PHY 712 Electrodynamics
10-10:50 AM MWF Online

Discussion for Lecture 29:

Continue reading Chap. 14 –

Radiation by moving charges

1. Angular dependence of radiation from an accelerating particle
2. Spectral analysis of radiation
3. Detailed analysis of synchrotron radiation

28	Fri: 04/09/2021	Chap. 14	Radiation from accelerating charged particles	#20	04/12/2021
29	Mon: 04/12/2021	Chap. 14	Synchrotron radiation	#21	04/14/2021
30	Wed: 04/14/2021	Chap. 14	Synchrotron radiation		
31	Fri: 04/16/2021	Chap. 15	Radiation from collisions of charged particles		
32	Mon: 04/19/2021	Chap. 15	Radiation from collisions of charged particles		
33	Wed: 04/21/2021	Chap. 13	Cherenkov radiation		
34	Fri: 04/23/2021		Special topic: E & M aspects of superconductivity		
35	Mon: 04/26/2021		Special topic: E & M aspects of superconductivity		
	Wed: 04/28/2021		Presentations I		
	Fri: 04/30/2021		Presentations II		
36	Mon: 05/03/2021		Review		
37	Wed: 05/05/2021		Review		

PHY 712 -- Assignment #21

April 12, 2021

Continue reading Chap. 14 in **Jackson** . This problem is designed to demonstrate Parseval's theorem using the definitions given in the lecture notes and on Page 674 in **Jackson**. We will use the example

$A(t)=K\, e^{-(t/T)^2}$,

where K and T are positive constants.

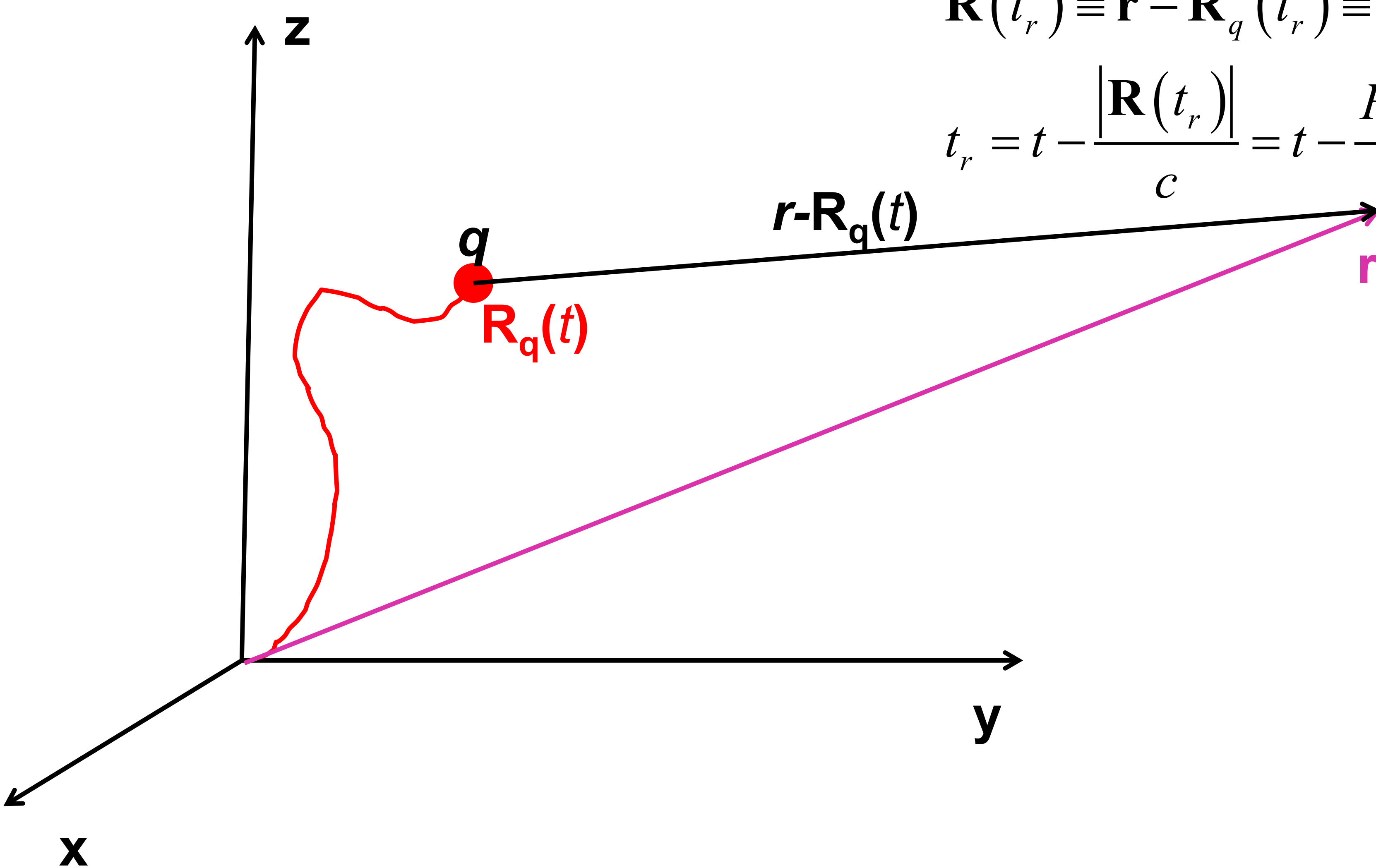
1. Find the Fourier transform of $A(t)$.
2. *Evaluate the integral of the squared modulus of $A(t)$ between $-\infty \le t \le \infty$.*
3. *Evaluate the integral of the squared modulus of the Fourier transform of $A(t)$ between $-\infty \le \omega \le \infty$.*

Radiation from a moving charged particle

Variables (notation): $\dot{\mathbf{R}}_q(t_r) \equiv \dfrac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$$

$$t_r = t - \frac{|\mathbf{R}(t_r)|}{c} = t - \frac{R(t_r)}{c}$$



Electric field far from source:

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left\{\mathbf{R}\times\left[\left(\mathbf{R}-\frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t)=\frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{R}$$

Note that all of the variables on the right hand side of the equations depend on t_r .

