PHY 712 Electrodynamics
10-10:50 AM MWF Online

Plan for Lecture 28:

Start reading Chap. 14 –

Radiation by moving charges

1. **Motion in a line**
2. **Motion in a circle**
3. **Spectral analysis of radiation**

In this lecture, we will continue discussing the material presented in Chap. 14 of Jackson's textbook on the subject of radiation from moving charged particles.

21	Mon: 03/22/2021	Chap. 8	EM waves in wave guides		
22	Wed: 03/24/2021	Chap. 9	Radiation from localized oscillating sources	#15	03/26/2021
23	Fri: 03/26/2021	Chap. 9	Radiation from oscillating sources	#16	03/29/2021
24	Mon: 03/29/2021	Chap. 9 & 10	Radiation and scattering	#17	03/31/2021
25	Wed: 03/31/2021	Chap. 11	Special Theory of Relativity	#18	04/05/2021
26	Fri: 04/02/2021	Chap. 11	Special Theory of Relativity		
27	Mon: 04/05/2021	Chap. 11	Special Theory of Relativity	#19	04/09/2021
	Wed: 04/07/2021	No class	*Holiday*		
28	Fri: 04/09/2021	Chap. 14	Radiation from accelerating charged particles	#20	04/12/2021
29	Mon: 04/12/2021	Chap. 14	Synchrotron radiation		

PHY 712 -- Assignment #20

April 9, 2021

Start reading Chap. 14 in **Jackson** .

1. Consider an electron moving at constant speed $\beta c \approx c$ in a circular trajectory of radius ρ. Its total energy is $E= \gamma m c^2$. Determine the ratio of the energy lost during one full cycle to its total energy. Evaluate the expression for an electron with total energy 400 GeV in a synchroton of radius $\rho=10^3$ m.

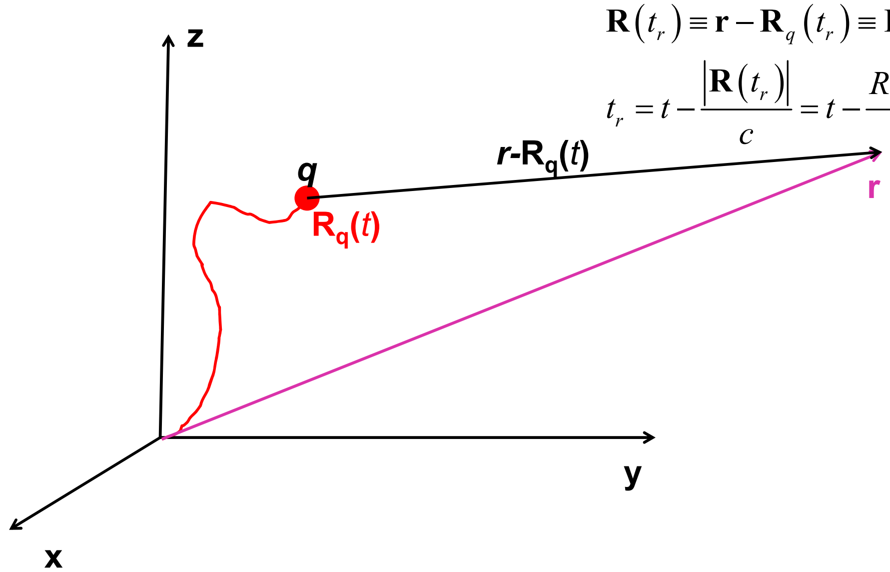
The homework problem for this time asks you to estimate the power radiated by a particle moving in a circular trajectory.

Radiation from a moving charged particle

Variables (notation): $\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$$

$$t_r = t - \frac{|\mathbf{R}(t_r)|}{c} = t - \frac{R(t_r)}{c}$$



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Here is the general diagram we have been using to denote the field point r and the trajectory R_q(t).

Liénard-Wiechert fields (cgs Gaussian units):

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R} \times \left\{\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \times \frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right]. \tag{19}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right]. \tag{20}$$

In this case, the electric and magnetic fields are related according to

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{R}. \tag{21}$$

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v} \qquad \mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R} \qquad \dot{\mathbf{v}} \equiv \frac{d^2\mathbf{R}_q(t_r)}{dt_r^{\;2}}$$

Review of the E and B fields produced by the moving charged particle.

