

PHY 712 Electrodynamics

10-10:50 AM MWF online

Discussion for Lecture 32:

Start reading Chap. 15 –

Radiation from collisions of charged particles

- 1. Overview**
- 2. X-ray tube**
- 3. Radiation from Rutherford scattering**
- 4. Other collision models**

28	Fri: 04/09/2021	Chap. 14	Radiation from accelerating charged particles	#20	04/12/2021
29	Mon: 04/12/2021	Chap. 14	Synchrotron radiation	#21	04/14/2021
30	Wed: 04/14/2021	Chap. 14	Synchrotron radiation	#22	04/19/2021
31	Fri: 04/16/2021	Chap. 15	Radiation from collisions of charged particles	#23	04/21/2021
32	Mon: 04/19/2021	Chap. 15	Radiation from collisions of charged particles		
33	Wed: 04/21/2021	Chap. 13	Cherenkov radiation		
34	Fri: 04/23/2021		Special topic: E & M aspects of superconductivity		
35	Mon: 04/26/2021		Special topic: E & M aspects of superconductivity		
36	Wed: 04/28/2021		Review		
37	Fri: 04/30/2021		Review		
	Mon: 05/03/2021		Presentations I		
	Wed: 05/05/2021		Presentations II		

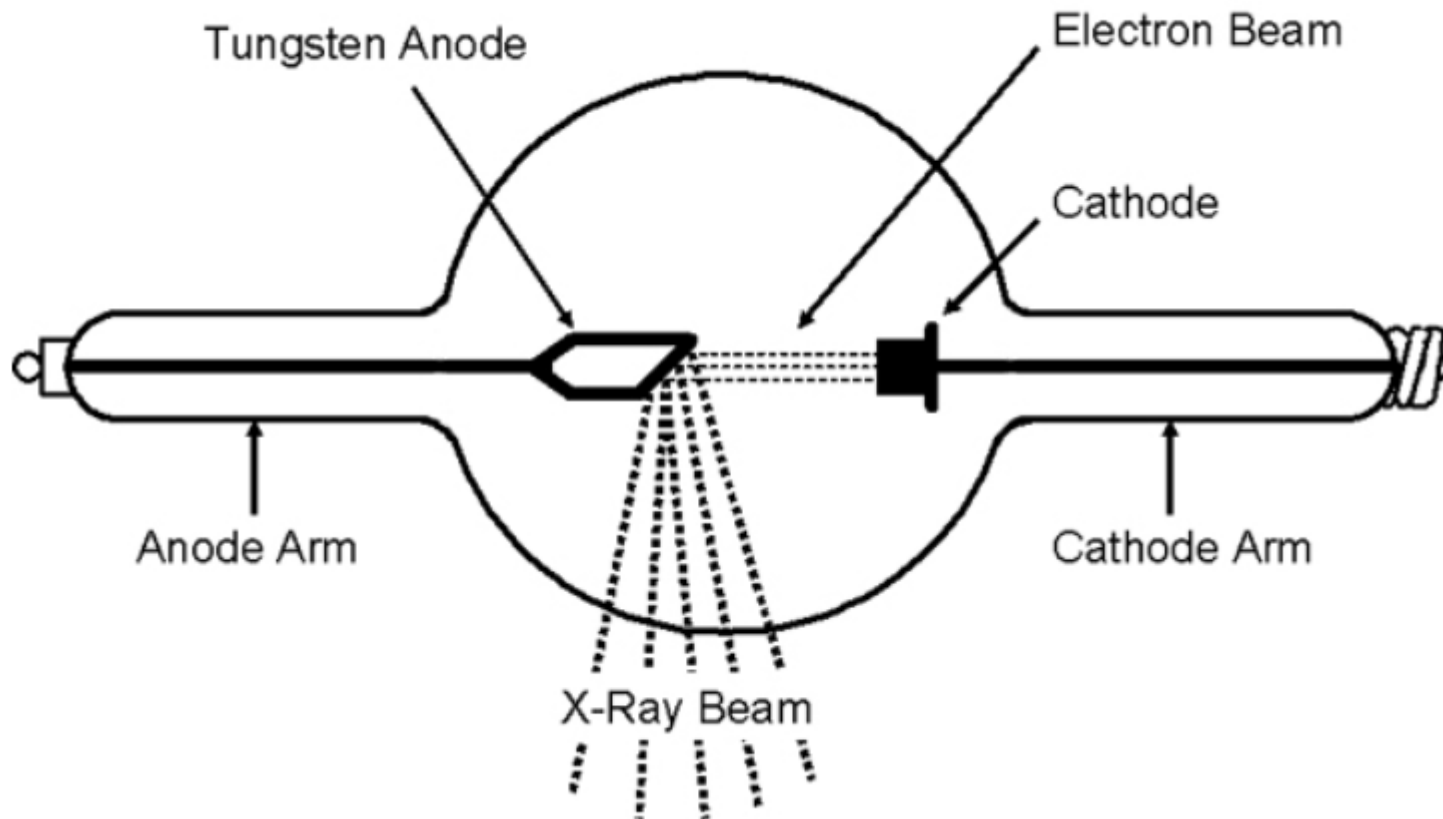
Your questions –

From Gao -- 1. Do we lose the [generality] if we take $\delta\beta$ perpendicular to β ? If so, why do we still want to do it?

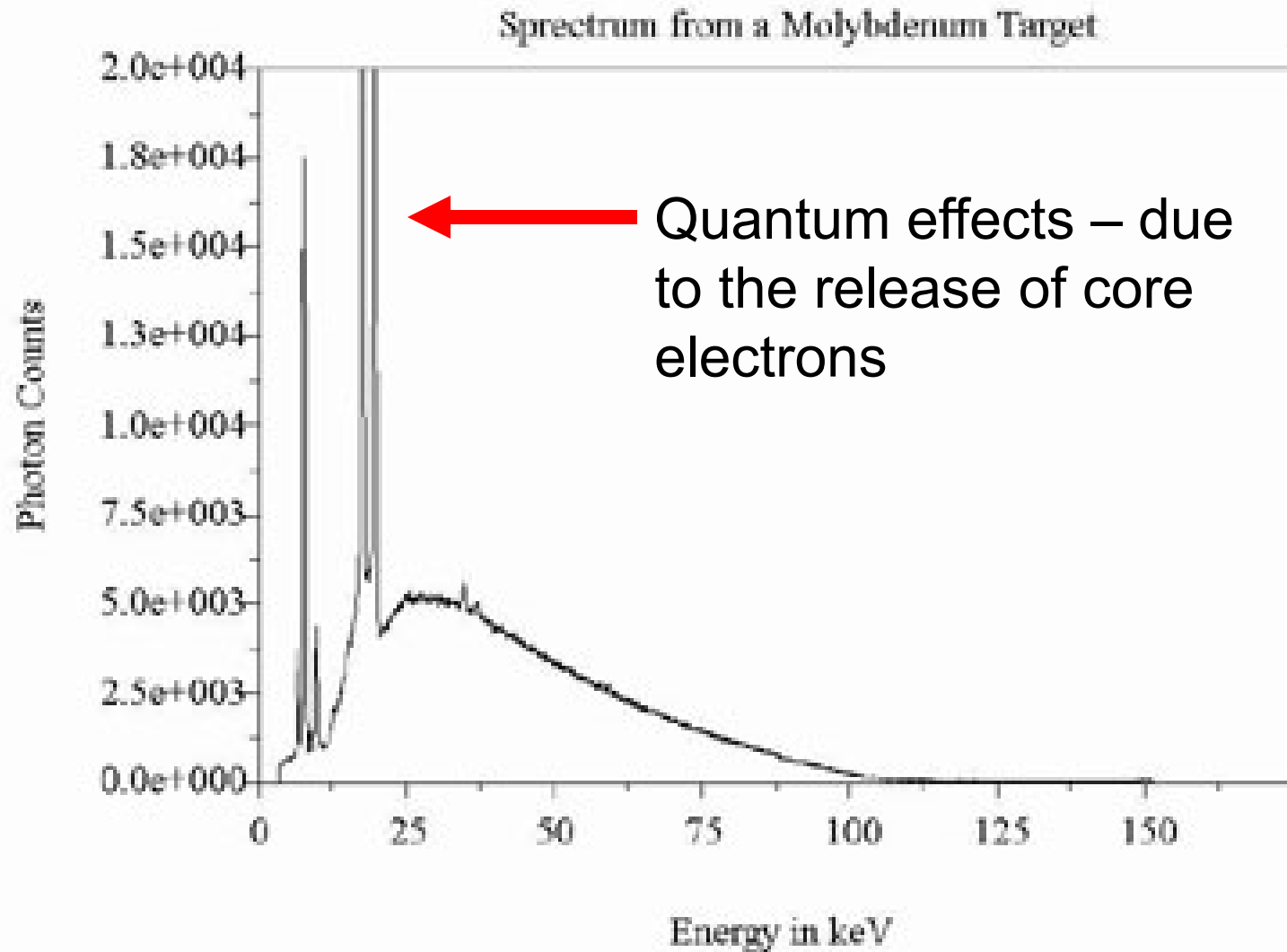
2. What convenience will we have to use momentum transfer instead of velocity?

