PHY 712 Electrodynamics
10-10:50 AM MWF Online

Discussion for Lecture 27:

Finish Chap. 11 and begin Chap. 14

A. Electromagnetic field transformations & corresponding analysis of Liénard-Wiechert potentials for constant velocity sources

B. Radiation by moving charged particles

In this lecture we will continue to discuss the electromagnetic fields produced by a moving charged particle using the Lienard-Wiechert potentials. First we need to make sure that we obtain consistent results with Lecture 26. Then we will start to discuss the results from more general trajectories.

21	Mon: 03/22/2021	Chap. 8	EM waves in wave guides		
22	Wed: 03/24/2021	Chap. 9	Radiation from localized oscillating sources	#15	03/26/2021
23	Fri: 03/26/2021	Chap. 9	Radiation from oscillating sources	#16	03/29/2021
24	Mon: 03/29/2021	Chap. 9 & 10	Radiation and scattering	#17	03/31/2021
25	Wed: 03/31/2021	Chap. 11	Special Theory of Relativity	#18	04/05/2021
26	Fri: 04/02/2021	Chap. 11	Special Theory of Relativity		
27	Mon: 04/05/2021	Chap. 11	Special Theory of Relativity	#19	04/09/2021
	Wed: 04/07/2021	No class	*Holiday*		
28	Fri: 04/09/2021	Chap. 14	Radiation from accelerating charged particles		
29	Mon: 04/12/2021	Chap. 14	Synchrotron radiation		

PHY 712 -- Assignment #19

April 05, 2021

Continue reading Chapter 11 in **Jackson** .

1. Supply some of the intermediate steps for deriving the **E** and **B** fields resulting from a particle of charge q moving along the x-axis at constant speed v, measured at a point a distance b along the y-axis.

The homework from today's lecture involves deriving some of the details of today's lecture.

Comment: Some of you have been looking at textbooks (such as Zangwill) and sources available on the internet and finding different equations from those presented in these lecture notes and in Jackson. That is a good thing in general, however please be aware that there are different units (SI for example) and different conventions for 4-vectors (some using different ordering of space and time, some using imaginary (i) for the time-like portion). Since we are using Jackson for now, it will good to make sure that you are OK with those equations as well.

PHYSICS COLLOQUIUM

4 PM

THURSDAY • APRIL 8, 2021

"Can Next-Generation 6G Mobile Communications Above 100 GHz Find a Way to Coexist with Passive Satellites Used for Weather and Environmental Sensing?"

Radio frequencies above 100 GHz presently have little actual use except for passive systems used for radio astronomy and for satellite-based sensing of weather data and pollution monitoring. But new technology and the growing demands for terrestrial telecom such as smart phones has resulted in growing needs for capacity in 5G and 6G systems. Some of this capacity is expected to be above 100 GHz for policy decisions in the 1980s and 90s set aside many blocks of spectrum for purely passive systems to a much greater degree than in lower frequencies. A major challenge is thus how can we shoehorn both uses into the same spectrum. Fortunately the quirky nature of radio propagation above 100 GHz offers some possible paths as does the small wavelengths here that permit novel antenna designs. The talk will review possible building blocks of such a solution and

Dr. Michael J. Marcus
Marcus Spectrum Solutions, LLC
Washington, DC

4:00 pm
Via Video Conference

Your questions –

From Gao: Have we learned Liénard-Wiechert potentials before. Where do these formulas come from? If so, Could you offer some hints?

What follows are some of the slides from Lectures 15 & 16..

Solution of Maxwell's equations in the Lorentz gauge -- continued

Liènard-Wiechert potentials and fields --
Determination of the scalar and vector potentials for a moving point particle (also see Landau and Lifshitz ***The Classical Theory of Fields***, Chapter 8.)

Consider the fields produced by the following source: a point charge *q* moving on a trajectory $R_q(t)$.