Comment --

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R} \times \left\{\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \times \frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right]. \tag{19}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right]. \tag{20}$$

In this case, the electric and magnetic fields are related according to

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{R}. \tag{21}$$

Note that (21) can be demonstrated by evaluating ***R x E(r,t)***

Other helpful identities: ***ax(bxc)=b(a·c)-c(a·b)***
a·(bxc) = b·(cxa) = c·(axb)

Electric field far from source:

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3} \left\{ \mathbf{R} \times \left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c} \right) \times \frac{\dot{\mathbf{v}}}{c^2} \right] \right\}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R} \times \mathbf{E}(\mathbf{r},t)}{R}$$

Let $\hat{\mathbf{R}} \equiv \frac{\mathbf{R}}{R} \qquad \boldsymbol{\beta} \equiv \frac{\mathbf{v}}{c} \qquad \dot{\boldsymbol{\beta}} \equiv \frac{\dot{\mathbf{v}}}{c}$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{cR\left(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^3} \left\{ \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right\}$$

$$\mathbf{B}(\mathbf{r},t) = \hat{\mathbf{R}} \times \mathbf{E}(\mathbf{r},t)$$

Specializing the equations to fields in the radiation zone.

Poynting vector:

$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}(\mathbf{E}\times\mathbf{B})$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{cR\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^3}\left\{\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t) = \hat{\mathbf{R}}\times\mathbf{E}(\mathbf{r},t)$$

$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}\hat{\mathbf{R}}\left|\mathbf{E}(\mathbf{r},t)\right|^2 = \frac{q^2}{4\pi cR^2}\hat{\mathbf{R}}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

Note: We have used the fact that

$\hat{\mathbf{R}}\cdot\mathbf{E}(\mathbf{r},t) = 0$ which follows from the vector identities.

Evaluating the Poynting vector for the radiation zone.

Power radiated

$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}\hat{\mathbf{R}}\left|\mathbf{E}(\mathbf{r},t)\right|^2 = \frac{q^2}{4\pi c R^2}\hat{\mathbf{R}}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

$$\frac{dP}{d\Omega} = \mathbf{S}\cdot\hat{\mathbf{R}}R^2 = \frac{q^2}{4\pi c}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

In the non-relativistic limit: $\beta << 1$

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c}\left|\hat{\mathbf{R}}\times\left[\hat{\mathbf{R}}\times\dot{\boldsymbol{\beta}}\right]\right|^2 = \frac{q^2}{4\pi c^3}\left|\dot{\mathbf{v}}\right|^2\sin^2\Theta$$

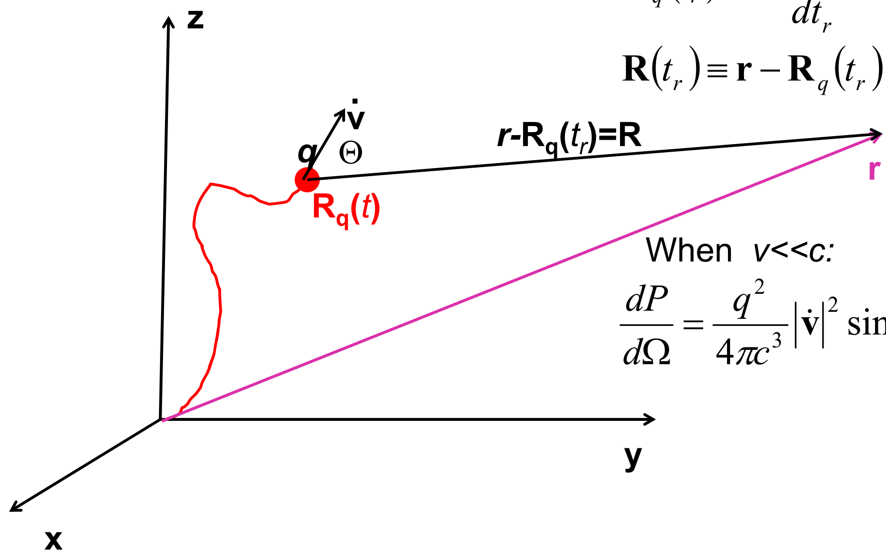
The general expression for the power per unit solid angle. The last expression represents the result in the non-relativistic limit.