Let $\hat{\mathbf{R}}\equiv\frac{\mathbf{R}}{R}$ $\quad\boldsymbol{\beta}\equiv\frac{\mathbf{v}}{c}$ $\quad\dot{\boldsymbol{\beta}}\equiv\frac{\dot{\mathbf{v}}}{c}$

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{cR\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^3}\left\{\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t)=\hat{\mathbf{R}}\times\mathbf{E}(\mathbf{r},t)$$

Poynting vector:

$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}(\mathbf{E}\times\mathbf{B})$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{cR\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^3}\left\{\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t) = \hat{\mathbf{R}}\times\mathbf{E}(\mathbf{r},t)$$

$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}\hat{\mathbf{R}}\left|\mathbf{E}(\mathbf{r},t)\right|^2 = \frac{q^2}{4\pi cR^2}\hat{\mathbf{R}}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

Power radiated

$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}\hat{\mathbf{R}}\left|\mathbf{E}(\mathbf{r},t)\right|^2 = \frac{q^2}{4\pi c R^2}\hat{\mathbf{R}}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

$$\frac{dP}{d\Omega} = \mathbf{S}\cdot\hat{\mathbf{R}}R^2 = \frac{q^2}{4\pi c}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

Spectral composition of electromagnetic radiation

Previously we determined the power distribution from a charged particle:

$$\frac{dP(t)}{d\Omega} = \mathbf{S}\cdot\hat{\mathbf{R}}R^2 = \frac{q^2}{4\pi c}\left.\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}\right|_{t_r=t-R/c}$$

$$\equiv \left|\boldsymbol{a}(t)\right|^2$$

where $$\boldsymbol{a}(t) \equiv \sqrt{\frac{q^2}{4\pi c}}\left.\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^3}\right|_{t_r=t-R/c}$$

Time integrated power per solid angle:

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt\,\frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt\left|\boldsymbol{a}(t)\right|^2 = \int_{-\infty}^{\infty} d\omega\left|\tilde{\boldsymbol{a}}(\omega)\right|^2$$

Spectral composition of electromagnetic radiation -- continued

Time integrated power per solid angle:

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt \, \frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt \left| \mathbf{a}(t) \right|^2 = \int_{-\infty}^{\infty} d\omega \left| \tilde{\mathbf{a}}(\omega) \right|^2$$

Fourier amplitude:

$$\tilde{\mathbf{a}}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, \mathbf{a}(t) \, e^{i\omega t} \qquad \mathbf{a}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \, \tilde{\mathbf{a}}(\omega) \, e^{-i\omega t}$$

Parseval's theorem

Marc-Antoine Parseval des Chênes 1755-1836

http://www-history.mcs.st-andrews.ac.uk/Biographies/Parseval.html

Spectral composition of electromagnetic radiation -- continued

Consequences of Parseval's analysis:

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt \frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt \left|\mathbf{a}(t)\right|^2 = \int_{-\infty}^{\infty} d\omega \left|\tilde{\mathbf{a}}(\omega)\right|^2$$

Note that: $\tilde{\mathbf{a}}(\omega) = \tilde{\mathbf{a}}^*(-\omega)$

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} d\omega \left|\tilde{\mathbf{a}}(\omega)\right|^2 = \int_{0}^{\infty} d\omega \left(\left|\tilde{\mathbf{a}}(\omega)\right|^2 + \left|\tilde{\mathbf{a}}(-\omega)\right|^2\right) \equiv \int_{0}^{\infty} d\omega \frac{\partial^2 I}{\partial\Omega\partial\omega}$$

$$\frac{\partial^2 I}{\partial\Omega\partial\omega} \equiv 2\left|\tilde{\mathbf{a}}(\omega)\right|^2$$

Spectral composition of electromagnetic radiation -- continued

For our case: $$\mathcal{a}(t) \equiv \sqrt{\frac{q^2}{4\pi c}} \left. \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^3} \right|_{t_r = t - R/c}$$

Fourier amplitude:

$$\tilde{\mathcal{a}}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\ e^{i\omega t}\ \mathcal{a}(t)$$

$$= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt\ e^{i\omega t} \left. \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^3} \right|_{t_r = t - R/c}$$

Spectral composition of electromagnetic radiation -- continued

Fourier amplitude:

$$\tilde{\boldsymbol{a}}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, \boldsymbol{a}(t)\, e^{i\omega t}$$

$$= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt \left. \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^3} \right|_{t_r = t - R/c} e^{i\omega t}$$

$$= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \, \frac{dt}{dt_r} \left. \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^3} \right|_{t_r = t - R/c} e^{i\omega\left(t_r + R(t_r)/c\right)}$$

$$= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \left. \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^2} \right|_{t_r = t - R/c} e^{i\omega\left(t_r + R(t_r)/c\right)}$$