Charge density: $\rho(\mathbf{r},t) = q\delta^3(\mathbf{r} - \mathbf{R}_q(t))$

Current density: $\mathbf{J}(\mathbf{r},t) = q\,\dot{\mathbf{R}}_q(t)\delta^3(\mathbf{r} - \mathbf{R}_q(t)),\quad \text{where}\quad \dot{\mathbf{R}}_q(t) \equiv \frac{d\mathbf{R}_q(t)}{dt}.$

Solution of Maxwell's equations in the Lorentz gauge -- continued

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int\int d^3r'dt' \frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}']|}\delta\left(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\right)$$

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0 c^2}\int\int d^3r'dt' \frac{\mathbf{J}(\mathbf{r}',\mathrm{t}')}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\right).$$

We performing the integrations over first d^3r' and then dt' making use of the fact that for any function of t',

$$\int_{-\infty}^{\infty} dt' \, f(t')\delta\left(t'-(t-|\mathbf{r}-\mathbf{R}_q(t')|/c)\right) = \frac{f(t_r)}{1-\frac{\dot{\mathbf{R}}_q(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c\,|\mathbf{r}-\mathbf{R}_q(t_r)|}},$$

where the ``retarded time'' is defined to be

$$t_r \equiv t - \frac{|\mathbf{r}-\mathbf{R}_q(t_r)|}{c}.$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

Resulting scalar and vector potentials:

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}},$$

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2}\frac{\mathbf{v}}{R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}},$$

Notation: $\mathbf{R} \equiv \mathbf{r} - \mathbf{R}_q(t_r)$

$$\mathbf{v} \equiv \dot{\mathbf{R}}_q(t_r), \qquad t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}.$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

In order to find the electric and magnetic fields, we need to evaluate

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t}$$

$$\mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t)$$

The trick of evaluating these derivatives is that the retarded time t_r depends on position **r** and on itself. We can show the following results using the shorthand notation:

$$\nabla t_r = -\frac{\mathbf{R}}{c\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)} \quad \text{and} \quad \frac{\partial t_r}{\partial t} = \frac{R}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)}.$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

$$-\nabla\Phi(\mathbf{r},t)=\frac{q}{4\pi\epsilon_0}\frac{1}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\mathbf{R}\left(1-\frac{v^2}{c^2}\right)-\frac{v}{c}\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)+\mathbf{R}\frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right],$$

$$-\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}=\frac{q}{4\pi\epsilon_0}\frac{1}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\frac{\mathbf{v}R}{c}\left(\frac{v^2}{c^2}-\frac{\mathbf{v}\cdot\mathbf{R}}{Rc}-\frac{\dot{\mathbf{v}}\cdot R}{c^2}\right)-\frac{\dot{\mathbf{v}}R}{c^2}\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)\right].$$

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{4\pi\epsilon_0}\frac{1}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R}-\frac{\mathbf{v}R}{c}\right)\left(1-\frac{v^2}{c^2}\right)+\left(\mathbf{R}\times\left\{\left(\mathbf{R}-\frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right].$$

$$\mathbf{B}(\mathbf{r},t)=\frac{q}{4\pi\epsilon_0c^2}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1-\frac{v^2}{c^2}+\frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right)-\frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right]=\frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{cR}$$

Note that this analysis is carried out in a single frame of reference. Now we resume our discussion about transforming values between two different inertial frames of reference.

Field strength tensor $F^{\alpha\beta} \equiv \left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right)$

$$F^{\alpha\beta} \equiv \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \qquad F'^{\alpha\beta} \equiv \begin{pmatrix} 0 & -E'_x & -E'_y & -E'_z \\ E'_x & 0 & -B'_z & B'_y \\ E'_y & B'_z & 0 & -B'_x \\ E'_z & -B'_y & B'_x & 0 \end{pmatrix}$$

Transformation of field strength tensor

$$F^{\alpha\beta} = \mathcal{L}_v{}^{\alpha\gamma} F'^{\gamma\delta} \mathcal{L}_v{}^{\delta\beta} \qquad \mathcal{L}_v = \begin{pmatrix} \gamma_v & \gamma_v\beta_v & 0 & 0 \\ \gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E'_x & -\gamma_v\left(E'_y+\beta_v B'_z\right) & -\gamma_v\left(E'_z-\beta_v B'_y\right) \\ E'_x & 0 & -\gamma_v\left(B'_z+\beta_v E'_y\right) & \gamma_v\left(B'_y-\beta_v E'_z\right) \\ \gamma_v\left(E'_y+\beta_v B'_z\right) & \gamma_v\left(B'_z+\beta_v E'_y\right) & 0 & -B'_x \\ \gamma_v\left(E'_z-\beta_v B'_y\right) & -\gamma_v\left(B'_y-\beta_v E'_z\right) & B'_x & 0 \end{pmatrix}$$

Lecture 26 introduced the field strength tensor.