Generation of X-rays in a Coolidge tube

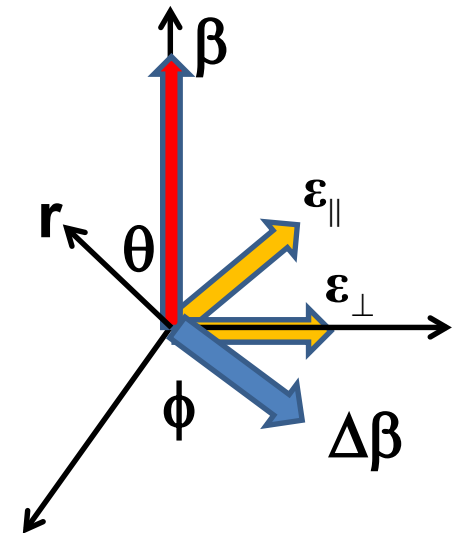
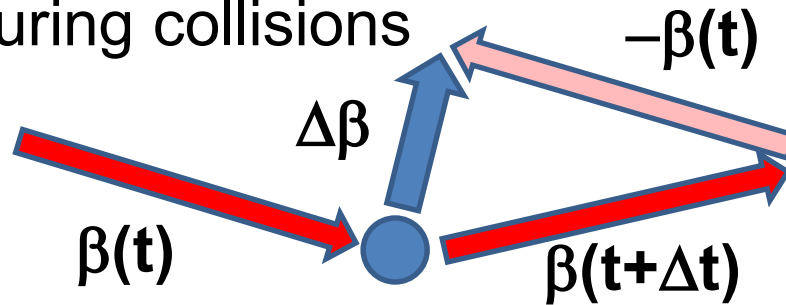
<https://www.orau.org/ptp/collection/xraytubescoolidge/coolidgeinformation.htm>



Invented in 1913. Associated with the German word “bremsstrahlung” – meaning breaking radiation.



Radiation during collisions



Results from previous analyses:
Intensity:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c} \left| \int dt e^{i\omega(t - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t)/c)} \frac{d}{dt} \left[\frac{\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \boldsymbol{\beta})}{1 - \hat{\mathbf{r}} \cdot \boldsymbol{\beta}} \right] \right|^2$$

Note that $\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \boldsymbol{\beta}) = \hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \boldsymbol{\beta}) - \boldsymbol{\beta} = -(\boldsymbol{\epsilon}_{\parallel} \cdot \boldsymbol{\beta})\boldsymbol{\epsilon}_{\parallel} - (\boldsymbol{\epsilon}_{\perp} \cdot \boldsymbol{\beta})\boldsymbol{\epsilon}_{\perp}$

For a collision of duration τ emitting radiation with polarization $\boldsymbol{\epsilon}$ and frequency $\omega \rightarrow 0$:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c} \left| \boldsymbol{\epsilon} \cdot \left(\frac{\boldsymbol{\beta}(t+\tau)}{1 - \hat{\mathbf{r}} \cdot \boldsymbol{\beta}(t+\tau)} - \frac{\boldsymbol{\beta}(t)}{1 - \hat{\mathbf{r}} \cdot \boldsymbol{\beta}(t)} \right) \right|^2$$

Note that in the following slides we are taking the limit $\omega \rightarrow 0$ but keeping the notation of the differential intensity....

Radiation during collisions -- continued

For a collision of duration τ emitting radiation with polarization $\boldsymbol{\varepsilon}$ and frequency $\omega \rightarrow 0$:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c} \left| \boldsymbol{\varepsilon} \cdot \left(\frac{\boldsymbol{\beta}(t+\tau)}{1 - \hat{\mathbf{r}} \cdot \boldsymbol{\beta}(t+\tau)} - \frac{\boldsymbol{\beta}(t)}{1 - \hat{\mathbf{r}} \cdot \boldsymbol{\beta}(t)} \right) \right|^2$$

We will evaluate this expression for two cases:

Non-relativistic limit:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c} \left| \boldsymbol{\varepsilon} \cdot (\Delta\boldsymbol{\beta}) \right|^2 \quad \Delta\boldsymbol{\beta} \equiv \boldsymbol{\beta}(t+\tau) - \boldsymbol{\beta}(t)$$

Relativistic collision with small $|\Delta\boldsymbol{\beta}| \equiv \boldsymbol{\beta}(t+\tau) - \boldsymbol{\beta}(t) :$

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c} \left| \boldsymbol{\varepsilon} \cdot \left(\frac{\Delta\boldsymbol{\beta} + \hat{\mathbf{r}} \times (\boldsymbol{\beta} \times \Delta\boldsymbol{\beta})}{(1 - \hat{\mathbf{r}} \cdot \boldsymbol{\beta})^2} \right) \right|^2$$

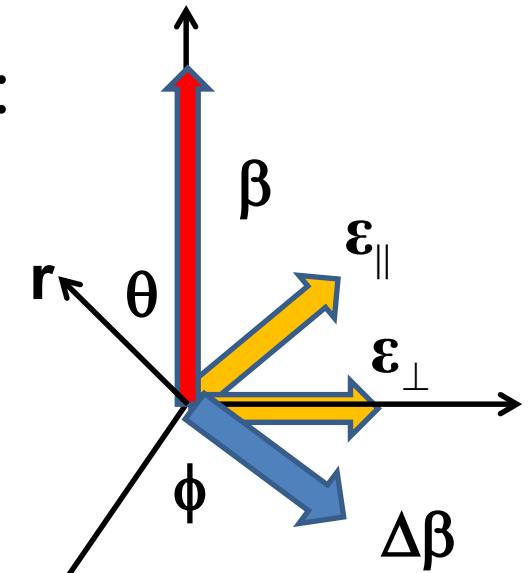
In the limit $\beta \rightarrow 0$, this is the same as the non-relativistic case.

Radiation during collisions -- continued

Relativistic collision with small $|\Delta\boldsymbol{\beta}|$:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c} \left| \boldsymbol{\varepsilon} \cdot \left(\frac{\Delta\boldsymbol{\beta} + \hat{\mathbf{r}} \times (\boldsymbol{\beta} \times \Delta\boldsymbol{\beta})}{(1 - \hat{\mathbf{r}} \cdot \boldsymbol{\beta})^2} \right) \right|^2$$

Also assume $\Delta\boldsymbol{\beta}$ is perpendicular to $\boldsymbol{\beta}$ direction

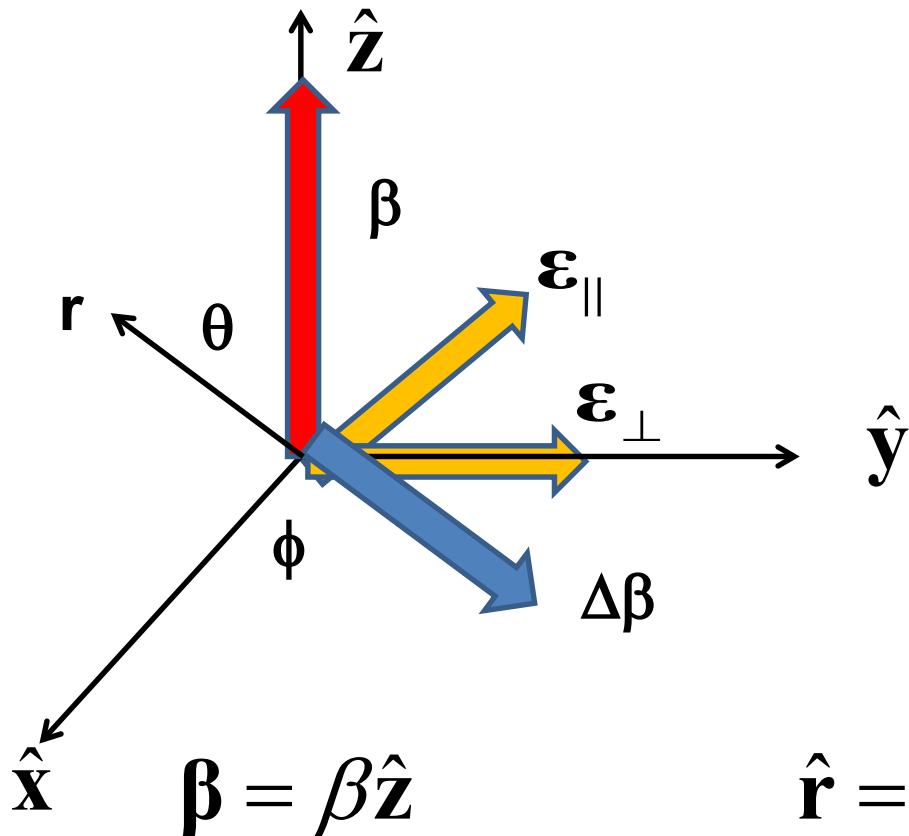


Expressions (averaging over φ) for $\parallel$ or $\perp$ polarization:

$$\frac{d^2 I_{\parallel}}{d\omega d\Omega} = \frac{q^2}{8\pi^2 c} |\Delta\boldsymbol{\beta}|^2 \frac{(\beta - \cos\theta)^2}{(1 - \beta \cos\theta)^4} \quad \text{polarization in } \mathbf{r} \text{ and } \boldsymbol{\beta} \text{ plane}$$

$$\frac{d^2 I_{\perp}}{d\omega d\Omega} = \frac{q^2}{8\pi^2 c} |\Delta\boldsymbol{\beta}|^2 \frac{1}{(1 - \beta \cos\theta)^2} \quad \text{polarization perpendicular to } \mathbf{r} \text{ and } \boldsymbol{\beta} \text{ plane}$$

Some details:



$$\boldsymbol{\beta} = \beta \hat{\mathbf{z}}$$

$$\hat{\mathbf{r}} = \sin \theta \hat{\mathbf{x}} + \cos \theta \hat{\mathbf{z}}$$

$$\boldsymbol{\varepsilon}_{\parallel} = -\cos \theta \hat{\mathbf{x}} + \sin \theta \hat{\mathbf{z}} \quad \boldsymbol{\varepsilon}_{\perp} = \hat{\mathbf{y}}$$

$$\Delta\boldsymbol{\beta} = \Delta\beta (\cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}})$$

Note: This is a wild assumption!

Your question: Why this assumption?

Comment: Presumably, the authors have found that contributions in the β direction are smaller.

Some details -- continued:

$$\hat{\mathbf{r}} = \sin \theta \hat{\mathbf{x}} + \cos \theta \hat{\mathbf{z}}$$

$$\boldsymbol{\varepsilon}_{\perp} = \hat{\mathbf{y}}$$

$$\boldsymbol{\varepsilon}_{\parallel} = -\cos \theta \hat{\mathbf{x}} + \sin \theta \hat{\mathbf{z}}$$

Consistent with
radiation from
charged
particles.

$$\boldsymbol{\beta} = \beta \hat{\mathbf{z}}$$

Convenient geometry

$$\Delta \boldsymbol{\beta} = \Delta \beta (\cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}}) \quad \text{Wild guess}$$

$$\Delta \boldsymbol{\beta} + \hat{\mathbf{r}} \times (\boldsymbol{\beta} \times \Delta \boldsymbol{\beta}) = \Delta \boldsymbol{\beta} (1 - \hat{\mathbf{r}} \cdot \boldsymbol{\beta}) + \boldsymbol{\beta} (\hat{\mathbf{r}} \cdot \Delta \boldsymbol{\beta})$$

$$\boldsymbol{\varepsilon}_{\perp} \cdot (\Delta \boldsymbol{\beta} + \hat{\mathbf{r}} \times (\boldsymbol{\beta} \times \Delta \boldsymbol{\beta})) = \Delta \beta \sin \phi (1 - \beta \cos \theta)$$

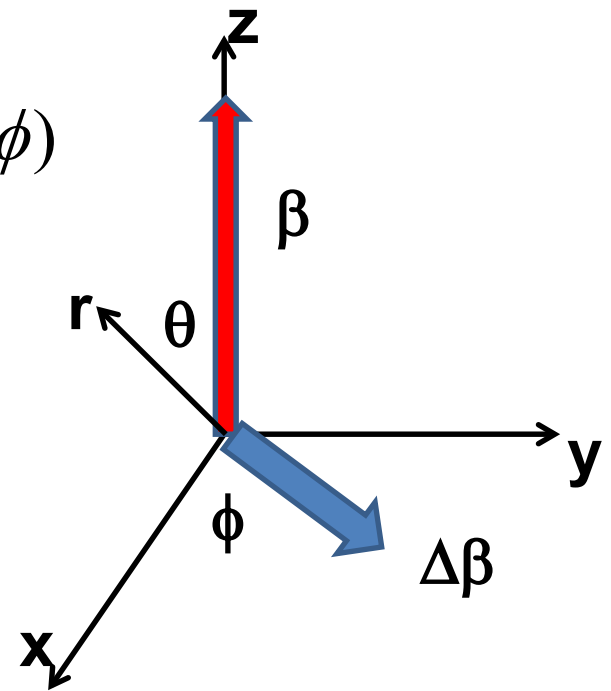
$$\boldsymbol{\varepsilon}_{\parallel} \cdot (\Delta \boldsymbol{\beta} + \hat{\mathbf{r}} \times (\boldsymbol{\beta} \times \Delta \boldsymbol{\beta})) = \Delta \beta \cos \phi (\beta - \cos \theta)$$