Spectral composition of electromagnetic radiation -- continued

Exact expression :

$$\tilde{\boldsymbol{a}}(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \left. \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^2} \right|_{t_r = t - R/c} e^{i\omega\left(t_r + R(t_r)/c\right)}$$

Recall: $\dot{\mathbf{R}}_q(t_r) \equiv \dfrac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v} \qquad \mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$

For $r >> R_q(t_r) \qquad R(t_r) \approx r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)$ where $\hat{\mathbf{r}} \equiv \dfrac{\mathbf{r}}{r}$

At the same level of approximation: $\hat{\mathbf{R}} \approx \hat{\mathbf{r}}$

Spectral composition of electromagnetic radiation -- continued

Exact expression:

$$\tilde{\mathbf{a}}(\omega)=\sqrt{\frac{q^2}{8\pi^2 c}}\int_{-\infty}^{\infty} dt_r \left.\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^2}\right|_{t_r=t-R/c} e^{i\omega\left(t_r+R(t_r)/c\right)}$$

Approximate expression:

$$\tilde{\mathbf{a}}(\omega)=\sqrt{\frac{q^2}{8\pi^2 c}}e^{i\omega(r/c)}\int_{-\infty}^{\infty} dt_r \left.\frac{\left|\hat{\mathbf{r}}\times\left[\left(\hat{\mathbf{r}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)^2}\right|_{t_r=t-R/c} e^{i\omega\left(t_r-\hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)}$$

Resulting spectral intensity expression:

$$\frac{\partial^2 I}{\partial\omega\partial\Omega}=\frac{q^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt_r \left.\frac{\left|\hat{\mathbf{r}}\times\left[\left(\hat{\mathbf{r}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)^2}\right|_{t_r=t-R/c} e^{i\omega\left(t_r-\hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)}\right|^2$$

Spectral composition of electromagnetic radiation -- continued

Alternative expression --

It can be shown that:

$$\frac{\hat{\mathbf{r}}\times\left[\left(\hat{\mathbf{r}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)^2}=\frac{d}{dt_r}\left(\frac{\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}\right)}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)}\right)$$

Integration by parts and assumptions about the integration limit behaviors shows that the spectral intensity depends on the following integral:

$$\frac{\partial^2 I}{\partial\omega\partial\Omega}=\frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty}dt_r\left[\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}\left(t_r\right)\right)\right]e^{i\omega\left(t_r-\hat{\mathbf{r}}\cdot\mathbf{R}_q\left(t_r\right)/c\right)}\right|^2$$

Some details --

Spectral intensity expression that needs to be evaluated:

$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2}{4\pi^2 c}\left| \int_{-\infty}^{\infty} dt_r \; e^{i\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)} \frac{\left|\hat{\mathbf{r}}\times\left[\left(\hat{\mathbf{r}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)^2}\right|_{t_r = t - R/c} \right|^2$$

It can be shown that: $$\frac{\hat{\mathbf{r}}\times\left[\left(\hat{\mathbf{r}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)^2} = \frac{d}{dt_r}\left(\frac{\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}\right)}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)}\right)$$

$$\int_{-\infty}^{\infty} dt_r \; e^{i\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)} \frac{\left|\hat{\mathbf{r}}\times\left[\left(\hat{\mathbf{r}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)^2}\right|_{t_r = t - R/c} = \int_{-\infty}^{\infty} dt_r \; e^{i\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)} \frac{d}{dt_r}\left(\frac{\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}(t_r)\right)}{\left(1-\boldsymbol{\beta}(t_r)\cdot\hat{\mathbf{r}}\right)}\right)$$

$$= \int_{-\infty}^{\infty} dt_r \; \frac{d}{dt_r}\left(e^{i\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)} \left(\frac{\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}(t_r)\right)}{\left(1-\boldsymbol{\beta}(t_r)\cdot\hat{\mathbf{r}}\right)}\right)\right) - i\omega\int_{-\infty}^{\infty} dt_r \; \left(e^{i\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)} \left(\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}(t_r)\right)\right)\right)$$

0

Spectral composition of electromagnetic radiation -- continued

When the dust clears, the spectral intensity depends on the following integral:

$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt_r \; e^{i\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)} \left[\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}\left(t_r\right)\right)\right]\right|^2$$

Recall that the spectral intensity is related to the time integrated power:

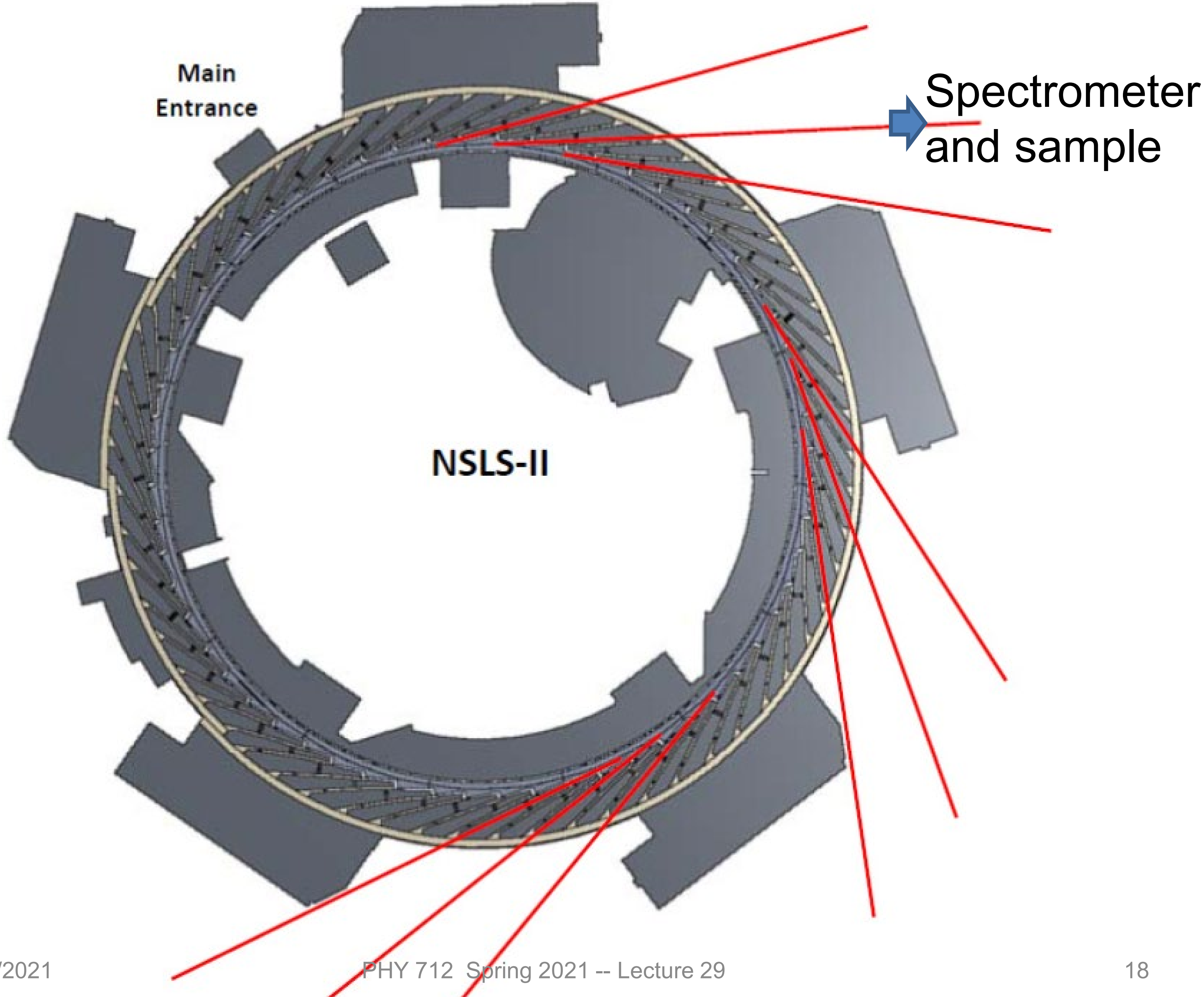
$$\int_{-\infty}^{\infty} dt \frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} d\omega \frac{\partial^2 I}{\partial\omega\partial\Omega}$$

Synchrotron radiation light source installations

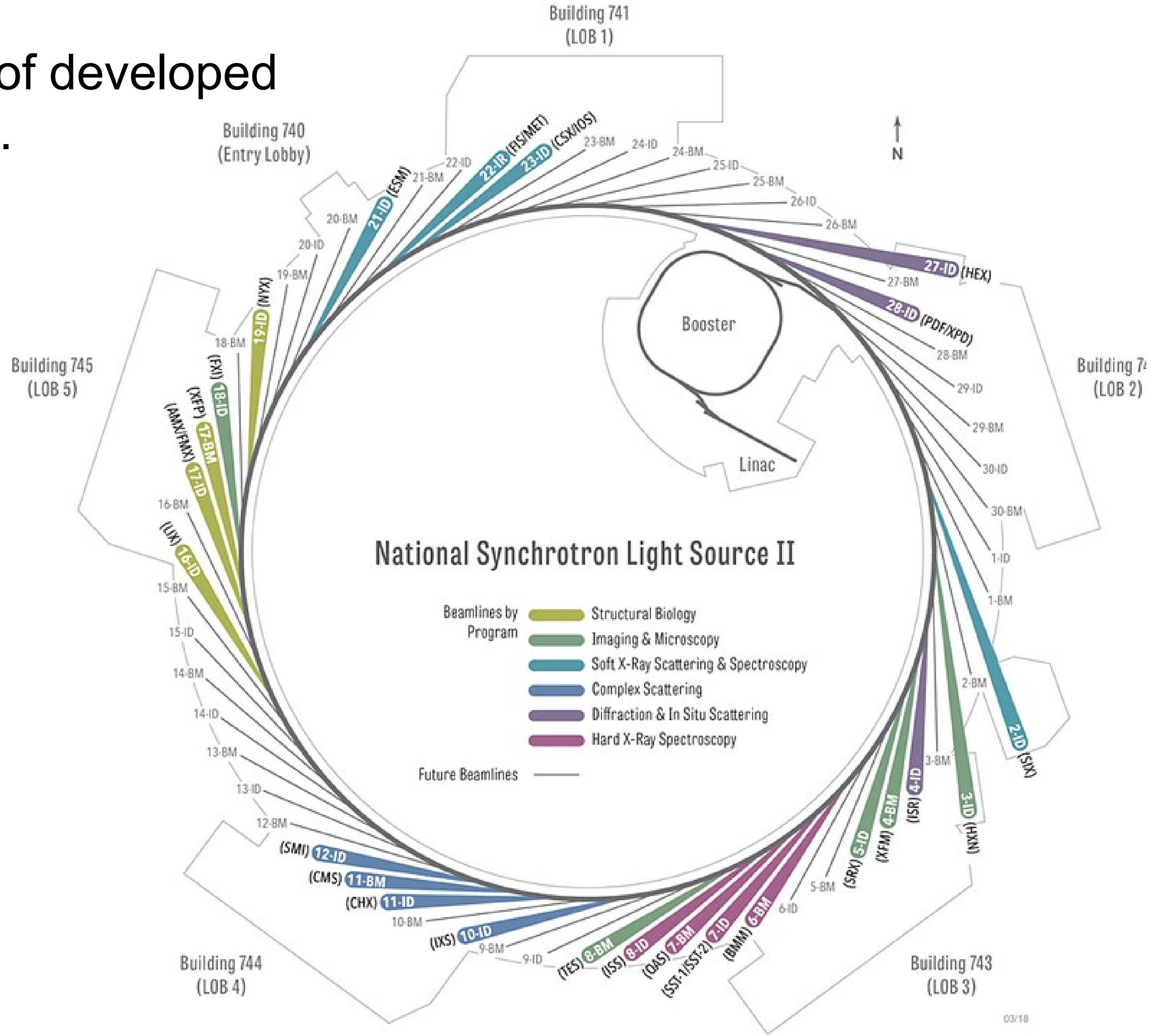
Synchrotron at Brookhaven National Lab, NY

