 - \frac{v^2}{c^2}\right)$$

$$\Rightarrow \left(1 - \frac{u^2}{c^2}\right)\left(1 + \frac{u_x v}{c^2}\right)^2 = \left(1 - \frac{u'^2}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)$$

Some identities that can be proven…

Significance of 4-velocity vector: $\begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix}$

Introduce the "rest" mass *m* of particle characterized by velocity **u**:

$$mc\begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix} = \begin{pmatrix} \gamma_u mc^2 \\ \gamma_u mu_x c \\ \gamma_u mu_y c \\ \gamma_u mu_z c \end{pmatrix} = \begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix}$$

Properties of energy-moment 4-vector:

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \mathcal{L}\begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} \qquad \begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} = \mathcal{L}^{-1}\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} \qquad \text{Note: } E^2 - p^2c^2 = E'^2 - p'^2 c^2$$

When the dust clears, the related physical parameters are the energy-momentum 4 vector.

Properties of Energy-momentum 4-vector -- continued

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \begin{pmatrix} \gamma_u mc^2 \\ \gamma_u m u_x c \\ \gamma_u m u_y c \\ \gamma_u m u_z c \end{pmatrix}$$

$$\text{Note: } E^2 - p^2c^2 = \frac{(mc^2)^2}{1-\beta_u^2}\left(1 - \left(\frac{u_x}{c}\right)^2 - \left(\frac{u_y}{c}\right)^2 - \left(\frac{u_z}{c}\right)^2\right) = (mc^2)^2 = E'^2 - p'^2 c^2$$

Notion of "rest energy": For $\mathbf{p} \equiv 0, \quad E = mc^2$

Define kinetic energy: $E_K \equiv E - mc^2 = \sqrt{p^2c^2 + m^2c^4} - mc^2$

Non-relativistic limit: If $\frac{p}{mc} << 1, \quad E_K = mc^2\left(\sqrt{1+\left(\frac{p}{mc}\right)^2} - 1\right)$

$$\approx \frac{p^2}{2m}$$

In order to relate the equations to the non-relativistic treatments, we must use the same zero of energy for both. The kinetic energy of a relativistic free particle is related to the energy E-mc^2.

Summary of relativistic energy relationships

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \begin{pmatrix} \gamma_u mc^2 \\ \gamma_u m u_x c \\ \gamma_u m u_y c \\ \gamma_u m u_z c \end{pmatrix}$$

$$E = \sqrt{p^2c^2 + m^2c^4} = \gamma_u mc^2$$

$$\text{Check}: \quad \sqrt{p^2c^2 + m^2c^4} = mc^2\sqrt{\gamma_u{}^2\beta_u{}^2 + 1} = \gamma_u mc^2$$

Example: for an electron $m\mathrm{c}^2 = 0.5\,\mathrm{MeV}$

for $E = 200\,\mathrm{GeV}$

$$\gamma_u = \frac{E}{mc^2} = 4\times 10^5$$

$$\beta_u = \sqrt{1 - \frac{1}{\gamma_u{}^2}} \approx 1 - \frac{1}{2\gamma_u{}^2} \approx 1 - 3\times 10^{-12}$$

This slide gives some numerical relationships for a highly accelerated electron.

Special theory of relativity and Maxwell's equations

Continuity equation: $$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

Lorenz gauge condition: $$\frac{1}{c} \frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{A} = 0$$

Potential equations: $$\frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = 4\pi\rho$$

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \frac{4\pi}{c}\mathbf{J}$$

Field relations: $$\mathbf{E} = -\nabla \Phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

All of the previous equations represent relativistic mechanics. Now we want to relate the ideas to electromagnetic theory. We have said that Maxwell's equations already are consistent with the theory of relativity. But we still have some work to do in order to relate the measured fields and sources in two different reference frames. The idea is to guess the correct 4 vectors.

More 4-vectors:

$$\text{Time and position :} \quad \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \Rightarrow x^\alpha$$

$$\text{Charge and current :} \quad \begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix} \Rightarrow J^\alpha$$

$$\text{Vector and scalar potentials :} \quad \begin{pmatrix} \Phi \\ A_x \\ A_y \\ A_z \end{pmatrix} \Rightarrow A^\alpha$$

Here are our guesses.

Lorentz transformations

$$\mathcal{L}_v = \begin{pmatrix} \gamma_v & \gamma_v\beta_v & 0 & 0 \\ \gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Time and space: $x^\alpha = \mathcal{L}_v x'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} x'^\beta$

Charge and current: $J^\alpha = \mathcal{L}_v J'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} J'^\beta$

Vector and scalar potential: $A^\alpha = \mathcal{L}_v A'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} A'^\beta$

These 4 vectors obey the Lorentz transformations. Here we use the notation that repeated indices should be summed over the 4 components. In this case beta is the summed index. Next time we will see how the E and B fields are represented in terms of the Lorentz transformations.

Summary of results --

$$\text{Time and space:} \quad x^\alpha = \mathcal{L}_v x'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} x'^\beta$$

$$\text{Charge and current:} \quad J^\alpha = \mathcal{L}_v J'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} J'^\beta$$

$$\text{Vector and scalar potential:} \quad A^\alpha = \mathcal{L}_v A'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} A'^\beta$$

Here, the notation varies among the textbooks.
In general, it is convenient to use the matrix multiplication conventions to multiply our 4×4 matrices and 4 vectors

For example: $\mathcal{L}_v^{\alpha\beta} x'^\beta = \sum_{\beta=1}^{4} \mathcal{L}_v^{\alpha\beta} x'^\beta = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$