Radiation from a moving charged particle

Variables (notation):

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$$



When $v \ll c$:

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \sin^2 \Theta$$

Diagram showing geometry of previous equations.

Radiation power in non-relativistic case -- continued

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \sin^2 \Theta$$

$$P = \int d\Omega \frac{dP}{d\Omega} = \frac{2}{3} \frac{q^2}{c^3} |\dot{\mathbf{v}}|^2$$

Integrating the expression for the power over solid angle gives the total power. On this slide, the non-relativistic expressions are given..

Radiation distribution in the relativistic case

$$\frac{dP}{d\Omega} = \mathbf{S}\cdot\hat{\mathbf{R}}R^2 = \frac{q^2}{4\pi c}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}\Bigg|_{t_r=t-R/c}$$

This expression gives us the energy per unit field time *t.* We are often interested in the power per unit retarded time $t_r=t-R/c$:

$$\frac{dP_r(t)}{d\Omega} = \frac{dP(t)}{d\Omega}\frac{dt}{dt_r} \qquad \frac{dt}{dt_r} = 1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}$$

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^5}\Bigg|_{t_r=t-R/c}$$

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What happens to the complete expression, particularly when the relativistic effects are numerically significant? For this, we follow Jackson's approach and measure the power with respect to the retarded time. Please make sure that you check the derivation of the equations on this slide.

Some details –

The power derived from the Poynting vector in terms of the field times is given by:

$$\frac{dP}{d\Omega} = \mathbf{S}\cdot\hat{\mathbf{R}}R^2 = \frac{q^2}{4\pi c}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}\Bigg|_{t_r=t-R/c}$$

The integrated power would be given by

$$W = \int dt\,\frac{dP(t)}{d\Omega} = \int dt_r \frac{dt}{dt_r}\frac{dP(t)}{d\Omega} \rightarrow \frac{dP_r(t_r)}{d\Omega}$$

More comments

$$t_r = t - \frac{\left|\mathbf{r} - \mathbf{R}_q(t_r)\right|}{c}$$

$$t = t_r + \frac{\left|\mathbf{r} - \mathbf{R}_q(t_r)\right|}{c}$$

$$\frac{dt}{dt_r} = 1 + \left(-\frac{d\mathbf{R}_q(t_r)}{c dt_r} \right) \cdot \frac{\mathbf{r} - \mathbf{R}_q(t_r)}{\left|\mathbf{r} - \mathbf{R}_q(t_r)\right|} = 1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}$$

➔ $$\frac{dP_r\left(t_r\right)}{d\Omega} = \frac{q^2}{4\pi c} \left. \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^5} \right|_{t_r = t - R/c}$$

Why do you think it useful to measure the power as energy per unit retarded time P_r?

1. Jackson likes to torture us.
2. There should be no difference.
3. ???

What do you think?

Radiation distribution in the relativistic case -- continued

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^5}\Bigg|_{t_r = t - R/c}$$

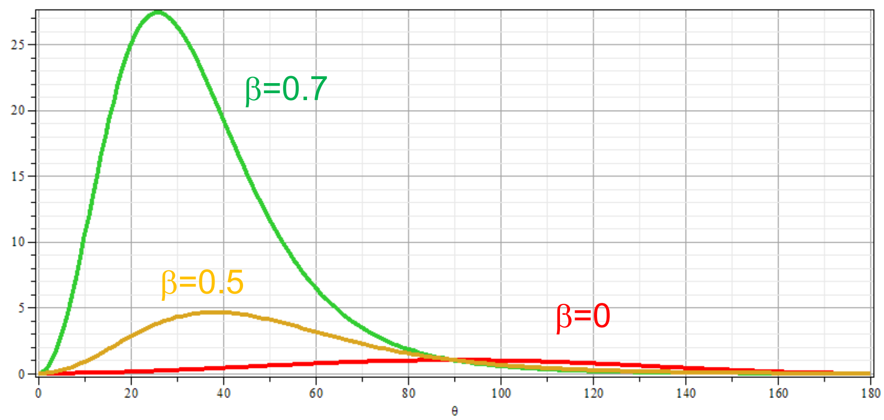
For linear acceleration: $\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}} = 0$

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{\left|\hat{\mathbf{R}} \times \left(\hat{\mathbf{R}} \times \dot{\boldsymbol{\beta}}\right)\right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^5}\Bigg|_{t_r = t - R/c} = \frac{q^2}{4\pi c^3} \left|\dot{\mathbf{v}}\right|^2 \frac{\sin^2\theta}{\left(1 - \beta\cos\theta\right)^5}$$

First we will consider the case of linear acceleration. Since the velocity of the particle and its acceleration are in the same direction, the cross product is 0. The retarded time power distribution can be shown to have the form given in the last equation of the slide.