Inverse transformation of field strength tensor

$$F'^{\alpha\beta} = \mathcal{L}_v^{-1\alpha\gamma} F^{\gamma\delta} \mathcal{L}_v^{-1\delta\beta} \qquad \mathcal{L}_v^{-1} = \begin{pmatrix} \gamma_v & -\gamma_v\beta_v & 0 & 0 \\ -\gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F'^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -\gamma_v\left(E_y - \beta_v B_z\right) & -\gamma_v\left(E_z + \beta_v B_y\right) \\ E_x & 0 & -\gamma_v\left(B_z - \beta_v E_y\right) & \gamma_v\left(B_y + \beta_v E_z\right) \\ \gamma_v\left(E_y - \beta_v B_z\right) & \gamma_v\left(B_z - \beta_v E_y\right) & 0 & -B_x \\ \gamma_v\left(E_z + \beta_v B_y\right) & -\gamma_v\left(B_y + \beta_v E_z\right) & B_x & 0 \end{pmatrix}$$

Summary of results:

$$E'_x = E_x \qquad B'_x = B_x$$

$$E'_y = \gamma_v\left(E_y - \beta_v B_z\right) \qquad B'_y = \gamma_v\left(B_y + \beta_v E_z\right)$$

$$E'_z = \gamma_v\left(E_z + \beta_v B_y\right) \qquad B'_z = \gamma_v\left(B_z - \beta_v E_y\right)$$

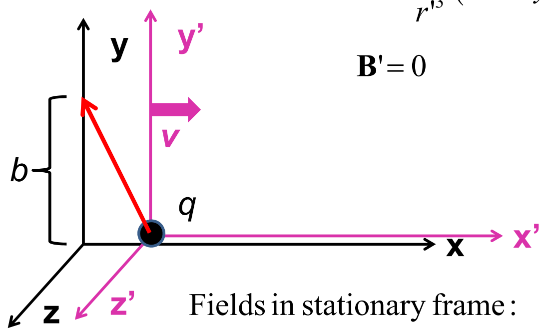
Review of the Lorentz transformation for the field strength tensor --

Example:

Fields in moving frame:

$$\mathbf{E}' = \frac{q}{r'^3}\left(x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}\right) = \frac{q\left(-vt'\hat{\mathbf{x}} + b\hat{\mathbf{y}}\right)}{\left(\left(-vt'\right)^2 + b^2\right)^{3/2}}$$

$$\mathbf{B}' = 0$$



Fields in stationary frame:

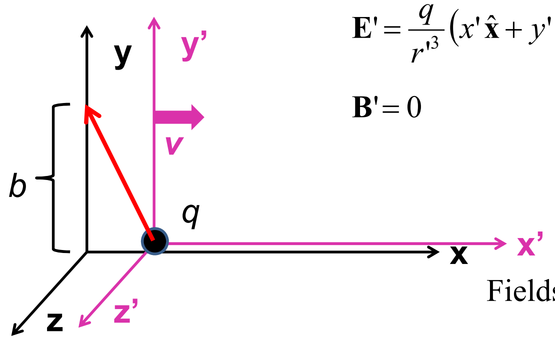
$$E_x = E'_x \qquad B_x = B'_x$$

$$E_y = \gamma_v\left(E'_y + \beta_v B'_z\right) \qquad B_y = \gamma_v\left(B'_y - \beta_v E'_z\right)$$

$$E_z = \gamma_v\left(E'_z - \beta_v B'_y\right) \qquad B_z = \gamma_v\left(B'_z + \beta_v E'_y\right)$$

This is the example that we have been studying from Lecture 26.

