PHY 712 Electrodynamics
10-10:50 AM MWF Online

Discussion for Lecture 26:

Continue reading Chap. 11 –

Theory of Special Relativity

A. Lorentz transformation relations

B. Electromagnetic field transformations

C. Connection to Liénard-Wiechert potentials for constant velocity sources

In this lecture, we will continue our discussion of Special Relativity. In particular, we will discuss how the E and B fields transform between two relatively moving reference frame. Using a particular example, we will be able to show that our results for transformed fields are consistent with the results we obtain using the analysis using the Lienard-Wiechert potentials discussed earlier.

21	Mon: 03/22/2021	Chap. 8	EM waves in wave guides		
22	Wed: 03/24/2021	Chap. 9	Radiation from localized oscillating sources	#15	03/26/2021
23	Fri: 03/26/2021	Chap. 9	Radiation from oscillating sources	#16	03/29/2021
24	Mon: 03/29/2021	Chap. 9 & 10	Radiation and scattering	#17	03/31/2021
25	Wed: 03/31/2021	Chap. 11	Special Theory of Relativity	#18	04/05/2021
26	Fri: 04/02/2021	Chap. 11	Special Theory of Relativity		

PHY 712 -- Assignment #18

March 31, 2021

Begin reading Chapter 11 in **Jackson** .

1. Derive the relationships between the component of the electric and magnetic field components E_1, E_2, E_3, B_1, B_2, and B_3 as measured in the stationary frame of reference and the components E'_1, E'_2, E'_3, B'_1, B'_2, and B'_3 measured in the moving frame of reference. Note that the reverse relationships are given in Eq. 11.148.

The homework #20 assigned last lecture is due on Friday. No new homework has been assigned.

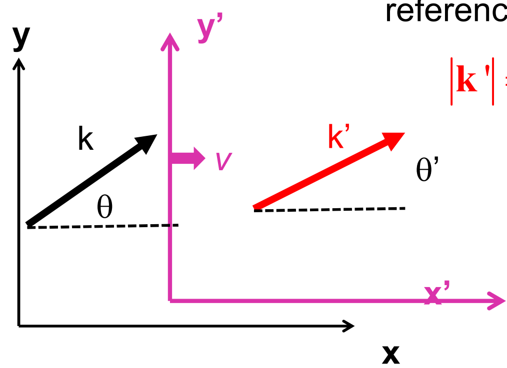
Your questions – (from lecture 25)

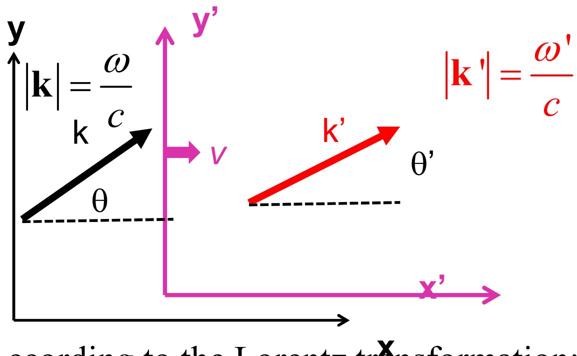
From Nick -- What exactly does \theta represent in the slides (especially circa the doppler effect). Can you go over the proof of the relationships on slide 22 lecture 25? I'm not sure where you are starting from and also what is the point of those particular relationships?

$$|\mathbf{k}| = \frac{\omega}{c}$$

Wave vector and frequencies in two reference frames

$$|\mathbf{k}'| = \frac{\omega'}{c}$$





According to the Lorentz transformation:

$$\omega' = \gamma\omega(1-\beta\cos\theta)$$

$$\omega'\cos\theta' = \gamma\omega(\cos\theta-\beta)$$

$$\omega'\sin\theta' = \omega\sin\theta$$

$$\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta-\beta)}$$

For $\theta = 0 = \theta'$

$$\omega' = \gamma\omega(1-\beta)$$

$$\omega' = \omega\sqrt{\frac{1-\beta}{1+\beta}}$$

Velocity relationships

Consider: $u_x = \dfrac{u'_x + v}{1 + vu'_x / c^2} \qquad u_y = \dfrac{u'_y}{\gamma_v\left(1 + vu'_x / c^2\right)} \qquad u_z = \dfrac{u'_z}{\gamma_v\left(1 + vu'_x / c^2\right)}.$