Power from linearly accelerating particle

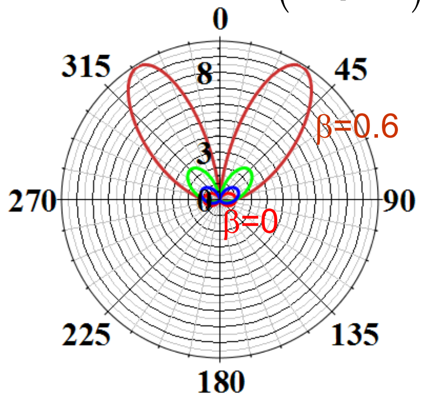
$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c}\left.\frac{\left|\hat{\mathbf{R}}\times\left(\hat{\mathbf{R}}\times\dot{\boldsymbol{\beta}}\right)\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^5}\right|_{t_r=t-R/c} = \frac{q^2}{4\pi c^3}\left|\dot{\mathbf{v}}\right|^2\frac{\sin^2\theta}{\left(1-\beta\cos\theta\right)^5}$$

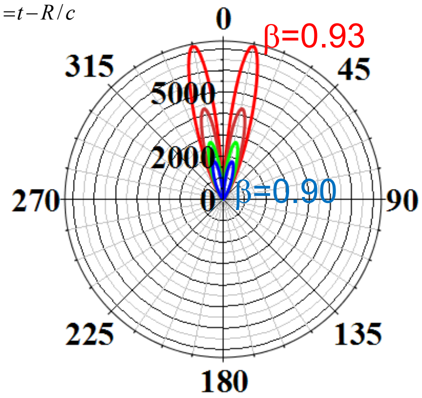


This plot illustrates the sensitivity of the retarded time power distribution to the value of beta.

Polar plots:

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{\left|\hat{\mathbf{R}} \times \left(\hat{\mathbf{R}} \times \dot{\boldsymbol{\beta}}\right)\right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^5}\Bigg|_{t_r = t - R/c} = \frac{q^2}{4\pi c^3} \left|\dot{\mathbf{v}}\right|^2 \frac{\sin^2\theta}{\left(1 - \beta\cos\theta\right)^5}$$





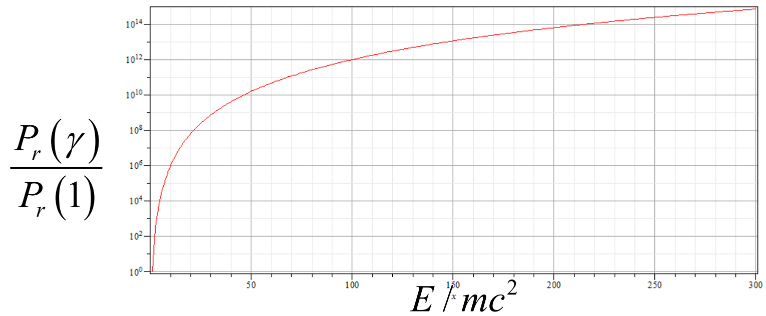
Note – two separate plots are introduced in order to see the drastic change of scale at values of β close to 1.

Polar plot of the previous results.