Example:



Fields in moving frame:

$$\mathbf{E}' = \frac{q}{r'^3}\left(x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}\right) = \frac{q\left(-vt'\hat{\mathbf{x}} + b\hat{\mathbf{y}}\right)}{\left((-vt')^2 + b^2\right)^{3/2}}$$

$$\mathbf{B}' = 0$$

Fields in stationary frame:

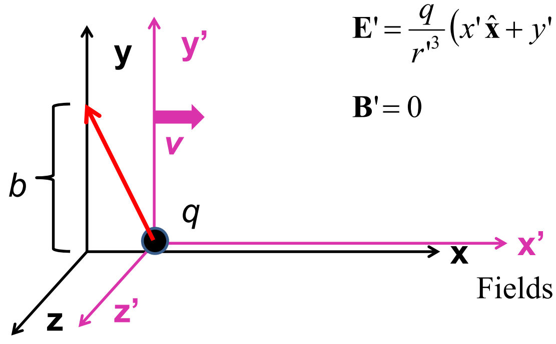
$$E_x = E'_x = \frac{q(-vt')}{\left((-vt')^2 + b^2\right)^{3/2}}$$

$$E_y = \gamma_v\left(E'_y\right) = \frac{q(\gamma_v b)}{\left((-vt')^2 + b^2\right)^{3/2}}$$

$$B_z = \gamma_v\left(\beta_v E'_y\right) = \frac{q(\gamma_v \beta_v b)}{\left((-vt')^2 + b^2\right)^{3/2}}$$

Using the fields from the moving frame, we can write the expressions for the fields in the stationary frame.

Example:



Fields in moving frame:

$$\mathbf{E}' = \frac{q}{r'^3}\left(x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}\right) = \frac{q\left(-vt'\hat{\mathbf{x}} + b\hat{\mathbf{y}}\right)}{\left(\left(-vt'\right)^2 + b^2\right)^{3/2}}$$

$$\mathbf{B}' = 0$$

Fields in stationary frame:

$$E_x = E'_x = \frac{q\left(-v\gamma_v t\right)}{\left(\left(-v\gamma_v t\right)^2 + b^2\right)^{3/2}}$$

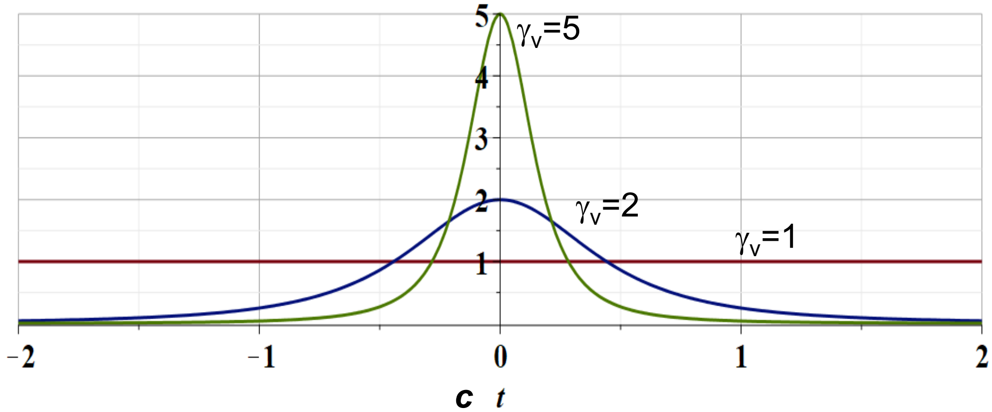
$$E_y = \gamma_v\left(E'_y\right) = \frac{q\left(\gamma_v b\right)}{\left(\left(-v\gamma_v t\right)^2 + b^2\right)^{3/2}}$$

$$B_z = \gamma_v\left(\beta_v E'_y\right) = \frac{q\left(\gamma_v \beta_v b\right)}{\left(\left(-v\gamma_v t\right)^2 + b^2\right)^{3/2}}$$

Expression in terms of consistent coordinates

Here the fields measured in the stationary frame are expressed in terms of the time t measured in the stationary frame.

$$E_y = \frac{q(\gamma_v b)}{\left((-v\gamma_v t)^2 + b^2\right)^{3/2}} = \frac{q(\gamma_v b)}{\left((\gamma_v^2 - 1)c^2 t^2 + b^2\right)^{3/2}} = B_z / (\gamma_v \beta_v)$$



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This is a plot shown in Lecture 26 of E_y as a function of time.