Radiation during collisions -- continued

Intensity expressions: (averaging over ϕ)

$$\frac{d^2 I_{\parallel}}{d\omega d\Omega} = \frac{q^2}{8\pi^2 c} |\Delta\boldsymbol{\beta}|^2 \frac{(\beta - \cos\theta)^2}{(1 - \beta \cos\theta)^4}$$

$$\frac{d^2 I_{\perp}}{d\omega d\Omega} = \frac{q^2}{8\pi^2 c} |\Delta\boldsymbol{\beta}|^2 \frac{1}{(1 - \beta \cos\theta)^2}$$



Relativistic collision at low ω and with small

$|\Delta\boldsymbol{\beta}|$ and $\Delta\boldsymbol{\beta}$ perpendicular to plane of $\hat{\mathbf{r}}$ and $\boldsymbol{\beta}$,

as a function of θ where $\hat{\mathbf{r}} \cdot \boldsymbol{\beta} = \beta \cos\theta$;

Integrating over solid angle:

$$\frac{dI}{d\omega} = \int d\Omega \left(\frac{d^2 I_{\parallel}}{d\omega d\Omega} + \frac{d^2 I_{\perp}}{d\omega d\Omega} \right) = \frac{2}{3\pi} \frac{q^2}{c} \gamma^2 |\Delta\boldsymbol{\beta}|^2$$

Some more details:

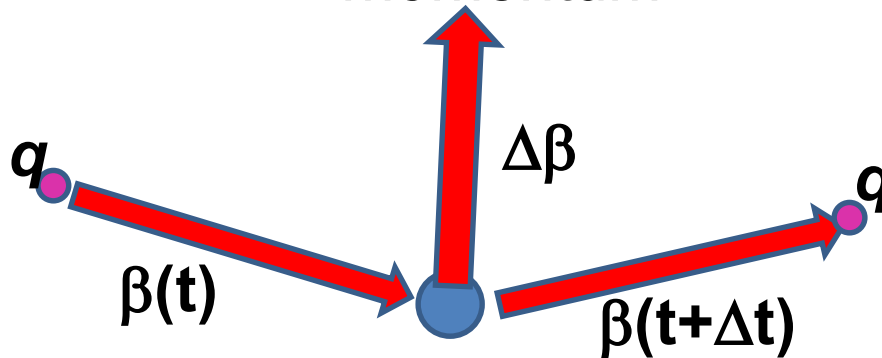
$$\begin{aligned}\int d\Omega \frac{d^2 I_{\parallel}}{d\omega d\Omega} &= \frac{q^2}{8\pi^2 c} |\Delta\boldsymbol{\beta}|^2 2\pi \int_{-1}^1 d\cos\theta \frac{(\beta - \cos\theta)^2}{(1 - \beta\cos\theta)^4} \\ &= \frac{q^2}{4\pi c} |\Delta\boldsymbol{\beta}|^2 \frac{2}{3} \frac{1}{(1 - \beta^2)}\end{aligned}$$

$$\begin{aligned}\int d\Omega \frac{d^2 I_{\perp}}{d\omega d\Omega} &= \frac{q^2}{8\pi^2 c} |\Delta\boldsymbol{\beta}|^2 \int_{-1}^1 d\cos\theta \frac{1}{(1 - \beta\cos\theta)^2} \\ &= \frac{q^2}{4\pi c} |\Delta\boldsymbol{\beta}|^2 \frac{2}{(1 - \beta^2)}\end{aligned}$$

$$\frac{dI}{d\omega} = \int d\Omega \left(\frac{d^2 I_{\parallel}}{d\omega d\Omega} + \frac{d^2 I_{\perp}}{d\omega d\Omega} \right) = \frac{2}{3\pi} \frac{q^2}{c} \gamma^2 |\Delta\boldsymbol{\beta}|^2$$