Power from linearly accelerating particle

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c}\left.\frac{\left|\hat{\mathbf{R}}\times\left(\hat{\mathbf{R}}\times\dot{\boldsymbol{\beta}}\right)\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^5}\right|_{t_r=t-R/c} = \frac{q^2}{4\pi c^3}\left|\dot{\mathbf{v}}\right|^2\frac{\sin^2\theta}{\left(1-\beta\cos\theta\right)^5}$$

$$P_r(t_r) = \int\frac{dP_r(t_r)}{d\Omega}d\Omega = \frac{2}{3}\frac{q^2}{c^3}\left|\dot{\mathbf{v}}\right|^2\gamma^6 \quad \text{where } \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$



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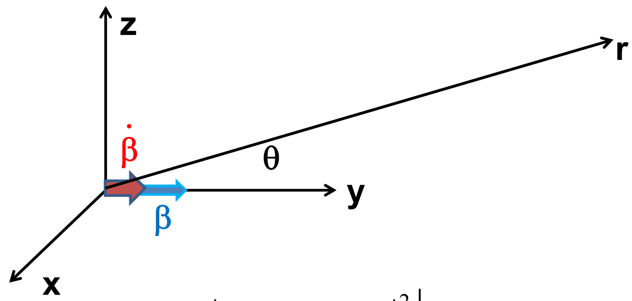
Integrating over solid angle, we obtain the total retarded time power radiated, finding it to vary as gamma^6. The logarithmic plot shows the gamma dependence.

Power distribution for linear acceleration -- continued

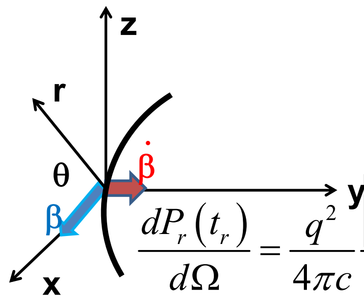


$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \left. \frac{\left|\hat{\mathbf{R}} \times \left(\hat{\mathbf{R}} \times \dot{\boldsymbol{\beta}}\right)\right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^5} \right|_{t_r = t - R/c} = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \frac{\sin^2\theta}{(1 - \beta\cos\theta)^5}$$

$$P_r(t_r) = \int \frac{dP_r(t_r)}{d\Omega} d\Omega = \frac{2}{3} \frac{q^2}{c^3} |\dot{\mathbf{v}}|^2 \gamma^6 \quad \text{where } \gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}$$

Summary of results for the linear acceleration case.

Power distribution for circular acceleration



$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^5}\Bigg|_{t_r = t - R/c}$$

$$= \frac{q^2}{4\pi c} \frac{\left|\dot{\boldsymbol{\beta}}\right|^2 \left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^2 - \left(\hat{\mathbf{R}} \cdot \dot{\boldsymbol{\beta}}\right)^2 \left(1 - \beta^2\right)}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^5}\Bigg|_{t_r = t - R/c}$$

$$P_r(t_r) = \int d\Omega \frac{dP_r(t_r)}{d\Omega} = \frac{2}{3} \frac{q^2}{c^3} \left|\dot{\mathbf{v}}\right|^2 \gamma^4$$

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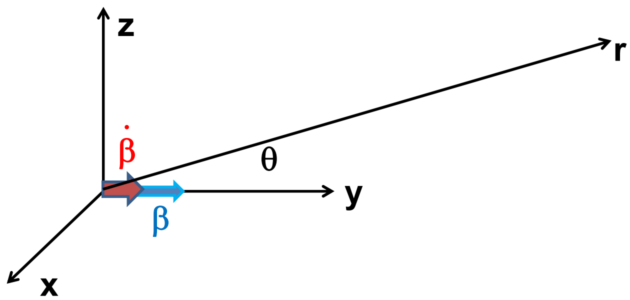
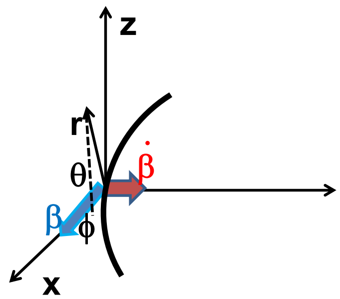
Now consider the case where the acceleration is perpendicular to the instantaneous velocity as in the case of circular motion. In this case, the retarded time power depends on gamma^4. Check whether you agree with this result (or not). Note that in this diagram the polar angle is not the conventional one.