E_c= 3 GeV   X-ray radiation

https://www.bnl.gov/ps/

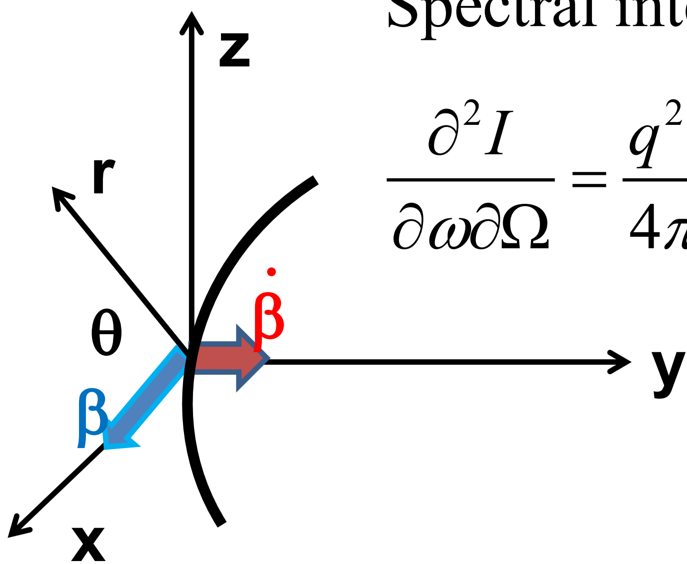

Main
Entrance
Spectrometer
and sample
NSLS-II

Overview of developed beamlines.



Advanced photon source, Argonne National Laboratory

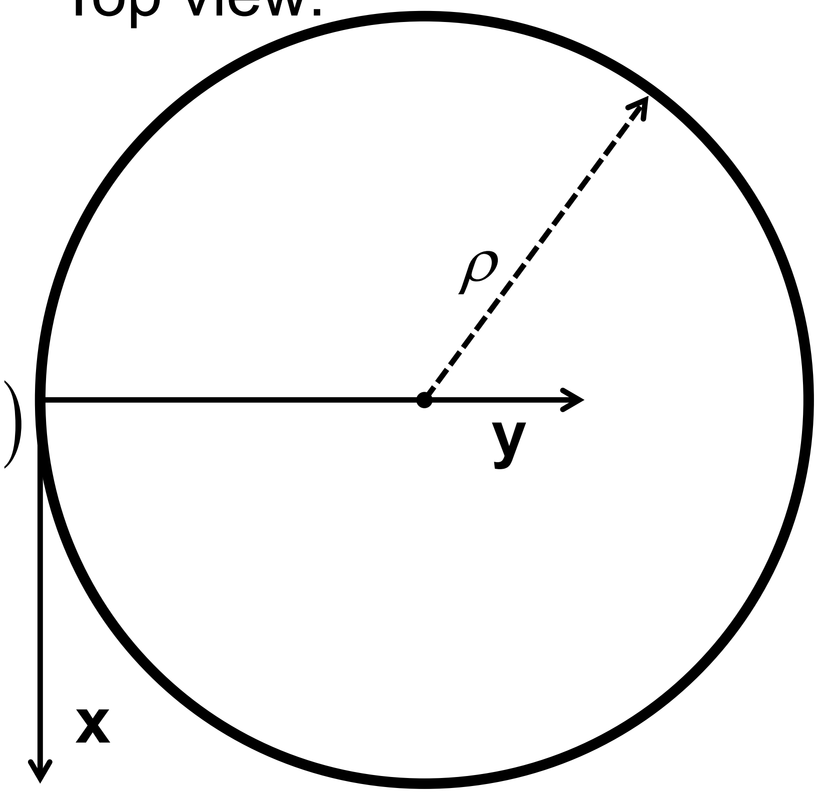
https://www.aps.anl.gov/



Spectral intensity relationship:

$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt_r\ e^{i\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)}\ \left[\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}(t_r)\right)\right]\right|^2$$

Top view:

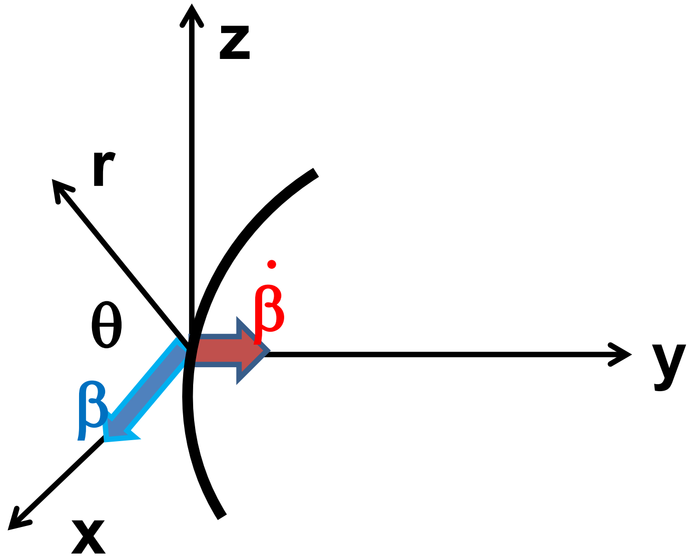


$$\mathbf{R}_q(t_r) = \rho\hat{\mathbf{x}}\sin(vt_r/\rho)$$
$$+\ \rho\hat{\mathbf{y}}\left(1-\cos(vt_r/\rho)\right)$$

$$\boldsymbol{\beta}(t_r) = \beta\left(\hat{\mathbf{x}}\cos(vt_r/\rho)+\hat{\mathbf{y}}\sin(vt_r/\rho)\right)$$

For convenience, choose:

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{z}}\sin\theta$$



$$\mathbf{R}_q(t_r) = \rho\hat{\mathbf{x}}\sin(vt_r/\rho) + \rho\hat{\mathbf{y}}\left(1-\cos(vt_r/\rho)\right)$$

$$\boldsymbol{\beta}(t_r) = \beta\left(\hat{\mathbf{x}}\cos(vt_r/\rho) + \hat{\mathbf{y}}\sin(vt_r/\rho)\right)$$

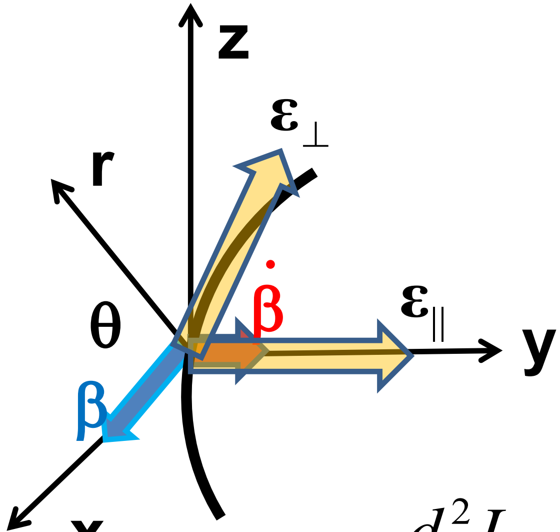
For convenience, choose:

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{z}}\sin\theta$$

Note that we have previous shown that in the radiation zone, the Poynting vector is in the $\hat{\mathbf{r}}$ direction; we can then choose to analyze two orthogonal polarization directions:

$$\boldsymbol{\varepsilon}_{\|} = \hat{\mathbf{y}} \qquad \boldsymbol{\varepsilon}_{\perp} = -\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta$$

$$\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\boldsymbol{\beta}) = \beta\left(-\boldsymbol{\varepsilon}_{\|}\sin(vt_r/\rho) + \boldsymbol{\varepsilon}_{\perp}\sin\theta\cos(vt_r/\rho)\right)$$



$$\boldsymbol{\varepsilon}_\parallel = \hat{\mathbf{y}} \qquad \boldsymbol{\varepsilon}_\perp = -\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta$$

$$\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}\right) =$$

$$\beta\left(-\boldsymbol{\varepsilon}_\parallel \sin\left(vt_r/\rho\right) + \boldsymbol{\varepsilon}_\perp \sin\theta\cos\left(vt_r/\rho\right)\right)$$

$$\frac{d^2I}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty}\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\beta)\mathrm{e}^{i\omega(t-\hat{\mathbf{r}}\cdot\mathbf{R}_q(t)/c)}dt\right|^2$$

$$\frac{d^2I}{d\omega d\Omega} = \frac{q^2\omega^2\beta^2}{4\pi^2 c}\left\{|C_\parallel(\omega)|^2 + |C_\perp(\omega)|^2\right\}$$

$$C_\parallel(\omega) = \int_{-\infty}^{\infty} dt\sin(vt/\rho)\mathrm{e}^{i\omega(t-\frac{\rho}{c}\cos\theta\sin(vt/\rho))}$$

$$C_\perp(\omega) = \int_{-\infty}^{\infty} dt\sin\theta\cos(vt/\rho)\mathrm{e}^{i\omega(t-\frac{\rho}{c}\cos\theta\sin(vt/\rho))}$$

