

PHY 712 Electrodynamics

10-10:50 AM MWF Online

Discussion for Lecture 25:

Start reading Chap. 11

- A. Equations in cgs (Gaussian) units**
- B. Special theory of relativity**
- C. Lorentz transformation relations**

PHYSICS COLLOQUIUM

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THURSDAY

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APRIL 1, 2021

ZOOM link

“From Fukushima to the Future: Lessons Learned and New Developments”

The day before a huge tsunami hit the coast of Japan on March 11, 2011, nuclear power appeared to be poised for a “renaissance” in much of the world. However, the tsunami resulted in a major accident at the Fukushima nuclear power plant, causing the world to hit the pause button on nuclear power development. In the 9 years since that accident, the industry has focused on understanding the underlying causes of the accident and modifying current nuclear plants and operations based on the lessons learned. Now, new nuclear power plants are being built in several countries, and more are being planned. This talk will address the accident and its aftermath, including major changes that have been made at existing plants, as well as the status of nuclear power today in different countries, and how advanced nuclear reactor concepts might affect the future of nuclear power.



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21	Mon: 03/22/2021	Chap. 8	EM waves in wave guides		
22	Wed: 03/24/2021	Chap. 9	Radiation from localized oscillating sources	#15	03/26/2021
23	Fri: 03/26/2021	Chap. 9	Radiation from oscillating sources	#16	03/29/2021
24	Mon: 03/29/2021	Chap. 9 & 10	Radiation and scattering	#17	03/31/2021
25	Wed: 03/31/2021	Chap. 11	Special Theory of Relativity	#18	04/02/2021
26	Fri: 04/02/2021	Chap. 11	Special Theory of Relativity		

PHY 712 -- Assignment #18

March 31, 2021

Begin reading Chapter 11 in **Jackson** .

1. Derive the relationships between the component of the electric and magnetic field components $E_1, E_2, E_3, B_1, B_2,$ and B_3 as measured in the stationary frame of reference and the components $E'_1, E'_2, E'_3, B'_1, B'_2,$ and B'_3 measured in the moving frame of reference. Note that the reverse relationships are given in Eq. 11.148.

Your questions –

From Gao -- In Lorentz transform, Why does it consider four vectors together instead of three? Because actually these four vectors are the same vector?

Units - SI vs Gaussian

Below is a table comparing SI and Gaussian unit systems. The fundamental units for each system are so labeled and are used to define the derived units.

Variable	SI		Gaussian		SI/Gaussian
	Unit	Relation	Unit	Relation	
length	m	fundamental	cm	fundamental	100
mass	kg	fundamental	gm	fundamental	1000
time	s	fundamental	s	fundamental	1
force	N	$kg \cdot m^2/s$	$dyne$	$gm \cdot cm^2/s$	10^5
current	A	fundamental	$statampere$	$statcoulomb/s$	$\frac{1}{10c}$
charge	C	$A \cdot s$	$statcoulomb$	$\sqrt{dyne \cdot cm^2}$	$\frac{1}{10c}$

Basic equations of electrodynamics

CGS (Gaussian)	SI
$\nabla \cdot \mathbf{D} = 4\pi\rho$	$\nabla \cdot \mathbf{D} = \rho$
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
$\nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t}$	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$
$\mathbf{F} = q(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})$	$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$
$u = \frac{1}{8\pi} (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$	$u = \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$
$\mathbf{S} = \frac{c}{4\pi} (\mathbf{E} \times \mathbf{H})$	$\mathbf{S} = (\mathbf{E} \times \mathbf{H})$

More relationships

CGS (Gaussian)

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = \epsilon\mathbf{E}$$

$$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} = \frac{1}{\mu}\mathbf{B}$$

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

MKS (SI)

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon\mathbf{E}$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} = \frac{1}{\mu}\mathbf{B}$$