Note that $\gamma_u \equiv \dfrac{1}{\sqrt{1-(u/c)^2}} = \dfrac{1 + vu'_x / c^2}{\sqrt{1-(u'/c)^2}\sqrt{1-(v/c)^2}} = \gamma_v \gamma_{u'}\left(1 + vu'_x / c^2\right)$

$\Rightarrow \gamma_u c = \gamma_v\left(\gamma_{u'} c + \beta_v \gamma_{u'} u'_x\right)$

$\Rightarrow \gamma_u u_x = \gamma_v\left(\gamma_{u'} u'_x + \gamma_{u'} v\right) = \gamma_v\left(\gamma_{u'} u'_x + \beta_v \gamma_{u'} c\right)$

$\Rightarrow \gamma_u u_y = \gamma_{u'} u'_y \qquad \gamma_u u_z = \gamma_{u'} u'_z$

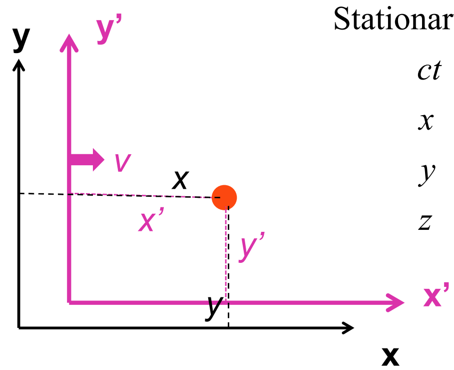
$$\Rightarrow \quad \begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix} = \mathcal{L}_v \begin{pmatrix} \gamma_{u'} c \\ \gamma_{u'} u'_x \\ \gamma_{u'} u'_y \\ \gamma_{u'} u'_z \end{pmatrix}$$

Lorentz transformations

Convenient notation :

$$\beta_v \equiv \frac{v}{c}$$

$$\gamma_v \equiv \frac{1}{\sqrt{1-\beta_v^{\,2}}}$$



Stationary frame		Moving frame
ct	$=$	$\gamma(ct'+\beta x')$
x	$=$	$\gamma(x'+\beta ct')$
y	$=$	y'
z	$=$	z'

We will continue to use the stationary and moving reference frames introduced in the previous lecture. In this case, the relative motion is along the x-axis. Of course, there is nothing special about this choice, but we will use it throughout this lecture.

Lorentz transformations -- continued

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\boldsymbol{\mathcal{L}}_v = \begin{pmatrix} \gamma_v & \gamma_v\beta_v & 0 & 0 \\ \gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{\mathcal{L}}_v^{-1} = \begin{pmatrix} \gamma_v & -\gamma_v\beta_v & 0 & 0 \\ -\gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \boldsymbol{\mathcal{L}}_v \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} \qquad \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \boldsymbol{\mathcal{L}}_v^{-1} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Notice:

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2$$

This slide reviews the transformations of the time and position 4-vector.

More questions –

From Gao -- How can we deduce that the field strength tensor transforms as a tensor with a Lorentz transformation sandwich?

Special theory of relativity and Maxwell's equations

Continuity equation: $$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

Lorenz gauge condition: $$\frac{1}{c}\frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{A} = 0$$

Potential equations: $$\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = 4\pi\rho$$

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \frac{4\pi}{c}\mathbf{J}$$

Field relations: $$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

This slide reviews the relevant equations for the continuity of our sources, and for Maxwell's equations in terms of the scalar and vector potentials, and for the relationship of the E and B fields to the scalar and vector potentials.

More 4-vectors: $\alpha = \{0,1,2,3\}$

Time and position : $\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \Rightarrow x^\alpha$

Charge and current : $\begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix} \Rightarrow J^\alpha$

Vector and scalar potentials : $\begin{pmatrix} \Phi \\ A_x \\ A_y \\ A_z \end{pmatrix} \Rightarrow A^\alpha$

Here we identify 4-vectors of time-position, charge and current sources, and scalar and vector potentials.