We will analyze this expression for two different cases. The first case, is appropriate for man-made synchrotrons used as light sources. In this case, the light is produced by short bursts of electrons moving close to the speed of light ($v \approx c(1-1/(2\gamma^2))$) passing a beam line port. In addition, because of the design of the radiation ports, $\theta \approx 0$, and the relevant integration times t are close to $t \approx 0$. This results in the form shown in Eq. 14.79 of your text. It is convenient to rewrite this form in terms of a critical frequency $\omega_c \equiv \dfrac{3c\gamma^3}{2\rho}$.

$$\frac{d^2 I}{d\omega d\Omega} = \frac{3q^2\gamma^2}{4\pi^2 c}\left(\frac{\omega}{\omega_c}\right)^2 (1+\gamma^2\theta^2)^2 \left\{\left[K_{2/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2 \right.$$

$$\left. + \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}\left[K_{1/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2\right\}$$

Some details:

Modified Bessel functions

$$K_{1/3}(\xi) = \sqrt{3}\int_0^\infty dx \cos\left[\tfrac{3}{2}\xi\left(x + \tfrac{1}{3}x^3\right)\right] \qquad K_{2/3}(\xi) = \sqrt{3}\int_0^\infty dx\, x \sin\left[\tfrac{3}{2}\xi\left(x + \tfrac{1}{3}x^3\right)\right]$$

Exponential factor

$$\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right) = \omega\left(t_r - \frac{\rho}{c}\cos\theta\sin(vt_r/\rho)\right)$$

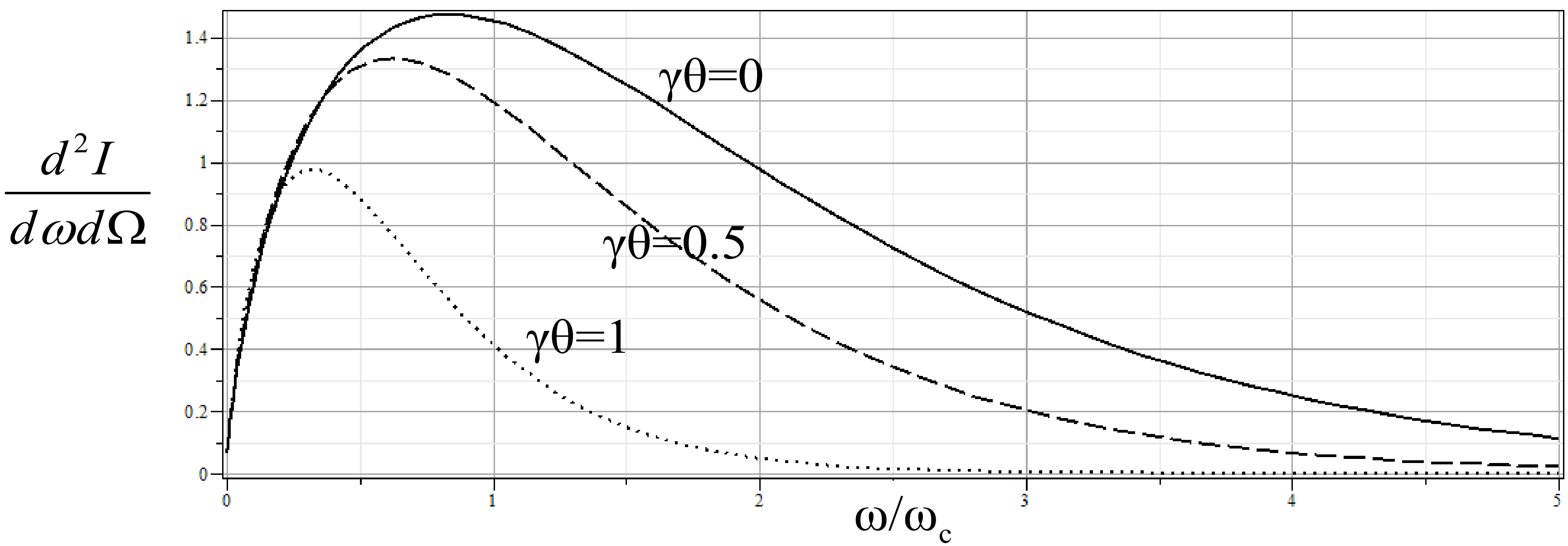
In the limit of $t_r \approx 0, \quad \theta \approx 0, \;\; v \approx c\left(1 - \dfrac{1}{2\gamma^2}\right)$

$$\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right) \approx \frac{\omega t_r}{2\gamma^2}\left(1+\gamma^2\theta^2\right) + \frac{\omega c^2 t_r{}^3}{6\rho^2} = \frac{3}{2}\xi\left(x + \frac{1}{3}x^3\right)$$

where $\quad \xi = \dfrac{\omega\rho}{3c\gamma^3}\left(1+\gamma^2\theta^2\right)^{3/2}$ and $x = \dfrac{c\gamma t_r}{\rho\left(1+\gamma^2\theta^2\right)^{1/2}}$

$$\frac{d^2 I}{d\omega d\Omega} = \frac{3q^2\gamma^2}{4\pi^2 c}\left(\frac{\omega}{\omega_c}\right)^2 (1+\gamma^2\theta^2)^2 \left\{\left[K_{2/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2 + \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}\left[K_{1/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2\right\}$$