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

ϵ

$\Leftrightarrow$

ϵ / ϵ_0

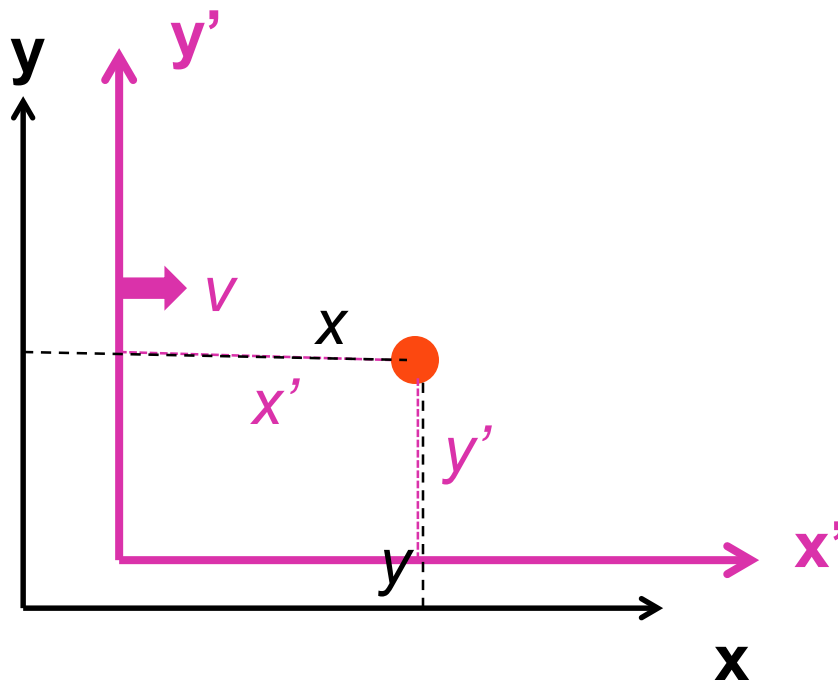
μ

$\Leftrightarrow$

μ / μ_0

Notions of special relativity

- The basic laws of physics are the same in all frames of reference (at rest or moving at constant velocity).
- The speed of light in vacuum c is the same in all frames of reference.

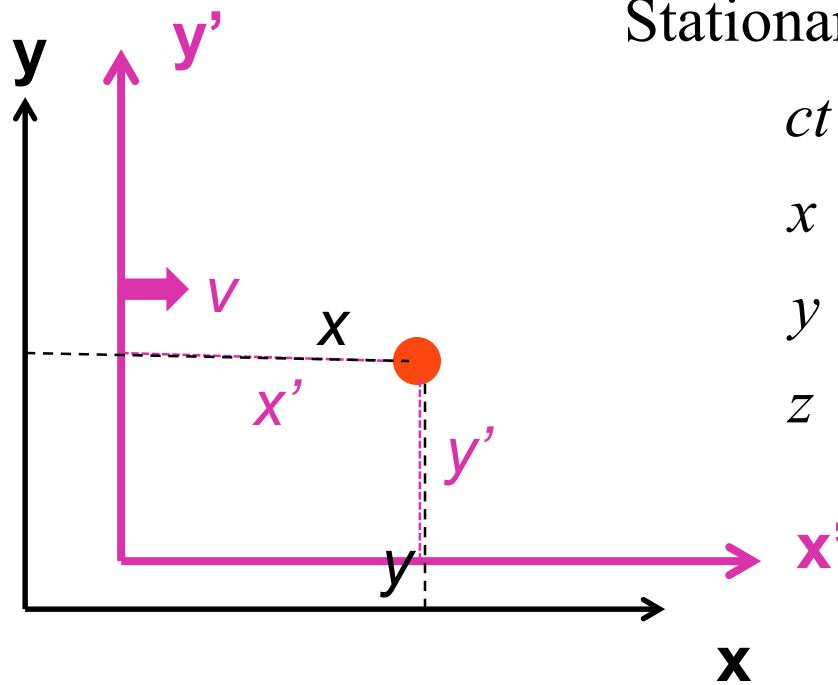


Lorentz transformations

Convenient notation :

$$\beta \equiv \frac{v}{c}$$

$$\gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}$$



Stationary frame

Moving frame

ct	$=$	$\gamma(ct' + \beta x')$
x	$=$	$\gamma(x' + \beta ct')$
y	$=$	y'
z	$=$	z'