Examination of this system from the viewpoint of the the Liénard-Wiechert potentials –(Gaussian units)

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R}\times\left\{\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right]$$

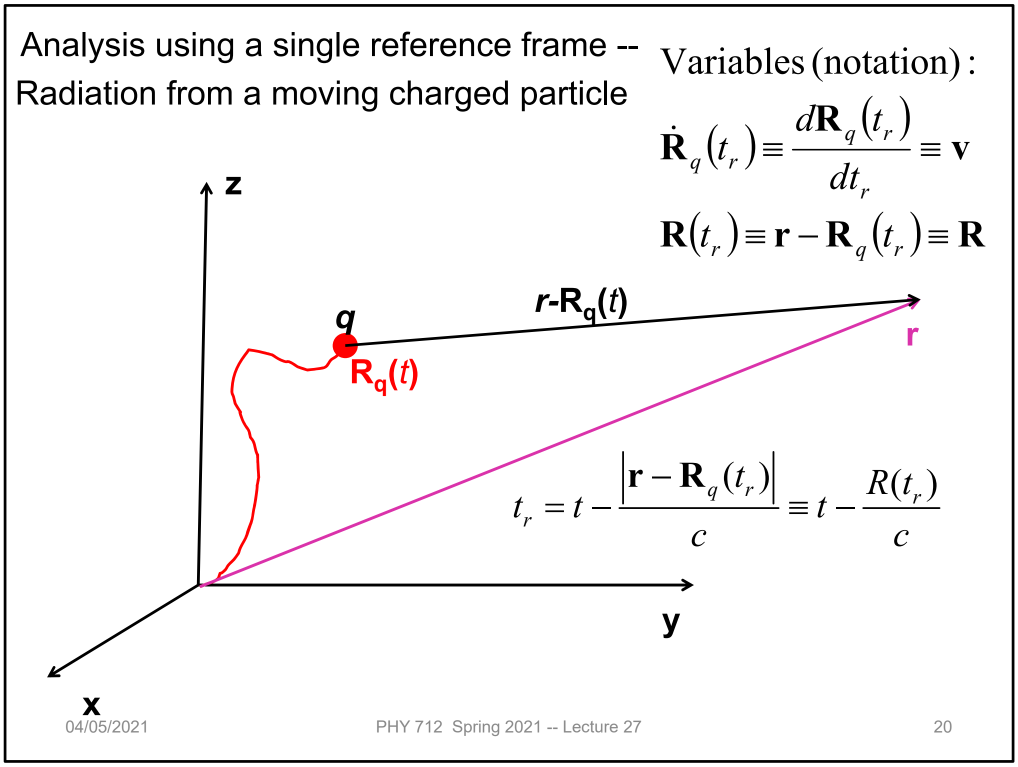
$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{R}.$$

Now we consider how we may arrive at the same result without changing reference frames by analyzing the EM fields produced by a moving charge using the Lienard-Wiechert analysis.

Question – Why would you want to use the Liénard-Wiechert potentials?

1. They are extremely complicated. It is best to avoid them at all costs?
2. The Lorentz transformations were bad enough?
3. There are some circumstances for which the Lorentz transformations do not simplify the analysis?



Here we consider a charged particle (charge q) moving along the red trajectory. The vector **r** indicates the point at which we will evaluate the fields. The retarded time t_r is defined here.

Examination of this system from the viewpoint of the the Liénard-Wiechert potentials –(Gaussian units)

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right)\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2}\right)\right]$$

Note that for our example there are no acceleration terms.

For our example:

$\mathbf{R}_q(t_r) = vt_r\hat{\mathbf{x}}$ $\quad$ $\mathbf{r} = b\hat{\mathbf{y}}$

$\mathbf{R} = b\hat{\mathbf{y}} - vt_r\hat{\mathbf{x}}$ $\quad$ $R = \sqrt{v^2t_r^2 + b^2}$

$\mathbf{v} = v\hat{\mathbf{x}}$ $\quad$ $t_r = t - \frac{R}{c}$

This should be equivalent to the result given in Jackson (11.152):

$$\mathbf{E}(x,y,z,t) = \mathbf{E}(0,b,0,t) = q\frac{-v\gamma t\hat{\mathbf{x}} + \gamma b\hat{\mathbf{y}}}{\left(b^2 + (v\gamma t)^2\right)^{3/2}}$$

$$\mathbf{B}(x,y,z,t) = \mathbf{B}(0,b,0,t) = q\frac{\gamma\beta b\hat{\mathbf{z}}}{\left(b^2 + (v\gamma t)^2\right)^{3/2}}$$

In our case, the trajectory of the moving particle is described as constant velocity along the x-axis while the fields are measured at the fixed point b along the y axis.

