PHY 712 Electrodynamics
10-10:50 AM MWF online

Notes for Lecture 32:

Start reading Chap. 15 –

Radiation from collisions of charged particles

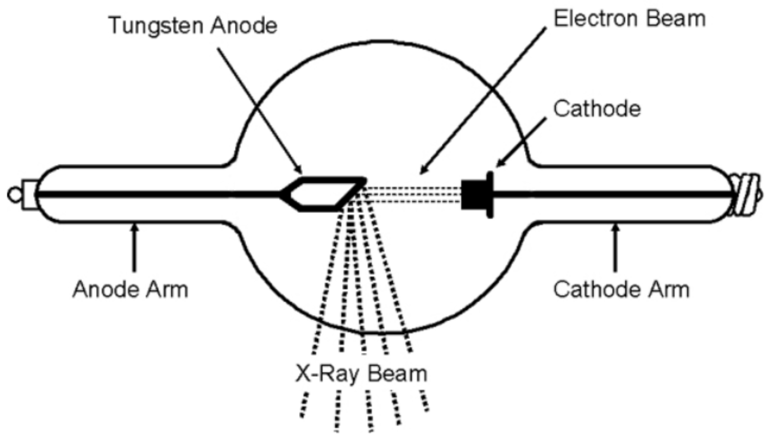
1. Overview
2. X-ray tube
3. Radiation from Rutherford scattering
4. Other collision models

In this lecture we will discuss some examples of radiation due to charged particles colliding. It is a complicated topic which quite a few famous physicists have worked on.

28	Fri: 04/09/2021	Chap. 14	Radiation from accelerating charged particles	#20	04/12/2021
29	Mon: 04/12/2021	Chap. 14	Synchrotron radiation	#21	04/14/2021
30	Wed: 04/14/2021	Chap. 14	Synchrotron radiation	#22	04/19/2021
31	Fri: 04/16/2021	Chap. 15	Radiation from collisions of charged particles	#23	04/21/2021
32	Mon: 04/19/2021	Chap. 15	Radiation from collisions of charged particles		
33	Wed: 04/21/2021	Chap. 13	Cherenkov radiation		
34	Fri: 04/23/2021		Special topic: E & M aspects of superconductivity		
35	Mon: 04/26/2021		Special topic: E & M aspects of superconductivity		
36	Wed: 04/28/2021		Review		
37	Fri: 04/30/2021		Review		
	Mon: 05/03/2021		Presentations I		
	Wed: 05/05/2021		Presentations II		

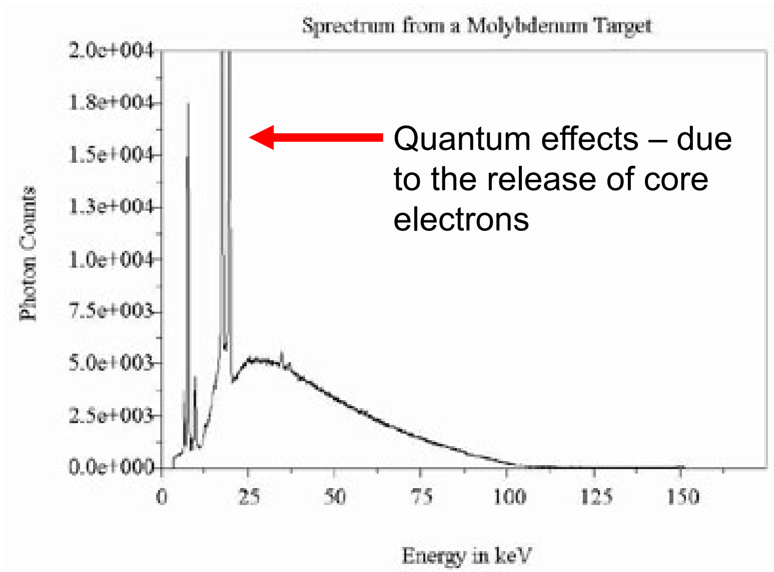
This is the revised schedule, subject to your input.