Lorentz transformations -- continued

$$\beta \equiv \frac{v}{c} \quad \gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}$$

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\mathcal{L} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{L}^1 = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \mathcal{L} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \mathcal{L}^1 \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Notice:

$$c^2 t^2 - x^2 - y^2 - z^2 = c^2 t'^2 - x'^2 - y'^2 - z'^2$$

Examples of other 4-vectors applicable to the Lorentz transformation:

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\beta \equiv \frac{v}{c} \quad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

$$\mathcal{L} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{L}^{-1} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix} = \mathcal{L} \begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix}$$

$$\begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix}$$

Note: $\omega t - \mathbf{k} \cdot \mathbf{r} = \omega' t' - \mathbf{k}' \cdot \mathbf{r}'$
In free space:

$$\left(\frac{\omega}{c}\right)^2 - k^2 = \left(\frac{\omega'}{c}\right)^2 - k'^2 = 0$$

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \mathcal{L} \begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix}$$

$$\begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix}$$

Note: $E^2 - p^2 c^2 = E'^2 - p'^2 c^2$

The Doppler Effect

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\mathcal{L} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{L}^1 = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix} = \mathcal{L} \begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix}$$

$$\begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} = \mathcal{L}^1 \begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix}$$

Note: $\omega t - \mathbf{k} \cdot \mathbf{r} = \omega' t' - \mathbf{k}' \cdot \mathbf{r}'$

$$\omega'/c = \gamma(\omega/c - \beta k_x)$$

$$k'_x = \gamma(k_x - \beta\omega/c)$$

$$k'_y = k_y$$

$$k'_z = k_z$$

The Doppler Effect -- continued

$$\omega' / c = \gamma(\omega / c - \beta k_x)$$

$$k'_x = \gamma(k_x - \beta \omega / c)$$

$$k'_y = k_y$$

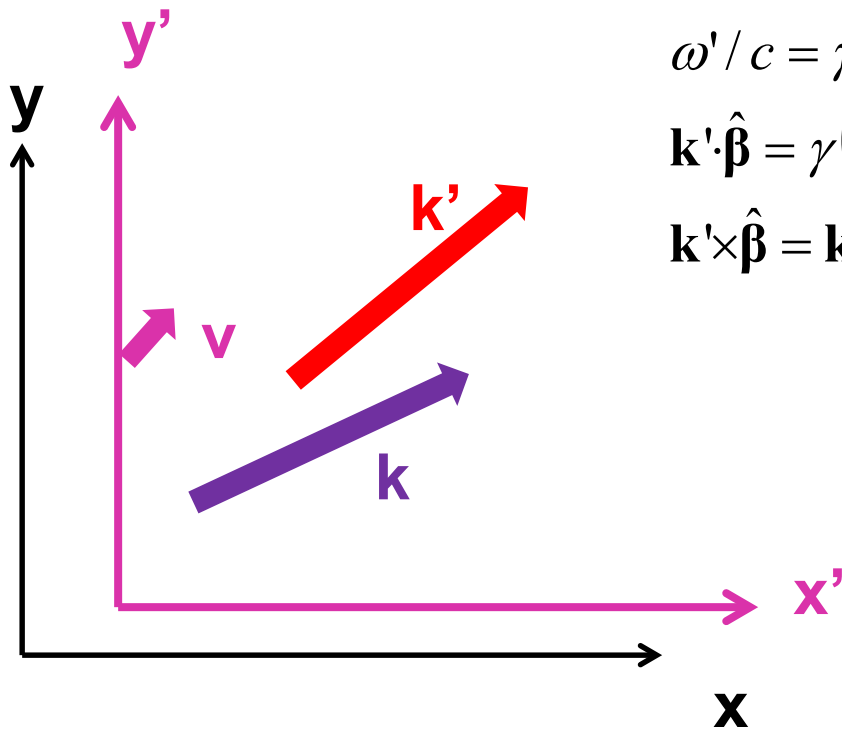
$$k'_z = k_z$$

More generally:

$$\omega' / c = \gamma(\omega / c - \boldsymbol{\beta} \cdot \mathbf{k}) \equiv \gamma(\omega / c - \beta k \cos \theta)$$

$$\mathbf{k}' \cdot \hat{\boldsymbol{\beta}} = \gamma(\mathbf{k} \cdot \hat{\boldsymbol{\beta}} - \beta \omega / c) \equiv k' \cos \theta' = \gamma(k \cos \theta - \beta \omega / c)$$

$$\mathbf{k}' \times \hat{\boldsymbol{\beta}} = \mathbf{k} \times \hat{\boldsymbol{\beta}}$$



For $\theta = 0$: ($k = \omega/c$)

$$\omega' = \omega \gamma(1 - \beta) \quad \Rightarrow \quad \omega' = \omega \sqrt{\frac{1 - \beta}{1 + \beta}}$$

For $\theta \neq 0$: ($k = \omega/c$)

$$\tan \theta' = \frac{\sin \theta}{\gamma(\cos \theta - \beta)}$$

Electromagnetic Doppler Effect ($\theta=0$)

$$\omega' = \omega \sqrt{\frac{1 - \beta}{1 + \beta}} \qquad \beta = \frac{v_{\text{source}} - v_{\text{detector}}}{c \left(1 - \frac{v_{\text{source}} v_{\text{detector}}}{c^2} \right)}$$

(details concerning velocities in the following slides.)

Sound Doppler Effect ($\theta=0$)

$$\omega' = \omega \left(\frac{1 \pm v_{\text{detector}} / c_s}{1 \mp v_{\text{source}} / c_s} \right)$$

Lorentz transformation of the velocity

Stationary frame

Moving frame

$$ct = \gamma(ct' + \beta x')$$

$$x = \gamma(x' + \beta ct')$$

$$y = y'$$

$$z = z'$$

For an infinitesimal increment:

Stationary frame

Moving frame

$$cdt = \gamma(cdt' + \beta dx')$$

$$dx = \gamma(dx' + \beta cdt')$$

$$dy = dy'$$

$$dz = dz'$$

Lorentz transformation of the velocity -- continued

Stationary frame

Moving frame

$$cdt = \gamma(cdt' + \beta dx')$$

$$dx = \gamma(dx' + \beta cdt')$$

$$dy = dy'$$

$$dz = dz'$$

Define :

$$u_x \equiv \frac{dx}{dt} \quad u_y \equiv \frac{dy}{dt} \quad u_z \equiv \frac{dz}{dt}$$
$$u'_x \equiv \frac{dx'}{dt'} \quad u'_y \equiv \frac{dy'}{dt'} \quad u'_z \equiv \frac{dz'}{dt'}$$

$$\frac{dx}{dt} = \frac{\gamma(dx' + \beta cdt')}{\gamma(dt' + \beta dx' / c)} = \frac{u'_x + v}{1 + vu'_x / c^2} = u_x$$

$$\frac{dy}{dt} = \frac{dy'}{\gamma(dt' + \beta dx' / c)} = \frac{u'_y}{\gamma(1 + vu'_x / c^2)} = u_y$$