Lorentz transformations

$$\mathcal{L}_v = \begin{pmatrix} \gamma_v & \gamma_v\beta_v & 0 & 0 \\ \gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Time and space: $x^\alpha = \mathcal{L}_v x'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} x'^\beta$

Charge and current: $J^\alpha = \mathcal{L}_v J'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} J'^\beta$

Vector and scalar potential: $A^\alpha = \mathcal{L}_v A'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} A'^\beta$

Notation: $\mathcal{L}_v^{\alpha\beta} x'^\beta \equiv \sum_{\beta=0}^{3} \mathcal{L}_v^{\alpha\beta} x'^\beta$

Repeated index summation convention

It is reasonable to postulate that each of these three 4-vectors transform from one reference frame to another with the Lorentz transformation.

4-vector relationships

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \Leftrightarrow \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} \Leftrightarrow \left(A^0, \mathbf{A}\right): \text{ upper index 4-vector } \quad A^\alpha \text{ for } \left(\alpha = 0,1,2,3\right)$$

Keeping track of signs -- lower index 4-vector $A_\alpha = \left(A^0, -\mathbf{A}\right)$

Derivative operators (defined with different sign convention):

$$\partial^\alpha = \left(\frac{\partial}{c\partial t}, -\nabla\right) \qquad \partial_\alpha = \left(\frac{\partial}{c\partial t}, \nabla\right)$$

In addition to the 4-vectors we have defined up to now, which are written with an upper index alpha, we will also need to define a lower index version of the 4-vector which just means that the space part is taken with a minus sign. We also need a notation for derivatives with respect to time and space given with the partial symbol. It turns out that for consistency, the upper and lower signs needed for the derivative operator, the upper and lower signs must be given as indicated. While Jackson's conventions are consistent throughout his text, other textbooks may use other sign conventions.

Special theory of relativity and Maxwell's equations

Continuity equation: $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \quad \rightarrow \quad \partial_\alpha J^\alpha = 0$

Lorenz gauge condition: $\frac{1}{c}\frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{A} = 0 \quad \rightarrow \quad \partial_\alpha A^\alpha = 0$

Potential equations:

$$\left.\begin{array}{l} \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = 4\pi\rho \\ \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \frac{4\pi}{c}\mathbf{J} \end{array}\right\} \partial_\alpha \partial^\alpha A^\beta = \frac{4\pi}{c} J^\beta$$

Field relations:

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$\rightarrow$ **??**

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Here we exercise our new notation to write the important equations. I have to admit the new notation looks quite compact, (pretty, intriguing?) But what about the E and B fields, how does the new notation work for them?

From the scalar and vector potentials, we can determine the E and B fields and then relate them to 4-vectors, finding --

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \qquad E_x = -\frac{\partial \Phi}{\partial x} - \frac{\partial A_x}{c\partial t} = -\left(\partial^0 A^1 - \partial^1 A^0\right)$$

$$E_y = -\frac{\partial \Phi}{\partial y} - \frac{\partial A_y}{c\partial t} = -\left(\partial^0 A^2 - \partial^2 A^0\right)$$

$$E_z = -\frac{\partial \Phi}{\partial z} - \frac{\partial A_z}{c\partial t} = -\left(\partial^0 A^3 - \partial^3 A^0\right)$$

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -\left(\partial^2 A^3 - \partial^3 A^2\right)$$

$$B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = -\left(\partial^3 A^1 - \partial^1 A^3\right)$$

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = -\left(\partial^1 A^2 - \partial^2 A^1\right)$$

Writing out the 6 equations for all of the E and B field components, we see that the new notation has a very nice pattern, but each field component has two indices!!! We can thus conclude that the 6 E and B field components are part of a 4x4 matrix or tensor.

Field strength tensor $F^{\alpha\beta} \equiv \left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right)$

For stationary frame

$$F^{\alpha\beta} \equiv \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

For moving frame

$$F'^{\alpha\beta} \equiv \begin{pmatrix} 0 & -E'_x & -E'_y & -E'_z \\ E'_x & 0 & -B'_z & B'_y \\ E'_y & B'_z & 0 & -B'_x \\ E'_z & -B'_y & B'_x & 0 \end{pmatrix}$$

Therefore we can define the field strength tensor and assign each of the 6 field components and their negative values to an entry in the 4x4 field strength tensor. From this logic, we can then deduce that the field strength tensor transforms as a tensor with a Lorentz transformation sandwich. Evaluating the multiplication of the three matrices, we obtain the result given on the last line. This is related to your homework problem due Friday.