Generation of X-rays in a Coolidge tube
https://www.orau.org/ptp/collection/xraytubescoolidge/coolidgeinformation.htm

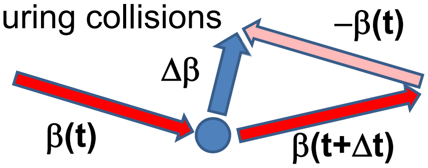


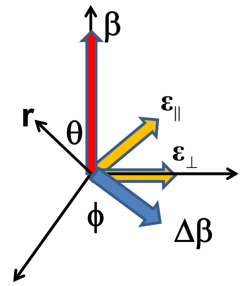
Invented in 1913. Associated with the German word "bremsstrahlung" – meaning breaking radiation.

http://www.ndt-ed.org/EducationResources/CommunityCollege/Radiography/Physics/xrays.htm



Radiation during collisions





Results from previous analyses:

Intensity:

$$\frac{d^2I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\int dt\ e^{i\omega\left(t-\hat{\mathbf{r}}\cdot\mathbf{R}_q(t)/c\right)}\frac{d}{dt}\left[\frac{\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}\right)}{1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta}}\right]\right|^2$$

Note that $\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}\right)=\hat{\mathbf{r}}\left(\hat{\mathbf{r}}\cdot\boldsymbol{\beta}\right)-\boldsymbol{\beta}=-\left(\boldsymbol{\varepsilon}_{\parallel}\cdot\boldsymbol{\beta}\right)\boldsymbol{\varepsilon}_{\parallel}-\left(\boldsymbol{\varepsilon}_{\perp}\cdot\boldsymbol{\beta}\right)\boldsymbol{\varepsilon}_{\perp}$

For a collision of duration τ emitting radiation with polarization $\boldsymbol{\varepsilon}$ and frequency $\omega\rightarrow 0$:

$$\frac{d^2I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\boldsymbol{\varepsilon}\cdot\left(\frac{\boldsymbol{\beta}(t+\tau)}{1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta}(t+\tau)}-\frac{\boldsymbol{\beta}(t)}{1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta}(t)}\right)\right|^2$$

Starting from the intensity analysis for radiation due to a charged particle moving in a trajectory with beta representing its velocity/c. We will consider the velocity changing due to a collision process and analyze the radiation at small frequencies.

Note that in the following slides we are taking the limit $\omega \rightarrow 0$ but keeping the notation of the differential intensity….

Radiation during collisions -- continued

For a collision of duration τ emitting radiation with polarization $\boldsymbol{\varepsilon}$ and frequency $\omega \to 0$:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\boldsymbol{\varepsilon}\cdot\left(\frac{\boldsymbol{\beta}(t+\tau)}{1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta}(t+\tau)} - \frac{\boldsymbol{\beta}(t)}{1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta}(t)}\right)\right|^2$$

We will evaluate this expression for two cases:

Non-relativistic limit:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\boldsymbol{\varepsilon}\cdot(\Delta\boldsymbol{\beta})\right|^2 \qquad \Delta\boldsymbol{\beta} \equiv \boldsymbol{\beta}(t+\tau) - \boldsymbol{\beta}(t)$$

Relativistic collision with small $|\Delta\boldsymbol{\beta}| \equiv \boldsymbol{\beta}(t+\tau) - \boldsymbol{\beta}(t)$:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\boldsymbol{\varepsilon}\cdot\left(\frac{\Delta\boldsymbol{\beta} + \hat{\mathbf{r}}\times(\boldsymbol{\beta}\times\Delta\boldsymbol{\beta})}{(1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta})^2}\right)\right|^2$$

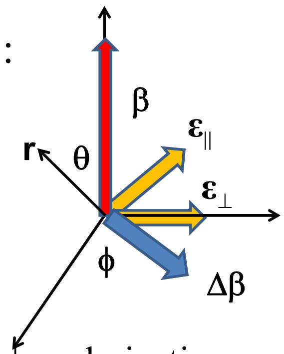
In the limit $\beta \rightarrow 0$, this is the same as the non-relativistic case.