Summary of velocity relationships

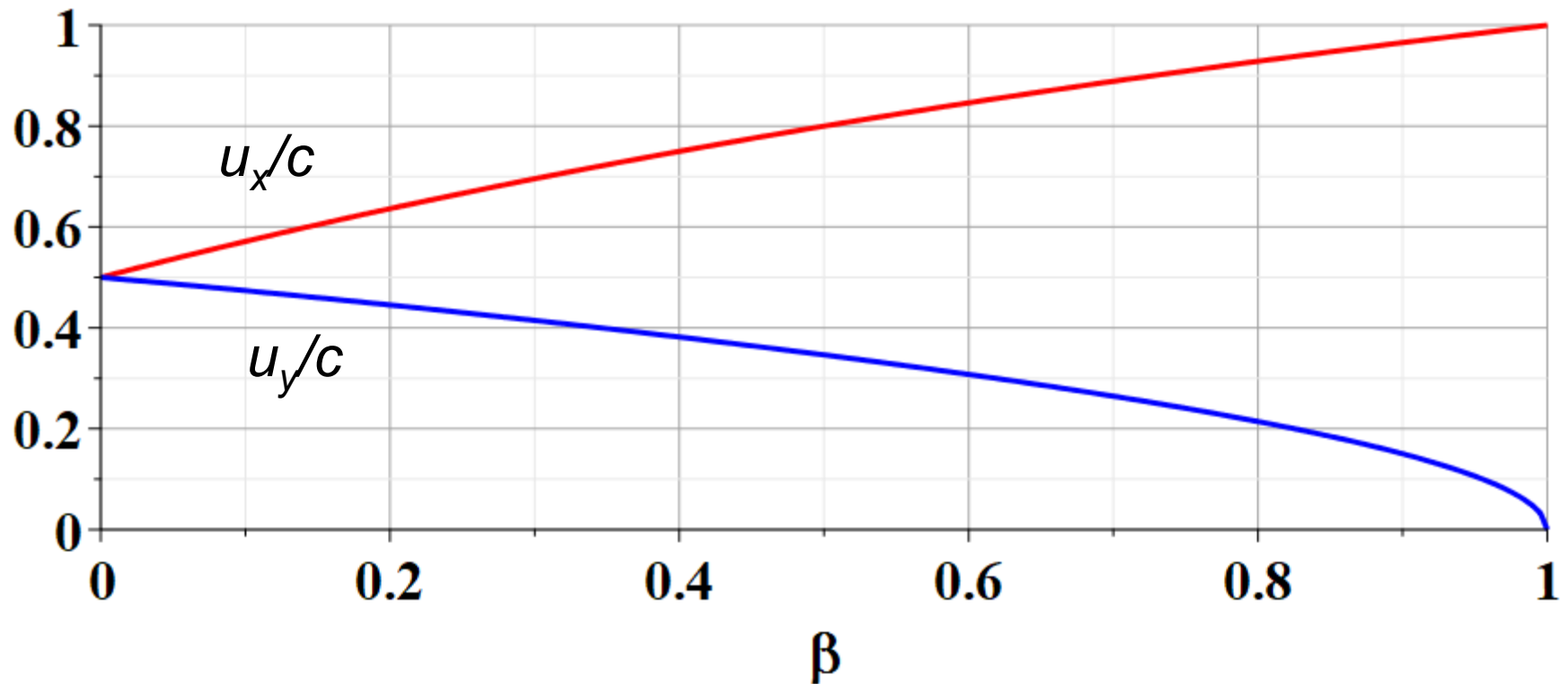
$$u_x = \frac{u'_x + v}{1 + vu'_x / c^2}$$

$$u_y = \frac{u'_y}{\gamma \left(1 + vu'_x / c^2\right)} \equiv \frac{u'_y}{\gamma_v \left(1 + vu'_x / c^2\right)}$$

$$u_z = \frac{u'_z}{\gamma \left(1 + vu'_x / c^2\right)} \equiv \frac{u'_z}{\gamma_v \left(1 + vu'_x / c^2\right)}$$

Example of velocity variation with β :

$$(u'_x/c = u'_y/c = 0.5)$$



Extention to tranformation of acceleration

$$\mathbf{a}_{\parallel} = \frac{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}{\left(1 + \frac{\mathbf{v} \cdot \mathbf{u}'}{c^2}\right)^3} \mathbf{a}'_{\parallel}$$

$$\mathbf{a}_{\perp} = \frac{\left(1 - \frac{v^2}{c^2}\right)}{\left(1 + \frac{\mathbf{v} \cdot \mathbf{u}'}{c^2}\right)^3} \left(\mathbf{a}'_{\perp} + \frac{\mathbf{v}}{c^2} \times (\mathbf{a}' \times \mathbf{u}') \right)$$

Comment –

The acceleration equations are obtained by taking the infinitesimal derivative of the velocity relationships and simplifying the expressions. (See Jackson Problem 11.5.)