Summary --

Field strength tensor $F^{\alpha\beta} \equiv \left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right)$

$$F^{\alpha\beta} \equiv \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \qquad F'^{\alpha\beta} \equiv \begin{pmatrix} 0 & -E'_x & -E'_y & -E'_z \\ E'_x & 0 & -B'_z & B'_y \\ E'_y & B'_z & 0 & -B'_x \\ E'_z & -B'_y & B'_x & 0 \end{pmatrix}$$

➔ This analysis shows that the E and B fields must be treated as components of the field strength tensor and that in order to transform between inertial frames, we need to use the tensor transformation relationships:

Transformation of field strength tensor

$$F^{\alpha\beta} = \mathcal{L}_v^{\alpha\gamma} F'^{\gamma\delta} \mathcal{L}_v^{\delta\beta} \qquad \mathcal{L}_v = \begin{pmatrix} \gamma_v & \gamma_v\beta_v & 0 & 0 \\ \gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E'_x & -\gamma_v\left(E'_y+\beta_v B'_z\right) & -\gamma_v\left(E'_z-\beta_v B'_y\right) \\ E'_x & 0 & -\gamma_v\left(B'_z+\beta_v E'_y\right) & \gamma_v\left(B'_y-\beta_v E'_z\right) \\ \gamma_v\left(E'_y+\beta_v B'_z\right) & \gamma_v\left(B'_z+\beta_v E'_y\right) & 0 & -B'_x \\ \gamma_v\left(E'_z-\beta_v B'_y\right) & -\gamma_v\left(B'_y-\beta_v E'_z\right) & B'_x & 0 \end{pmatrix}$$

Therefore we can define the field strength tensor and assign each of the 6 field components and their negative values to an entry in the 4x4 field strength tensor. From this logic, we can then deduce that the field strength tensor transforms as a tensor with a Lorentz transformation sandwich. Evaluating the multiplication of the three matrices, we obtain the result given on the last line. This is related to your homework problem due Monday.

Inverse transformation of field strength tensor

$$F'^{\alpha\beta} = \mathcal{L}_v^{-1\alpha\gamma} F^{\gamma\delta} \mathcal{L}_v^{-1\delta\beta} \qquad \mathcal{L}_v^{-1} = \begin{pmatrix} \gamma_v & -\gamma_v\beta_v & 0 & 0 \\ -\gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F'^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -\gamma_v\left(E_y - \beta_v B_z\right) & -\gamma_v\left(E_z + \beta_v B_y\right) \\ E_x & 0 & -\gamma_v\left(B_z - \beta_v E_y\right) & \gamma_v\left(B_y + \beta_v E_z\right) \\ \gamma_v\left(E_y - \beta_v B_z\right) & \gamma_v\left(B_z - \beta_v E_y\right) & 0 & -B_x \\ \gamma_v\left(E_z + \beta_v B_y\right) & -\gamma_v\left(B_y + \beta_v E_z\right) & B_x & 0 \end{pmatrix}$$

Summary of results:

$$E'_x = E_x \qquad B'_x = B_x$$

$$E'_y = \gamma_v\left(E_y - \beta_v B_z\right) \qquad B'_y = \gamma_v\left(B_y + \beta_v E_z\right)$$

$$E'_z = \gamma_v\left(E_z + \beta_v B_y\right) \qquad B'_z = \gamma_v\left(B_z - \beta_v E_y\right)$$

Using the same logic, it is possible to evaluate the inverse transformation. The last result is the same as given in Jackson Eq. 11.148.

Comparison of the two transformations

$$F^{\alpha\beta} = \mathcal{L}_v^{\alpha\gamma} F'^{\gamma\delta} \mathcal{L}_v^{\delta\beta} \qquad \mathcal{L}_v = \begin{pmatrix} \gamma_v & \gamma_v\beta_v & 0 & 0 \\ \gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