By plotting the intensity as a function of ω, we see that the intensity is largest near $\omega \approx \omega_c$. The plot below shows the intensity as a function of ω/ω_c for $\gamma\theta=0$, 0.5 and 1:

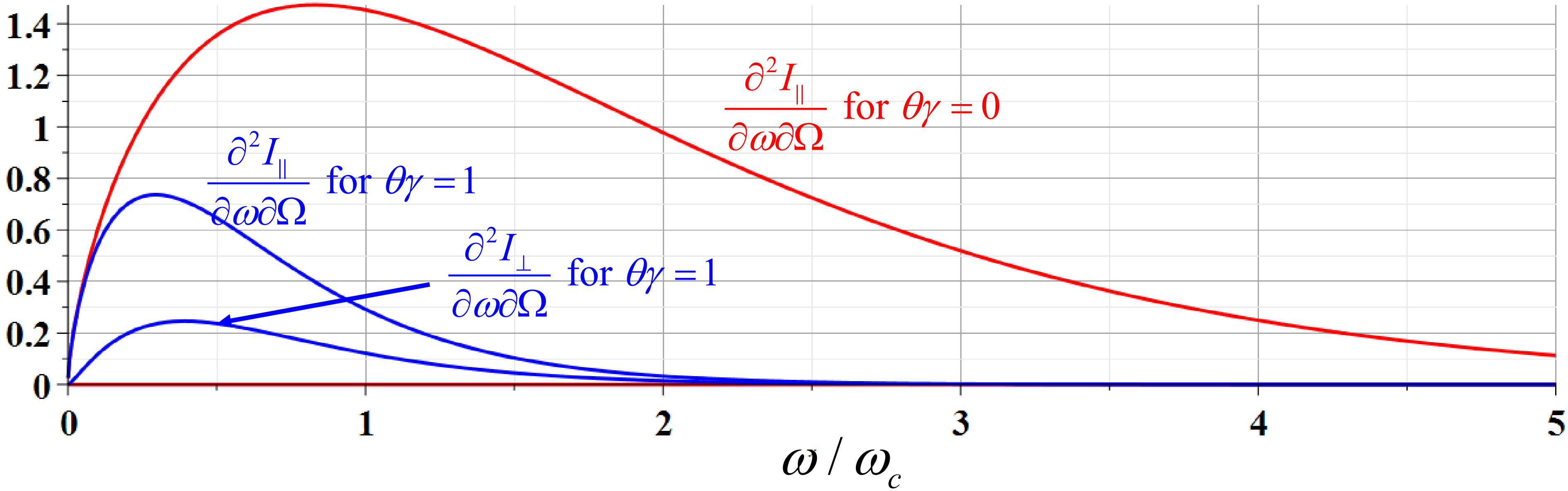


More details

$$\frac{d^2 I}{d\omega d\Omega} = \frac{d^2 I_{\parallel}}{d\omega d\Omega} + \frac{d^2 I_{\perp}}{d\omega d\Omega}$$

$$\frac{d^2 I_{\parallel}}{d\omega d\Omega} = \frac{3q^2\gamma^2}{4\pi^2 c}\left(\frac{\omega}{\omega_c}\right)^2 (1+\gamma^2\theta^2)^2 \left[K_{2/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2$$

$$\frac{d^2 I_{\perp}}{d\omega d\Omega} = \frac{3q^2\gamma^2}{4\pi^2 c}\left(\frac{\omega}{\omega_c}\right)^2 (1+\gamma^2\theta^2)^2 \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}\left[K_{1/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2$$



The above analysis applies to a class of man-made facilities dedicated to producing intense radiation in the continuous spectrum.   For more specific information on man-made synchrotron sources, the following web page is useful: http://www.als.lbl.gov/als/synchrotron_sources.html.

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On the Classical Radiation of Accelerated Electrons

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This paper is concerned with the properties of the radiation from a high energy accelerated electron, as recently observed in the General Electric synchrotron. An elementary derivation of the total rate of radiation is first presented, based on Larmor's formula for a slowly moving electron, and arguments of relativistic invariance. We then construct an expression for the instantaneous power radiated by an electron moving along an arbitrary, prescribed path. By casting this result into various forms, one obtains the angular distribution, the spectral distribution, or the combined angular and spectral distributions of the radiation. The method is based on an examination of the rate at which the electron irreversibly transfers energy to the electromagnetic field, as determined by half the difference of retarded and advanced electric field intensities. Formulas are obtained for an arbitrary charge-current distribution and then specialized to a point charge. The total radiated power and its angular distribution are obtained for an arbitrary trajectory. It is found that the direction of motion is a strongly preferred direction of emission at high energies. The spectral distribution of the radiation depends upon the detailed motion over a time interval large compared to the period of the radiation. However, the narrow cone of radiation generated by an energetic electron indicates that only a small part of the trajectory is effective in producing radiation observed in a given direction, which also implies that very high frequencies are emitted. Accordingly, we evaluate the spectral and angular distributions of the high frequency radiation by an energetic electron, in their dependence upon the parameters characterizing the instantaneous orbit. The average spectral distribution, as observed in the synchrotron measurements, is obtained by averaging the electron energy over an acceleration cycle. The entire spectrum emitted by an electron moving with constant speed in a circular path is also discussed. Finally, it is observed that quantum effects will modify the classical results here obtained only at extraordinarily large energies.

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