For beta<<1, we can neglect the denominator of the expression and obtain the non-relativistic expression. It is also convenient to analyze the relativistic case when the change in velocity is small.

Radiation during collisions -- continued

Relativistic collision with small $|\Delta\boldsymbol{\beta}|$:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\boldsymbol{\varepsilon}\cdot\left(\frac{\Delta\boldsymbol{\beta} + \hat{\mathbf{r}}\times(\boldsymbol{\beta}\times\Delta\boldsymbol{\beta})}{(1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta})^2}\right)\right|^2$$



Also assume $\Delta\boldsymbol{\beta}$ is perpendicular to $\boldsymbol{\beta}$ direction

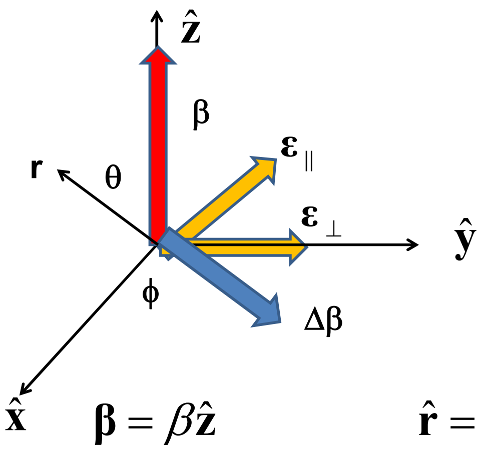
Expressions (averaging over φ) for $\parallel$ or $\perp$ polarization:

$$\frac{d^2 I_{\parallel}}{d\omega d\Omega} = \frac{q^2}{8\pi^2 c}|\Delta\boldsymbol{\beta}|^2 \frac{(\beta - \cos\theta)^2}{(1-\beta\cos\theta)^4}$$ polarization in ***r*** and ***β*** plane

$$\frac{d^2 I_{\perp}}{d\omega d\Omega} = \frac{q^2}{8\pi^2 c}|\Delta\boldsymbol{\beta}|^2 \frac{1}{(1-\beta\cos\theta)^2}$$ polarization perpendicular to ***r*** and ***β*** plane

It is convenient to consider two different polarizations of the radiation – parallel (meaning in the plane of the observation point r and the initial velocity of the particle) and perpendicular (meaning perpendicular to that plane).

Some details:



$$\boldsymbol{\beta} = \beta\hat{\mathbf{z}} \qquad \hat{\mathbf{r}} = \sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{z}}$$

$$\boldsymbol{\varepsilon}_{\parallel} = -\cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{z}} \qquad \boldsymbol{\varepsilon}_{\perp} = \hat{\mathbf{y}}$$

$$\boldsymbol{\Delta\beta} = \Delta\beta\left(\cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}\right)$$

Note: This is a wild assumption!

Showing the detailed geometry of the scattering process.

Some details -- continued:

$$\hat{\mathbf{r}} = \sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{z}}$$

$$\boldsymbol{\varepsilon}_{\perp} = \hat{\mathbf{y}} \qquad \boldsymbol{\varepsilon}_{\|} = -\cos\ \theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{z}}$$

Consistent with radiation from charged particles.

$$\boldsymbol{\beta} = \beta\hat{\mathbf{z}}$$

Convenient geometry

$$\Delta\boldsymbol{\beta} = \Delta\beta\left(\cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}\right)$$