Why take this example?

1. Complete waste of time since we already know the answer.
2. If we get the same answer as we did using the Lorentz transformation, we will feel more confident in applying this approach to study electromagnetic fields resulting from more complicated trajectories.

Note your homework for this lecture involves deriving for yourselves the details of the analysis.

Some details

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right)\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2}\right)\right]$$

For our example:

$\mathbf{R}_q(t_r) = vt_r\hat{\mathbf{x}}$ $\quad \mathbf{r} = b\hat{\mathbf{y}}$

$\mathbf{R} = b\hat{\mathbf{y}} - vt_r\hat{\mathbf{x}}$ $\quad R = \sqrt{v^2 t_r^2 + b^2}$

$\mathbf{v} = v\hat{\mathbf{x}}$ $\quad t_r = t - \frac{R}{c}$

t_r must be a solution to a quadratic equation:

$$t_r - t = -\frac{R}{c} \qquad \Rightarrow \qquad t_r^2 - 2\gamma^2 t t_r + \gamma^2 t^2 - \gamma^2 b^2 / c^2 = 0$$

with the physical solution:

$$t_r = \gamma\left(\gamma t - \frac{\sqrt{(v\gamma t)^2 + b^2}}{c}\right)$$

For your homework for this lecture, you are asked to review the evaluations here.

Some details continued:

Now we can express R as:

$$R = \gamma\left(-\beta v\gamma t + \sqrt{(v\gamma t)^2 + b^2}\right)$$

and the related quantities:

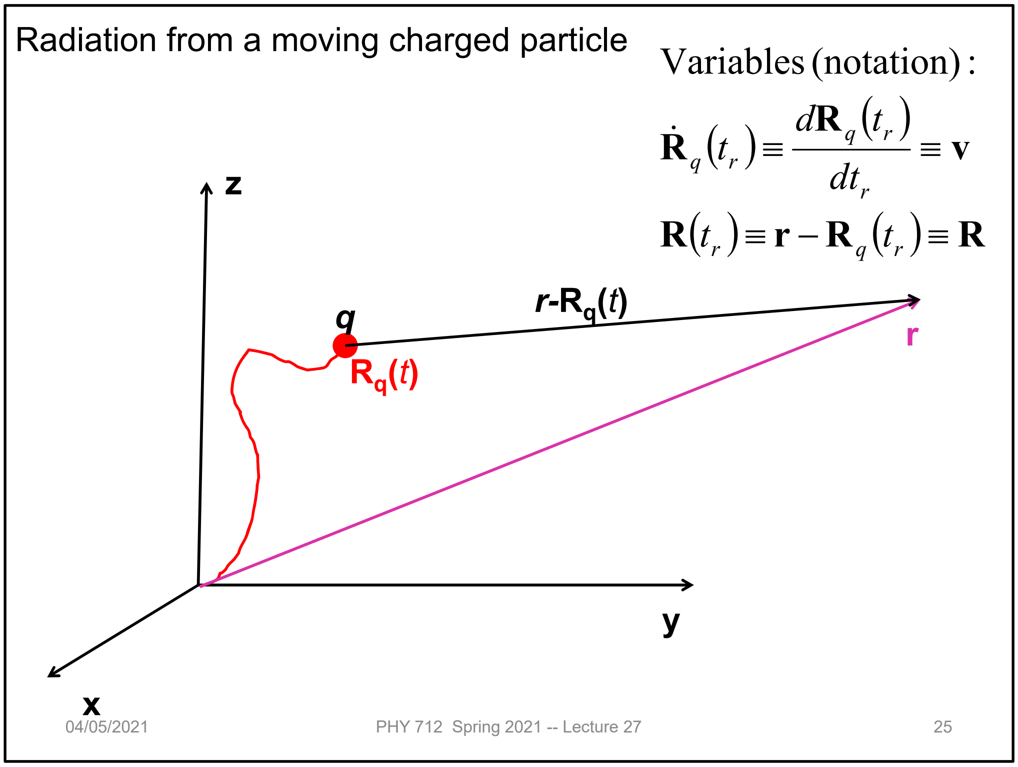
$$\mathbf{R} - \mathbf{v}R/c = -vt\hat{\mathbf{x}} + b\hat{\mathbf{y}}$$