Velocity transformations continued:

Consider: $u_x = \frac{u'_x + v}{1 + vu'_x / c^2} \quad u_y = \frac{u'_y}{\gamma_v (1 + vu'_x / c^2)} \quad u_z = \frac{u'_z}{\gamma_v (1 + vu'_x / c^2)}.$

Note that $\gamma_u \equiv \frac{1}{\sqrt{1 - (u/c)^2}} = \frac{1 + vu'_x / c^2}{\sqrt{1 - (u'/c)^2} \sqrt{1 - (v/c)^2}} = \gamma_v \gamma_{u'} (1 + vu'_x / c^2)$

$$\Rightarrow \gamma_u c = \gamma_v (\gamma_{u'} c + \beta_v \gamma_{u'} u'_x)$$

$$\Rightarrow \gamma_u u_x = \gamma_v (\gamma_{u'} u'_x + \gamma_{u'} v) = \gamma_v (\gamma_{u'} u'_x + \beta_v \gamma_{u'} c)$$

$$\Rightarrow \gamma_u u_y = \gamma_{u'} u'_y \quad \gamma_u u_z = \gamma_{u'} u'_z$$

Velocity 4-vector:

$$\begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix} = \mathcal{L}_v \begin{pmatrix} \gamma_{u'} c \\ \gamma_{u'} u'_x \\ \gamma_{u'} u'_y \\ \gamma_{u'} u'_z \end{pmatrix}$$

Some details:

$$\gamma_u = \gamma_v \gamma_{u'} \left(1 + v u'_x / c^2\right) \Rightarrow \left(1 - \frac{v^2}{c^2}\right) \left(1 - \frac{u'^2}{c^2}\right) = \left(1 - \frac{u^2}{c^2}\right) \left(1 + \frac{u_x v}{c^2}\right)^2$$

$$\text{where } u_x = \frac{u'_x + v}{1 + v u'_x / c^2} \quad u_y = \frac{u'_y}{\gamma_v (1 + v u'_x / c^2)} \quad u_z = \frac{u'_z}{\gamma_v (1 + v u'_x / c^2)}.$$

$$\left(\frac{u_x^2}{c^2} + \frac{u_y^2}{c^2} + \frac{u_z^2}{c^2}\right) \left(1 + \frac{u_x v}{c^2}\right)^2 = \left(\frac{u'_x}{c} + \frac{v}{c}\right)^2 + \left(\frac{u'^2_y}{c^2} + \frac{u'^2_z}{c^2}\right) \left(1 - \frac{v^2}{c^2}\right)$$

$$\frac{u^2}{c^2} \left(1 + \frac{u_x v}{c^2}\right)^2 = \frac{u'^2}{c^2} \left(1 - \frac{v^2}{c^2}\right) + \left(1 + \frac{u_x v}{c^2}\right)^2 - \left(1 - \frac{v^2}{c^2}\right)$$

$$\Rightarrow \left(1 - \frac{u^2}{c^2}\right) \left(1 + \frac{u_x v}{c^2}\right)^2 = \left(1 - \frac{u'^2}{c^2}\right) \left(1 - \frac{v^2}{c^2}\right)$$

Significance of 4-velocity vector:
$$\begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix}$$

Introduce the “rest” mass m of particle characterized by velocity $\mathbf{u}$:

$$m c \begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix} = \begin{pmatrix} \gamma_u m c^2 \\ \gamma_u m u_x c \\ \gamma_u m u_y c \\ \gamma_u m u_z c \end{pmatrix} = \begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix}$$

Properties of energy-moment 4-vector:

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \mathcal{L} \begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} \quad \begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} \quad \text{Note: } E^2 - p^2 c^2 = E'^2 - p'^2 c^2$$

Properties of Energy-momentum 4-vector -- continued

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \begin{pmatrix} \gamma_u m c^2 \\ \gamma_u m u_x c \\ \gamma_u m u_y c \\ \gamma_u m u_z c \end{pmatrix}$$

$$\text{Note: } E^2 - p^2 c^2 = \frac{(m c^2)^2}{1 - \beta_u^2} \left(1 - \left(\frac{u_x}{c} \right)^2 - \left(\frac{u_y}{c} \right)^2 - \left(\frac{u_z}{c} \right)^2 \right) = (m c^2)^2 = E'^2 - p'^2 c^2$$

Notion of "rest energy": For $\mathbf{p} \equiv 0$, $E = m c^2$

Define kinetic energy: $E_K \equiv E - m c^2 = \sqrt{p^2 c^2 + m^2 c^4} - m c^2$

Non-relativistic limit: If $\frac{p}{m c} \ll 1$, $E_K = m c^2 \left(\sqrt{1 + \left(\frac{p}{m c} \right)^2} - 1 \right)$

$$\approx \frac{p^2}{2m}$$

Summary of relativistic energy relationships

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \begin{pmatrix} \gamma_u m c^2 \\ \gamma_u m u_x c \\ \gamma_u m u_y c \\ \gamma_u m u_z c \end{pmatrix}$$

$$E = \sqrt{p^2 c^2 + m^2 c^4} = \gamma_u m c^2$$

$$\text{Check: } \sqrt{p^2 c^2 + m^2 c^4} = m c^2 \sqrt{\gamma_u^2 \beta_u^2 + 1} = \gamma_u m c^2$$

Example: for an electron $m c^2 = 0.5 \text{ MeV}$

for $E = 200 \text{ GeV}$

$$\gamma_u = \frac{E}{m c^2} = 4 \times 10^5$$

$$\beta_u = \sqrt{1 - \frac{1}{\gamma_u^2}} \approx 1 - \frac{1}{2\gamma_u^2} \approx 1 - 3 \times 10^{-12}$$

Special theory of relativity and Maxwell's equations

Continuity equation:
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

Lorenz gauge condition:
$$\frac{1}{c} \frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{A} = 0$$

Potential equations:
$$\frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = 4\pi\rho$$

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \frac{4\pi}{c} \mathbf{J}$$

Field relations:
$$\mathbf{E} = -\nabla \Phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

More 4-vectors:

Time and position :

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \Rightarrow x^\alpha$$

Charge and current :

$$\begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix} \Rightarrow J^\alpha$$

Vector and scalar potentials :

$$\begin{pmatrix} \Phi \\ A_x \\ A_y \\ A_z \end{pmatrix} \Rightarrow A^\alpha$$

Lorentz transformations

$$\mathcal{L}_v = \begin{pmatrix} \gamma_v & \gamma_v \beta_v & 0 & 0 \\ \gamma_v \beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Time and space :

$$x^\alpha = \mathcal{L}_v x'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} x'^\beta$$

Charge and current :

$$J^\alpha = \mathcal{L}_v J'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} J'^\beta$$

Vector and scalar potential : $A^\alpha = \mathcal{L}_v A'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} A'^\beta$

Summary of results --

Time and space :

$$x^\alpha = \mathcal{L}_v x'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} x'^\beta$$

Charge and current :

$$J^\alpha = \mathcal{L}_v J'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} J'^\beta$$

Vector and scalar potential : $A^\alpha = \mathcal{L}_v A'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} A'^\beta$

Here, the notation varies among the textbooks.

In general, it is convenient to use the matrix multiplication conventions to multiply our 4×4 matrices and 4 vectors

$$\text{For example: } \mathcal{L}_v^{\alpha\beta} x'^\beta = \sum_{\beta=1}^4 \mathcal{L}_v^{\alpha\beta} x'^\beta = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$