Wild guess

$$\Delta\boldsymbol{\beta} + \hat{\mathbf{r}}\times\left(\boldsymbol{\beta}\times\Delta\boldsymbol{\beta}\right) = \Delta\boldsymbol{\beta}\left(1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta}\right) + \boldsymbol{\beta}\left(\hat{\mathbf{r}}\cdot\Delta\boldsymbol{\beta}\right)$$

$$\boldsymbol{\varepsilon}_{\perp}\cdot\left(\Delta\boldsymbol{\beta} + \hat{\mathbf{r}}\times\left(\boldsymbol{\beta}\times\Delta\boldsymbol{\beta}\right)\right) = \Delta\beta\sin\phi\left(1-\beta\cos\theta\right)$$

$$\boldsymbol{\varepsilon}_{\|}\cdot\left(\Delta\boldsymbol{\beta} + \hat{\mathbf{r}}\times\left(\boldsymbol{\beta}\times\Delta\boldsymbol{\beta}\right)\right) = \Delta\beta\cos\phi\left(\beta-\cos\theta\right)$$

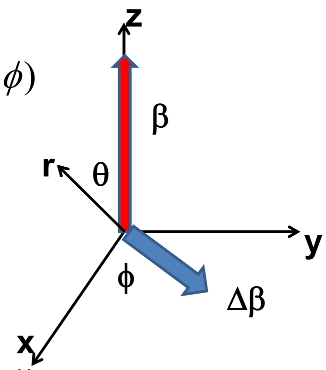
Evaluating the vectors.

Radiation during collisions -- continued

Intensity expressions: (averaging over ϕ)

$$\frac{d^2 I_{\parallel}}{d\omega d\Omega} = \frac{q^2}{8\pi^2 c}\left|\Delta\boldsymbol{\beta}\right|^2 \frac{(\beta-\cos\theta)^2}{(1-\beta\cos\theta)^4}$$

$$\frac{d^2 I_{\perp}}{d\omega d\Omega} = \frac{q^2}{8\pi^2 c}\left|\Delta\boldsymbol{\beta}\right|^2 \frac{1}{(1-\beta\cos\theta)^2}$$



Relativistic collision at low ω and with small $\left|\Delta\boldsymbol{\beta}\right|$ and $\Delta\boldsymbol{\beta}$ perpendicular to plane of $\hat{\mathbf{r}}$ and $\boldsymbol{\beta}$, as a function of θ where $\hat{\mathbf{r}}\cdot\boldsymbol{\beta} = \beta\cos\theta$;

Integrating over solid angle:

$$\frac{dI}{d\omega} = \int d\Omega\left(\frac{d^2 I_{\parallel}}{d\omega d\Omega} + \frac{d^2 I_{\perp}}{d\omega d\Omega}\right) = \frac{2}{3\pi}\frac{q^2}{c}\gamma^2 \left|\Delta\boldsymbol{\beta}\right|^2$$

It is possible to analytically integrate over all solid angles.

Some more details:

$$\int d\Omega \frac{d^2 I_{\parallel}}{d\omega d\Omega} = \frac{q^2}{8\pi^2 c}|\Delta\boldsymbol{\beta}|^2 \, 2\pi \int_{-1}^{1} d\cos\theta \frac{(\beta - \cos\theta)^2}{(1-\beta\cos\theta)^4}$$

$$= \frac{q^2}{4\pi c}|\Delta\boldsymbol{\beta}|^2 \frac{2}{3}\frac{1}{(1-\beta^2)}$$

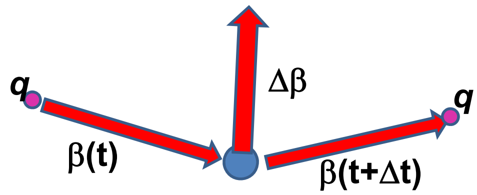
$$\int d\Omega \frac{d^2 I_{\perp}}{d\omega d\Omega} = \frac{q^2}{8\pi^2 c}|\Delta\boldsymbol{\beta}|^2 \int_{-1}^{1} d\cos\theta \frac{1}{(1-\beta\cos\theta)^2}$$

$$= \frac{q^2}{4\pi c}|\Delta\boldsymbol{\beta}|^2 \frac{2}{(1-\beta^2)}$$

$$\frac{dI}{d\omega} = \int d\Omega \left(\frac{d^2 I_{\parallel}}{d\omega d\Omega} + \frac{d^2 I_{\perp}}{d\omega d\Omega} \right) = \frac{2}{3\pi}\frac{q^2}{c}\gamma^2 \; |\Delta\boldsymbol{\beta}|^2$$

Some details of the analysis. With all of these considerations, we still need to estimate delta beta.