$$R - \mathbf{v}\cdot\mathbf{R}/c = \frac{\sqrt{(v\gamma t)^2 + b^2}}{\gamma}$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right)\right] = q\frac{-v\gamma t\hat{\mathbf{x}} + \gamma b\hat{\mathbf{y}}}{\left(b^2 + (v\gamma t)^2\right)^{3/2}}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2}\right)\right] = q\frac{\gamma\beta b\hat{\mathbf{z}}}{\left(b^2 + (v\gamma t)^2\right)^{3/2}}$$

When the dust clears, we do verify the E and B fields obtained using the Lorentz transformation.



With this success, we are motivated to apply this approach to more general particle trajectories.

Liénard-Wiechert fields (cgs Gaussian units):

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c})^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R} \times \left\{\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \times \frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right]. \tag{19}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c})^3}\left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c})^2}\right]. \tag{20}$$

In this case, the electric and magnetic fields are related according to

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{R}. \tag{21}$$

Notation:

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v} \qquad \mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R} \qquad \dot{\mathbf{v}} \equiv \frac{d^2\mathbf{R}_q(t_r)}{dt_r^{\,2}}$$

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Here we review the equations from the Lienard-Wiechert analysis. We particularly notice that for the fields very far from the particle positions, the dominant terms are those which involve the acceleration of the particle.

Electric field far from source:

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left\{\mathbf{R}\times\left[\left(\mathbf{R}-\frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t)=\frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{R}$$

Let $\hat{\mathbf{R}}\equiv\frac{\mathbf{R}}{R}$ $\quad \boldsymbol{\beta}\equiv\frac{\mathbf{v}}{c}$ $\quad \dot{\boldsymbol{\beta}}\equiv\frac{\dot{\mathbf{v}}}{c}$

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{cR\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^3}\left\{\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t)=\hat{\mathbf{R}}\times\mathbf{E}(\mathbf{r},t)$$

These acceleration terms are given here. These are the terms that we will focus on. Here we define a unit vector Rhat. Jackson calls this vector **n**. In principle, this unit vector varies in time, but at large enough distances from the source, it is an approximately constant unit vector.

Poynting vector:

$$\mathbf{S}(\mathbf{r},t)=\frac{c}{4\pi}(\mathbf{E}\times\mathbf{B})$$

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{cR\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^3}\left\{\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t)=\hat{\mathbf{R}}\times\mathbf{E}(\mathbf{r},t)$$

$$\mathbf{S}(\mathbf{r},t)=\frac{c}{4\pi}\hat{\mathbf{R}}\left|\mathbf{E}(\mathbf{r},t)\right|^2=\frac{q^2}{4\pi cR^2}\hat{\mathbf{R}}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

Note: We have used the fact that

$$\hat{\mathbf{R}}\cdot\mathbf{E}(\mathbf{r},t)=0$$

In addition to calculating the fields themselves, we will be interested in calculating the Poynting vector due to the fields in the radiation zone.