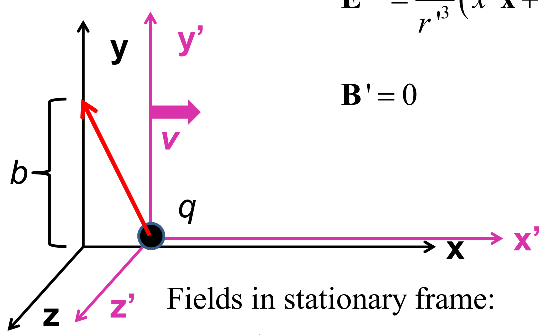
$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E'_x & -\gamma_v\left(E'_y + \beta_v B'_z\right) & -\gamma_v\left(E'_z - \beta_v B'_y\right) \\ E'_x & 0 & -\gamma_v\left(B'_z + \beta_v E'_y\right) & \gamma_v\left(B'_y - \beta_v E'_z\right) \\ \gamma_v\left(E'_y + \beta_v B'_z\right) & \gamma_v\left(B'_z + \beta_v E'_y\right) & 0 & -B'_x \\ \gamma_v\left(E'_z - \beta_v B'_y\right) & -\gamma_v\left(B'_y - \beta_v E'_z\right) & B'_x & 0 \end{pmatrix}$$

$$F'^{\alpha\beta} = \mathcal{L}_v^{-1\alpha\gamma} F^{\gamma\delta} \mathcal{L}_v^{-1\delta\beta} \qquad \mathcal{L}_v^{-1} = \begin{pmatrix} \gamma_v & -\gamma_v\beta_v & 0 & 0 \\ -\gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F'^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -\gamma_v\left(E_y - \beta_v B_z\right) & -\gamma_v\left(E_z + \beta_v B_y\right) \\ E_x & 0 & -\gamma_v\left(B_z - \beta_v E_y\right) & \gamma_v\left(B_y + \beta_v E_z\right) \\ \gamma_v\left(E_y - \beta_v B_z\right) & \gamma_v\left(B_z - \beta_v E_y\right) & 0 & -B_x \\ \gamma_v\left(E_z + \beta_v B_y\right) & -\gamma_v\left(B_y + \beta_v E_z\right) & B_x & 0 \end{pmatrix}$$

Comparing the various transformations.

Example:



Fields in moving frame:

$$\mathbf{E}' = \frac{q}{r'^3}\left(x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}\right) = \frac{q\left(-vt'\hat{\mathbf{x}} + b\hat{\mathbf{y}}\right)}{\left(\left(-vt'\right)^2 + b^2\right)^{3/2}}$$

$$\mathbf{B}' = 0$$

Fields in stationary frame:

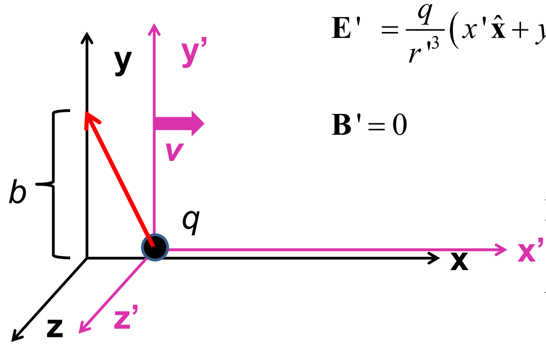
$$E_x = E'_x \qquad B_x = B'_x$$

$$E_y = \gamma_v\left(E'_y + \beta_v B'_z\right) \qquad B_y = \gamma_v\left(B'_y - \beta_v E'_z\right)$$

$$E_z = \gamma_v\left(E'_z - \beta_v B'_y\right) \qquad B_z = \gamma_v\left(B'_z + \beta_v E'_y\right)$$

Now, consider a particular example discussed in Section 11.10 of Jackson. A particle sits at the origin of the moving frame. The E and B fields are measured at the point b yhat in the stationary frame. What are the values of the fields measured in the stationary frame?

Example:



Fields in moving frame:

$$\mathbf{E}' = \frac{q}{r'^3}\left(x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}\right) = \frac{q\left(-vt'\hat{\mathbf{x}} + b\hat{\mathbf{y}}\right)}{\left(\left(-vt'\right)^2 + b^2\right)^{3/2}}$$

$$\mathbf{B}' = 0$$

Fields in stationary frame:

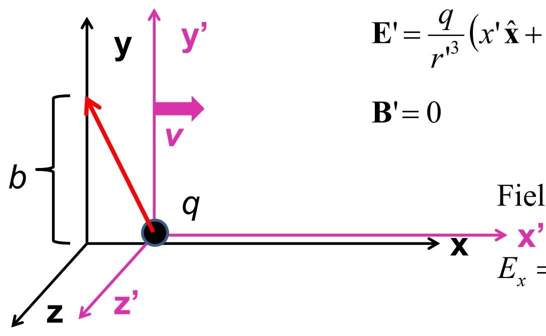
$$E_x = E'_x = \frac{q\left(-vt'\right)}{\left(\left(-vt'\right)^2 + b^2\right)^{3/2}}$$

$$E_y = \gamma_v\left(E'_y\right) = \frac{q\left(\gamma_v b\right)}{\left(\left(-vt'\right)^2 + b^2\right)^{3/2}}$$

$$B_z = \gamma_v\left(\beta_v E'_y\right) = \frac{q\left(\gamma_v \beta_v b\right)}{\left(\left(-vt'\right)^2 + b^2\right)^{3/2}}$$