Estimation of Δβ



Momentum transfer:

$$Qc \equiv \left|\mathbf{p}(t+\tau)-\mathbf{p}(t)\right|c \approx \gamma M c^2 \left|\Delta\boldsymbol{\beta}\right|$$

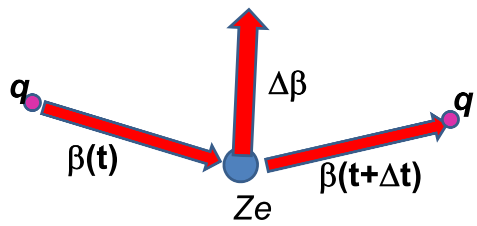
mass of particle having charge *q*

$$\frac{dI}{d\omega} = \frac{2}{3\pi}\,\frac{q^2}{c}\gamma^2\left|\Delta\boldsymbol{\beta}\right|^2 \approx \frac{2}{3\pi}\,\frac{q^2}{M^2c^3}Q^2$$

What are the conditions for the validity of this result?

What are possible sources for the momentum transfer Q?

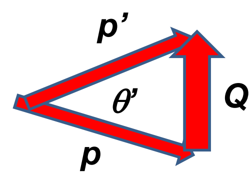
Estimation of $\Delta\beta$ or Q -- for the case of Rutherford scattering



Assume that target nucleus (charge Ze) has mass >>M;

Rutherford scattering cross-section in center of mass analysis:

$$\frac{d\sigma}{d\Omega} = \left(\frac{2Zeq}{pv}\right)^2 \frac{1}{\left(2\sin\left(\theta'/2\right)\right)^4}$$



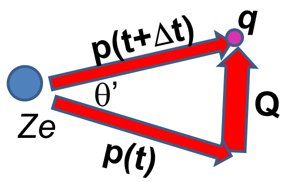
Assuming elastic scattering:

$$Q^2 = \left(2p\sin\left(\theta'/2\right)\right)^2 = 2p^2\left(1-\cos\theta'\right)$$

Delta beta will depend on the particular system. As an example, consider the case of Rutherford scattering.. Here are some of the equations we used in classical mechanics class.

Case of Rutherford scattering -- continued

Rutherford scattering cross-section:



$$\frac{d\sigma}{d\Omega} = \left(\frac{2Zeq}{pv}\right)^2 \frac{1}{\left(2\sin(\theta'/2)\right)^4}$$

$$\frac{d\sigma}{dQ} = \int_{\varphi'} \frac{d\sigma}{d\Omega}\left|\frac{d\Omega}{dQ}\right| d\varphi'$$

$$d\Omega = d\varphi' \, d\cos\theta'$$

$$Q^2 = \left(2p\sin(\theta'/2)\right)^2 = 2p^2\left(1-\cos\theta'\right)$$

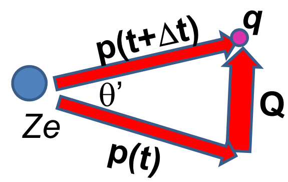
$$dQ = -\frac{p^2}{Q} d\cos\theta'$$

$$\Rightarrow \quad \frac{d\sigma}{dQ} = 8\pi\left(\frac{Zeq}{\beta c}\right)^2 \frac{1}{Q^3}$$

Does the algebra work out?

It is convenient to express the results in terms of the momentum transfer Q.