Estimation of $\Delta\beta$

Need to consider the mechanics of collision;
it is convenient to parameterize in terms of
momentum --



Momentum transfer:

$$Qc \equiv |\mathbf{p}(t + \tau) - \mathbf{p}(t)|c \approx \gamma M c^2 |\Delta\boldsymbol{\beta}|$$

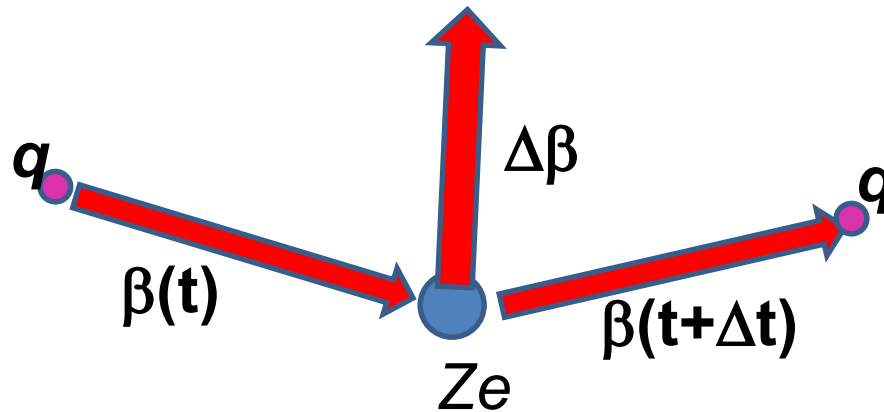
mass of particle
having charge q

$$\frac{dI}{d\omega} = \frac{2}{3\pi} \frac{q^2}{c} \gamma^2 |\Delta\boldsymbol{\beta}|^2 \approx \frac{2}{3\pi} \frac{q^2}{M^2 c^3} Q^2$$

What are the conditions for the validity of this result?

What are possible sources for the momentum transfer Q ?

Estimation of $\Delta\beta$ or Q -- for the case of Rutherford scattering

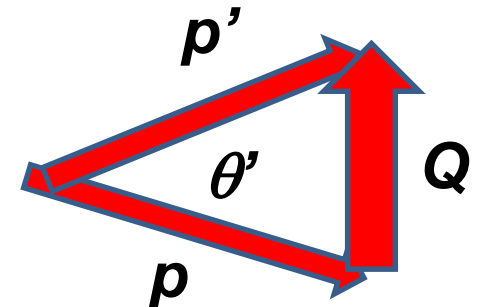


Assume that target nucleus (charge Ze) has mass $\gg M$;
Rutherford scattering cross-section in center of mass analysis:

$$\frac{d\sigma}{d\Omega} = \left(\frac{2Zeq}{pv} \right)^2 \frac{1}{(2\sin(\theta'/2))^4}$$

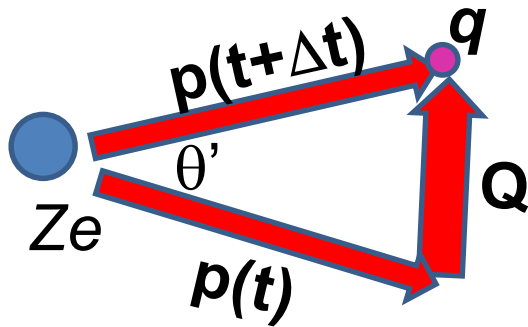
Assuming elastic scattering:

$$Q^2 = (2p\sin(\theta'/2))^2 = 2p^2(1 - \cos\theta')$$



Case of Rutherford scattering -- continued

Rutherford scattering cross-section:



$$\frac{d\sigma}{d\Omega} = \left(\frac{2Zeq}{pv} \right)^2 \frac{1}{\left(2 \sin(\theta'/2) \right)^4}$$

$$\frac{d\sigma}{dQ} = \int_{\varphi'} \frac{d\sigma}{d\Omega} \left| \frac{d\Omega}{dQ} \right| d\varphi'$$

$$d\Omega = d\varphi' d \cos \theta'$$

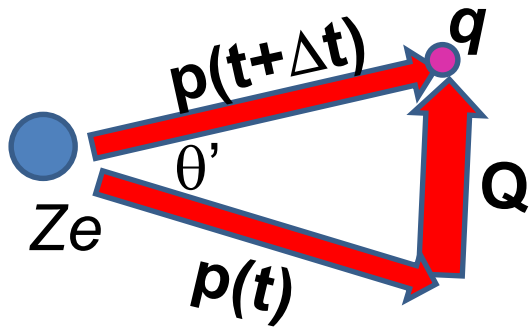
$$Q^2 = \left(2p \sin(\theta'/2) \right)^2 = 2p^2 (1 - \cos \theta')$$

$$dQ = -\frac{p^2}{Q} d \cos \theta'$$

$$\Rightarrow \frac{d\sigma}{dQ} = 8\pi \left(\frac{Zeq}{\beta c} \right)^2 \frac{1}{Q^3}$$