Summary of results --For linear acceleration --



$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{\left|\hat{\mathbf{R}} \times \left(\hat{\mathbf{R}} \times \dot{\boldsymbol{\beta}}\right)\right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^5}\Bigg|_{t_r = t - R/c} = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \frac{\sin^2\theta}{\left(1 - \beta\cos\theta\right)^5}$$

Power distribution for circular acceleration



$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \left. \frac{\left|\dot{\boldsymbol{\beta}}\right|^2 \left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^2 - \left(\hat{\mathbf{R}}\cdot\dot{\boldsymbol{\beta}}\right)^2 \left(1-\beta^2\right)}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^5} \right|_{t_r=t-R/c}$$

$$= \frac{q^2}{4\pi c^3} \frac{\left|\dot{\mathbf{v}}\right|^2}{(1-\beta\cos(\theta))^3} \left(1 - \frac{\cos^2\theta\sin^2\phi}{\gamma^2(1-\beta\cos(\theta))^2}\right)$$

Angular integrals for the two cases –

Linear acceleration

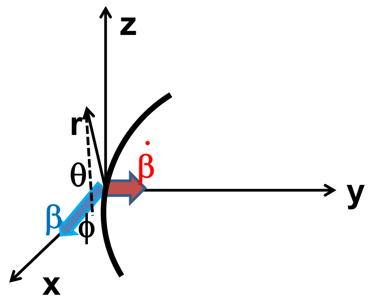
$$P_r(t_r) = \int \frac{dP_r(t_r)}{d\Omega} d\Omega = 2\pi \int \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \frac{\sin^2\theta \; d\sin\theta}{(1-\beta\cos\theta)^5} = \frac{2}{3}\frac{q^2}{c^3}|\dot{\mathbf{v}}|^2 \gamma^6$$

Circular acceleration

$$P_r(t_r) = \int \frac{dP_r(t_r)}{d\Omega} d\Omega = \int d\phi \; d\sin\theta \frac{q^2}{4\pi c^3} \frac{|\dot{\mathbf{v}}|^2}{(1-\beta\cos(\theta))^3}\left(1 - \frac{\cos^2\theta \sin^2\phi}{\gamma^2 (1-\beta\cos(\theta))^2}\right)$$

$$= \frac{2}{3}\frac{q^2}{c^3}|\dot{\mathbf{v}}|^2 \gamma^4$$

Power distribution for circular acceleration



$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \left. \frac{\left|\dot{\boldsymbol{\beta}}\right|^2 \left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^2 - \left(\hat{\mathbf{R}}\cdot\dot{\boldsymbol{\beta}}\right)^2 \left(1-\beta^2\right)}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^5} \right|_{t_r = t - R/c}$$

$$= \frac{q^2}{4\pi c^3} \frac{\left|\dot{\mathbf{v}}\right|^2}{\left(1-\beta\cos(\theta)\right)^3} \left(1 - \frac{\cos^2\theta \sin^2\phi}{\gamma^2\left(1-\beta\cos(\theta)\right)^2}\right)$$

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Some more details. This concludes the discussion of the geometry of the radiation. In the next several slides, we will start to discuss another aspect of the radiation, namely its spectral distribution.

Spectral composition of electromagnetic radiation

Previously we determined the power distribution from a charged particle:

$$\frac{dP(t)}{d\Omega} = \mathbf{S}\cdot\hat{\mathbf{R}}R^2 = \frac{q^2}{4\pi c}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}\Bigg|_{t_r=t-R/c}$$

$$\equiv\left|\boldsymbol{a}(t)\right|^2$$

$$\text{where}\qquad \boldsymbol{a}(t)\equiv\sqrt{\frac{q^2}{4\pi c}}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^3}\Bigg|_{t_r=t-R/c}$$

Time integrated power per solid angle:

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt\,\frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt\left|\boldsymbol{a}(t)\right|^2 = \int_{-\infty}^{\infty} d\omega\left|\tilde{\boldsymbol{a}}(\omega)\right|^2$$

Now we will return to the power measured with respect to the field time (as opposed to the retarded time). In this way will be able to use the beautiful mathematics of Fourier transforms to analyze the spectral properties of the radiation. Here we imagine that the radiation is measured at a given location for a long period of time so that we will want to evaluate the time integrated power W.