Case of Rutherford scattering -- continued



Differential radiation cross section:

$$\frac{d^2\chi}{d\omega dQ}=\frac{dI}{d\omega}\frac{d\sigma}{dQ}=\left(\frac{2}{3\pi}\frac{q^2}{M^2c^3}Q^2\right)\left(8\pi\left(\frac{Zeq}{\beta c}\right)^2\frac{1}{Q^3}\right)$$

$$=\frac{16}{3}\frac{(Ze)^2}{c}\left(\frac{q^2}{Mc^2}\right)^2\frac{1}{\beta^2}\frac{1}{Q}$$

It is of interest to estimate the probability of the radiation occurring which depends on the product of the radiation intensity for a given momentum transfer and the cross section as a function of momentum transfer.

Differential radiation cross section -- continued

Integrating over momentum transfer

$$\frac{d\chi}{d\omega} = \int_{Q_{\min}}^{Q_{\max}} dQ \frac{d^2\chi}{d\omega dQ} = \frac{16}{3}\frac{(Ze)^2}{c}\left(\frac{q^2}{Mc^2}\right)^2 \frac{1}{\beta^2} \ln\left(\frac{Q_{\max}}{Q_{\min}}\right)$$

How do the limits of Q occur?

Jackson suggests that these come from the limits of validity of the analysis.

1. Seems like cheating?
2. Perhaps fair?

But we are not done. Thinking of the case of the charged particle moving through the target material, there will be a range of momentum transfers that should be integrated as indicated here. Note that we have assumed that the frequency of the radiation is very small. Here we consider how frequency might ener this analysis.

Comment on frequency dependence --

Original expression for radiation intensity:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\int dt\; e^{i\omega\left(t - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t)/c\right)} \frac{d}{dt}\left[\frac{\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}\right)}{1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta}}\right]\right|^2$$

In the previous derivations, we have assumed that

$$\omega\left(t - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t)/c\right) << 1.$$

$$\omega\left(t - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t)/c\right) = \omega\left(t - \hat{\mathbf{r}}\cdot\int_0^t dt'\boldsymbol{\beta}(t')\right) \approx \omega\tau\left(1 - \hat{\mathbf{r}}\cdot\langle\boldsymbol{\beta}\rangle\right)$$

In the non-relativistic case, this means $\omega\tau \ll 1$.

Here τ is the effective collision time.

How to estimate the collision time?

Jackson uses the following analysis in terms of the impact parameter *b:*

Using classical mechanics and assuming $v \ll c$:

$$\tau \approx \frac{b}{v} \ll \frac{1}{\omega} \quad \text{and} \quad Q \approx \frac{2Zeq}{bv}$$

Assume that $Q_{\min} = \dfrac{2Zeq}{b_{\max} v} = \dfrac{2Zeq\omega}{v^2}$

Differential radiation cross section -- continued

Radiation cross section in terms of momentum transfer

$$\frac{d\chi}{d\omega} = \int_{Q_{\min}}^{Q_{\max}} dQ \frac{d^2\chi}{d\omega dQ} = \frac{16}{3}\frac{(Ze)^2}{c}\left(\frac{q^2}{Mc^2}\right)^2 \frac{1}{\beta^2}\ln\left(\frac{Q_{\max}}{Q_{\min}}\right)$$

Note that: $Q^2 = 2p^2(1-\cos\theta') \quad \Rightarrow Q_{\max} = 2p$

In general, $Q_{\min}$ is determined by the collision time condition $\omega\tau < 1 \quad \Rightarrow Q_{\min} \approx \frac{2Zeq\omega}{v^2}$

Radiation cross section for classical non-relativistic process

$$\frac{d\chi}{d\omega} = \frac{16}{3}\frac{(Ze)^2}{c}\left(\frac{q^2}{Mc^2}\right)^2 \frac{1}{\beta^2}\ln\left(\frac{\lambda M v^3}{Zeq\omega}\right)$$

λ= “fudge factor” of order unity

Hans Bethe considered this problem and also introduced a “correction” for quantum effects.

What could be the origin of the fudge factor?

What do you take away from this analysis

1. Disgust?
2. Admiration?
3. Motivation to avoid charged particles?