**Does the algebra
work out?**

Case of Rutherford scattering -- continued



Differential radiation cross section :

$$\begin{aligned}\frac{d^2\chi}{d\omega dQ} &= \frac{dI}{d\omega} \frac{d\sigma}{dQ} = \left(\frac{2}{3\pi} \frac{q^2}{M^2 c^3} Q^2 \right) \left(8\pi \left(\frac{Zeq}{\beta c} \right)^2 \frac{1}{Q^3} \right) \\ &= \frac{16}{3} \frac{(Ze)^2}{c} \left(\frac{q^2}{Mc^2} \right)^2 \frac{1}{\beta^2} \frac{1}{Q}\end{aligned}$$

Differential radiation cross section -- continued

Integrating over momentum transfer

$$\frac{d\chi}{d\omega} = \int_{Q_{\min}}^{Q_{\max}} dQ \frac{d^2\chi}{d\omega dQ} = \frac{16}{3} \frac{(Ze)^2}{c} \left(\frac{q^2}{Mc^2} \right)^2 \frac{1}{\beta^2} \ln \left(\frac{Q_{\max}}{Q_{\min}} \right)$$

How do the limits of Q occur?

Jackson suggests that these come from the limits of validity of the analysis.

1. Seems like cheating?
2. Perhaps fair?

Comment on frequency dependence --

Original expression for radiation intensity:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c} \left| \int dt e^{i\omega(t - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t)/c)} \frac{d}{dt} \left[\frac{\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \boldsymbol{\beta})}{1 - \hat{\mathbf{r}} \cdot \boldsymbol{\beta}} \right] \right|^2$$

In the previous derivations, we have assumed that

$$\omega(t - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t)/c) \ll 1.$$

$$\omega(t - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t)/c) = \omega \left(t - \hat{\mathbf{r}} \cdot \int_0^t dt' \boldsymbol{\beta}(t') \right) \approx \omega \tau (1 - \hat{\mathbf{r}} \cdot \langle \boldsymbol{\beta} \rangle)$$

In the non-relativistic case, this means $\omega \tau \ll 1$.

Here τ is the effective collision time.

How to estimate the collision time?

Jackson uses the following analysis in terms of the impact parameter b :

Using classical mechanics and assuming $v \ll c$:

$$\tau \approx \frac{b}{v} \ll \frac{1}{\omega} \quad \text{and} \quad Q \approx \frac{2Zeq}{bv}$$

$$\text{Assume that } Q_{\min} = \frac{2Zeq}{b_{\max} v} = \frac{2Zeq\omega}{v^2}$$

Differential radiation cross section -- continued

Radiation cross section in terms of momentum transfer

$$\frac{d\chi}{d\omega} = \int_{Q_{\min}}^{Q_{\max}} dQ \frac{d^2\chi}{d\omega dQ} = \frac{16}{3} \frac{(Ze)^2}{c} \left(\frac{q^2}{Mc^2} \right)^2 \frac{1}{\beta^2} \ln \left(\frac{Q_{\max}}{Q_{\min}} \right)$$

Note that: $Q^2 = 2p^2(1 - \cos\theta')$ $\Rightarrow Q_{\max} = 2p$

In general, $Q_{\min}$ is determined by the collision time

condition $\omega\tau < 1 \Rightarrow Q_{\min} \approx \frac{2Zeq\omega}{v^2}$

Radiation cross section for classical non - relativistic process

$$\frac{d\chi}{d\omega} = \frac{16}{3} \frac{(Ze)^2}{c} \left(\frac{q^2}{Mc^2} \right)^2 \frac{1}{\beta^2} \ln \left(\frac{\lambda M v^3}{Zeq\omega} \right)$$

$\lambda =$ “fudge factor”
of order unity

What could be the origin of the fudge factor?

What do you take away from this analysis

1. Disgust?
2. Admiration?
3. Motivation to avoid charged particles?