Spectral composition of electromagnetic radiation -- continued

Time integrated power per solid angle:

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt \frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt |\mathbf{a}(t)|^2 = \int_{-\infty}^{\infty} d\omega |\tilde{\mathbf{a}}(\omega)|^2$$

Fourier amplitude:

$$\tilde{\mathbf{a}}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, \mathbf{a}(t) e^{i\omega t} \qquad \mathbf{a}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, \tilde{\mathbf{a}}(\omega) e^{-i\omega t}$$

Parseval's theorem

Marc-Antoine Parseval des Chênes 1755-1836

http://www-history.mcs.st-andrews.ac.uk/Biographies/Parseval.html

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Here we make use of the Parseval's theorem which allows us to relate the time integral of the power to the frequency integral of its Fourier transform.

Spectral composition of electromagnetic radiation -- continued

Consequences of Parseval's analysis:

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt \frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt |\mathbf{a}(t)|^2 = \int_{-\infty}^{\infty} d\omega |\tilde{\mathbf{a}}(\omega)|^2$$

Note that: $\tilde{\mathbf{a}}(\omega) = \tilde{\mathbf{a}}^*(-\omega)$

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} d\omega |\tilde{\mathbf{a}}(\omega)|^2 = \int_{0}^{\infty} d\omega \left(|\tilde{\mathbf{a}}(\omega)|^2 + |\tilde{\mathbf{a}}(-\omega)|^2 \right) \equiv \int_{0}^{\infty} d\omega \frac{\partial^2 I}{\partial\Omega\partial\omega}$$

$$\frac{\partial^2 I}{\partial\Omega\partial\omega} \equiv 2|\tilde{\mathbf{a}}(\omega)|^2$$

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Mathematically, the theorem involves integrals over all frequencies, while physically negative frequencies are not measured. By using the fact that the power amplitude must be real (mathematically), we can then derive a formula for the intensity I as a function of frequency and solid angle.

What is the significance of $\frac{\partial^2 I}{\partial\Omega\partial\omega}$?

1. It is purely a mathematical construct
2. It can be measured

Spectral composition of electromagnetic radiation -- continued

For our case: $$\mathcal{a}(t) \equiv \sqrt{\frac{q^2}{4\pi c}} \left. \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^3} \right|_{t_r = t - R/c}$$

Fourier amplitude:

$$\tilde{\mathcal{a}}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\ e^{i\omega t}\ \mathcal{a}(t)$$

$$= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt\ e^{i\omega t} \left. \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^3} \right|_{t_r = t - R/c}$$

Here we analyze the power amplitude in order to take its Fourier transform. Apparently, if we can evaluate this integral, we can determine the intensity spectrum.

Spectral composition of electromagnetic radiation -- continued

Fourier amplitude:

$$\tilde{\boldsymbol{a}}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, \boldsymbol{a}(t)\, e^{i\omega t}$$

$$= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt \left. \frac{\left| \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^3} \right|_{t_r = t - R/c} e^{i\omega t}$$

$$= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \frac{dt}{dt_r} \left. \frac{\left| \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^3} \right|_{t_r = t - R/c} e^{i\omega \left(t_r + R(t_r)/c \right)}$$

$$= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \left. \frac{\left| \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^2} \right|_{t_r = t - R/c} e^{i\omega \left(t_r + R(t_r)/c \right)}$$

The integral must be performed over the field time, but the argument of the integral is expressed in terms of the retarded time. Fortunately, we can use the relationship between the two in order to perform the actual integral in terms of the retarded time.

Spectral composition of electromagnetic radiation -- continued

Exact expression :

$$\tilde{a}(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \left. \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^2} \right|_{t_r = t - R/c} e^{i\omega(t_r + R(t_r)/c)}$$

Recall: $\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v} \qquad \mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$

For $r >> R_q(t_r) \qquad R(t_r) \approx r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)$ where $\hat{\mathbf{r}} \equiv \frac{\mathbf{r}}{r}$

At the same level of approximation : $\hat{\mathbf{R}} \approx \hat{\mathbf{r}}$

Here we make use of some approximations valid far from the source.