Power radiated

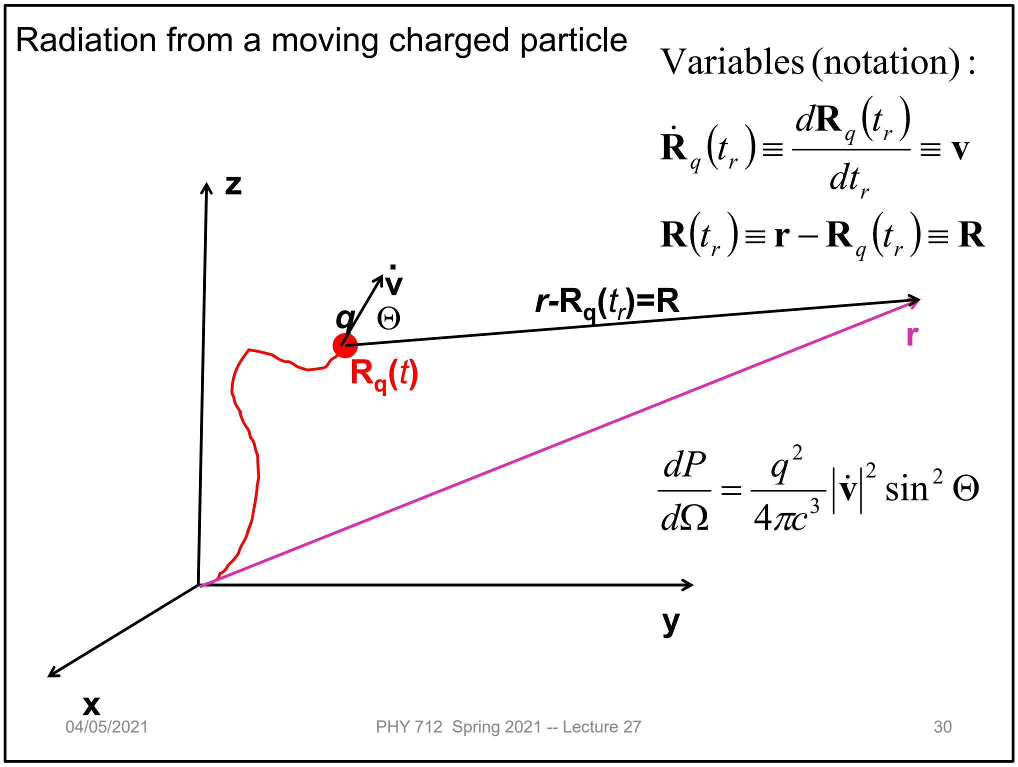
$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}\hat{\mathbf{R}}\left|\mathbf{E}(\mathbf{r},t)\right|^2 = \frac{q^2}{4\pi c R^2}\hat{\mathbf{R}}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

$$\frac{dP}{d\Omega} = \mathbf{S}\cdot\hat{\mathbf{R}}R^2 = \frac{q^2}{4\pi c}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

In the non-relativistic limit: $\beta << 1$

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c}\left|\hat{\mathbf{R}}\times\left[\hat{\mathbf{R}}\times\dot{\boldsymbol{\beta}}\right]\right|^2 = \frac{q^2}{4\pi c^3}\left|\dot{\mathbf{v}}\right|^2\sin^2\Theta$$

After some algebra, we arrive at the expression for the power radiated per unit solid angle. We will examine this result more in detail next time, but for now, we will consider the result in the non-relativistic limit when beta is nearly 0.

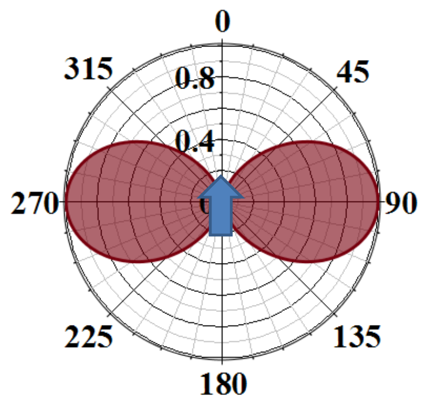


This slide attempts to show the geometry of the trajectory and fields.

Radiation power in non-relativistic case -- continued

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}|\dot{\mathbf{v}}|^2 \sin^2\Theta$$

$$P = \int d\Omega \frac{dP}{d\Omega} = \frac{2}{3}\frac{q^2}{c^3}|\dot{\mathbf{v}}|^2$$



Blue arrow indicates the particle acceleration direction

Here we illustrate the non-relativistic power distribution, showing that the radiation intensity is concentrated in the directions perpendicular to the particle acceleration. Next time we will see how relativistic effects change this radiation pattern.

What do you think will happen when the particle velocities become larger with respect to the speed of light in vacuum?

1. The radiation pattern will be essentially the same.
2. The radiation pattern will be quite different.