It is easy to write the fields in the moving frame, since the particle is stationary in that frame. Then we use the transformation equations to find the fields in the stationary frame. We are not quite done, because the expressions involve the time measured in the moving frame.

Example:



Fields in moving frame:

$$\mathbf{E}' = \frac{q}{r'^3}\left(x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}\right) = \frac{q\left(-vt'\hat{\mathbf{x}} + b\hat{\mathbf{y}}\right)}{\left((-vt')^2 + b^2\right)^{3/2}}$$

$$\mathbf{B}' = 0$$

Fields in stationary frame:

$$E_x = E'_x \quad = \frac{q\left(-v\gamma_v t\right)}{\left(\left(-v\gamma_v t\right)^2 + b^2\right)^{3/2}}$$

$$E_y = \gamma_v\left(E'_y\right) = \frac{q\left(\gamma_v b\right)}{\left(\left(-v\gamma_v t\right)^2 + b^2\right)^{3/2}}$$

$$B_z = \gamma_v\left(\beta_v E'_y\right) = \frac{q\left(\gamma_v \beta_v b\right)}{\left(\left(-v\gamma_v t\right)^2 + b^2\right)^{3/2}}$$

Expression in terms of consistent coordinates

$$t' = \gamma_v t$$

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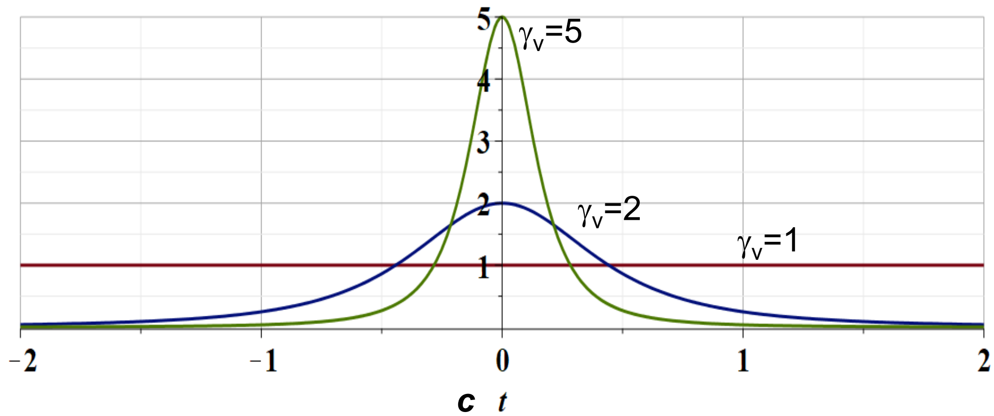
Using the time-coordinate transformation we can then write the fields measured in the stationary frame in terms of the time appropriate to that frame.

$$E_y = \frac{q(\gamma_v b)}{\left((-v\gamma_v t)^2 + b^2\right)^{3/2}} = \frac{q(\gamma_v b)}{\left((\gamma_v^2 - 1)c^2 t^2 + b^2\right)^{3/2}}$$



This plot shows the y component of the electric field as measured in the stationary frame plotted as a function of time. For large gamma, there is a large peak at t=0.