Spectral composition of electromagnetic radiation -- continued

Exact expression:

$$\tilde{a}(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \left. \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^2} \right|_{t_r = t - R/c} e^{i\omega\left(t_r + R(t_r)/c\right)}$$

Approximate expression:

$$\tilde{a}(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} e^{i\omega(r/c)} \int_{-\infty}^{\infty} dt_r \left. \frac{\left|\hat{\mathbf{r}} \times \left[\left(\hat{\mathbf{r}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}}\right)^2} \right|_{t_r = t - R/c} e^{i\omega\left(t_r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)/c\right)}$$

Resulting spectral intensity expression:

$$\frac{\partial^2 I}{\partial\omega \partial\Omega} = \frac{q^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} dt_r \left. \frac{\left|\hat{\mathbf{r}} \times \left[\left(\hat{\mathbf{r}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}}\right)^2} \right|_{t_r = t - R/c} e^{i\omega\left(t_r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)/c\right)} \right|^2$$

Summarizing the approximations.

Example – radiation from a collinear acceleration burst

$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt_r \left.\frac{\left|\hat{\mathbf{r}}\times\left[\left(\hat{\mathbf{r}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)^2}\right|_{t_r=t-R/c} e^{i\omega\left(t_r-\hat{\mathbf{r}}\cdot\mathbf{R}_q\left(t_r\right)/c\right)}\right|^2$$

Suppose that $\dot{\boldsymbol{\beta}} = \begin{cases} \dfrac{\hat{\boldsymbol{\beta}}\Delta v}{c\tau} & 0 < t_r < \tau \\ 0 & \text{otherwise} \end{cases}$

$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2}{4\pi^2 c^3}\left|\frac{\left|\hat{\mathbf{r}}\times\left[\hat{\mathbf{r}}\times\hat{\boldsymbol{\beta}}\right]\right|\Delta v}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)^2\tau}\right|^2\left|\int_0^{\tau} dt_r e^{i\omega\left(t_r-\hat{\mathbf{r}}\cdot\boldsymbol{\beta}t_r\right)}\right|^2 \quad \text{Let } \boldsymbol{\beta}\cdot\hat{\mathbf{r}} = \beta\cos\theta$$

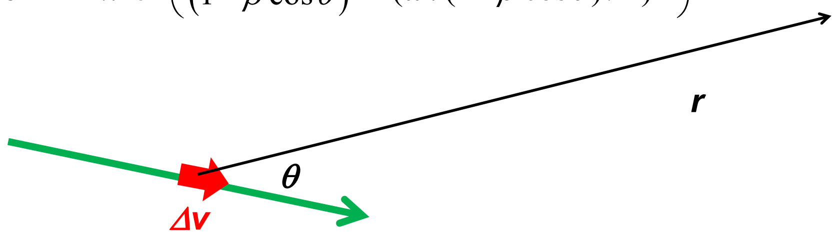
$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2}{4\pi^2 c^3}\left(\frac{\Delta v\sin\theta}{\left(1-\beta\cos\theta\right)^2}\frac{\sin(\omega\tau(1-\beta\cos\theta)/2)}{(\omega\tau(1-\beta\cos\theta)/2)}\right)^2$$

Here we consider an example of motion due to an abrupt collision. This example is actually discussed at the beginning of Chapter 15 of Jackson.

Example:

$$\text{Suppose that } \dot{\boldsymbol{\beta}} = \begin{cases} \dfrac{\hat{\boldsymbol{\beta}}\Delta v}{c\tau} & 0 < t_r < \tau \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2}{4\pi^2 c^3}\left(\frac{\Delta v \sin\theta}{\left(1-\beta\cos\theta\right)^2}\frac{\sin(\omega\tau(1-\beta\cos\theta)/2)}{(\omega\tau(1-\beta\cos\theta)/2)}\right)^2$$



Example: "Bremsstrahlung" radiation

This radiation is for example caused by a fast moving charged particle coming to an abrupt stop such as when it smashes into matter. The value of tau depends on the matter and the particle.

Spectral composition of electromagnetic radiation -- continued

Alternative expression --

It can be shown that:

$$\frac{\hat{\mathbf{r}}\times\left[\left(\hat{\mathbf{r}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)^2}=\frac{d}{dt_r}\left(\frac{\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}\right)}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)}\right)$$

Integration by parts and assumptions about the integration limit behaviors shows that the spectral intensity depends on the following integral:

$$\frac{\partial^2 I}{\partial\omega\partial\Omega}=\frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt_r\left[\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}(t_r)\right)\right]e^{i\omega\left(t_r-\hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)}\right|^2$$

Next time we will evaluate this expression for synchrotron radiation.