Examination of this system from the viewpoint of the the Liènard-Wiechert potentials (temporarily keeping SI units)

$$\rho(\mathbf{r},t) = q\delta^3(\mathbf{r}-\mathbf{R}_q(t)) \quad \mathbf{J}(\mathbf{r},t) = q\dot{\mathbf{R}}_q(t)\delta^3(\mathbf{r}-\mathbf{R}_q(t)) \quad \dot{\mathbf{R}}_q(t) = \frac{d\mathbf{R}_q(t)}{dt}$$

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int\int d^3r'\,dt'\,\frac{\rho(\mathbf{r},t')}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\right)$$

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0 c^2}\int\int d^3r'\,dt'\,\frac{\mathbf{J}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\right)$$

Evaluating integral over t':

$$\int_{-\infty}^{\infty} dt'\, f(t')\delta\left(t'-(t-|\mathbf{r}-\mathbf{R}_q(t')|/c)\right) = \frac{f(t_r)}{1-\dfrac{\mathbf{R}_q(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c|\mathbf{r}-\mathbf{R}_q(t_r)|}},$$

Do these results make sense? In order to check the results, we can calculate the fields directly in the stationary frame using the methods we discussed several lectures ago using the Lienard-Wiechert potentials. Here we review some of those equations.

Examination of this system from the viewpoint of the the Liènard-Wiechert potentials – continued (SI units)

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2}\frac{\mathbf{v}}{R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{R}_q(t_r)$ $\quad \mathbf{v} = \frac{d\mathbf{R}_q(t_r)}{dt_r}$

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}$$

$$\mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}(\mathbf{r},t)$$

More equations.

Examination of this system from the viewpoint of the the Liènard-Wiechert potentials – continued (SI units)

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R}-\frac{\mathbf{v}R}{c}\right)\left(1-\frac{v^2}{c^2}\right)+\left(\mathbf{R}\times\left\{\left(\mathbf{R}-\frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1-\frac{v^2}{c^2}+\frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right)-\frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{cR}.$$

Finally the E and B fields obtained from that analysis.

Examination of this system from the viewpoint of the the Liènard-Wiechert potentials – (**Gaussian units**)

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R}\times\left\{\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{R}.$$

Here are the equations in cgs Gaussian units that we are now using.

Examination of this system from the viewpoint of the the Liènard-Wiechert potentials – continued (**Gaussian units**)

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right)\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2}\right)\right]$$

For our example:

$\mathbf{R}_q(t_r) = vt_r\hat{\mathbf{x}}$ $\quad \mathbf{r} = b\hat{\mathbf{y}}$

$\mathbf{R} = b\hat{\mathbf{y}} - vt_r\hat{\mathbf{x}}$ $\quad R = \sqrt{v^2t_r^2 + b^2}$

$\mathbf{v} = v\hat{\mathbf{x}}$ $\quad t_r = t - \frac{R}{c}$

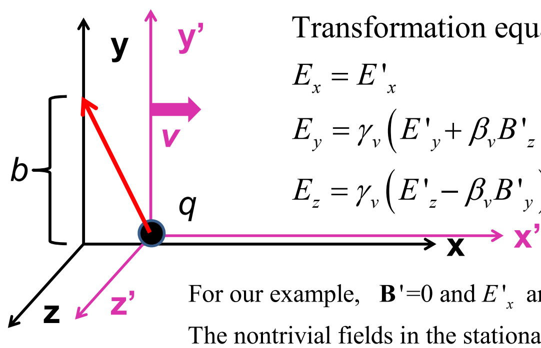
This should be equivalent to the result given in Jackson (11.152):

$$\mathbf{E}(x,y,z,t) = \mathbf{E}(0,b,0,t) = q\frac{-v\gamma t\hat{\mathbf{x}} + \gamma b\hat{\mathbf{y}}}{\left(b^2 + (v\gamma t)^2\right)^{3/2}}$$

$$\mathbf{B}(x,y,z,t) = \mathbf{B}(0,b,0,t) = q\frac{\gamma\beta b\hat{\mathbf{z}}}{\left(b^2 + (v\gamma t)^2\right)^{3/2}}$$

Now to evaluate the equations, we need to consider the constant velocity trajectory of our example. We will continue this discussion on Friday.

Summary --



Transformation equations:

$$E_x = E'_x \qquad B_x = B'_x$$

$$E_y = \gamma_v\left(E'_y + \beta_v B'_z\right) \qquad B_y = \gamma_v\left(B'_y - \beta_v E'_z\right)$$

$$E_z = \gamma_v\left(E'_z - \beta_v B'_y\right) \qquad B_z = \gamma_v\left(B'_z + \beta_v E'_y\right)$$

For our example, $\mathbf{B}'=0$ and E'_x and E'_y are nontrivial

The nontrivial fields in the stationary frame are

$$E_x = E'_x$$

$$E_y = \gamma_v E'_y$$

$$B_z = \gamma_v \beta_v E'_y$$