PHY 712 Electrodynamics
10-10:50 AM MWF Online

Class notes for Lecture 3:

Reading: Chapter 1 in JDJ

1. **Review of electrostatics with one-dimensional examples**
2. **Poisson and Laplace Equations**
3. **Green's Theorem and its use in electrostatics**

Your questions –

From Nick -- When we choose f and g to be Phi and G, where does that intuition come from in plugging into divergence theorem? On slide 16, how do we know which one of x and x' is the x_< term?

From Tim -- On page 10, for both the top and bottom electric potentials, you solved the Laplace Equation, correct? Because there is no charge density when x>a or x<-a.

From Gao -- We use the Green theorem to get the potential. In this formula, is phi (r prime) a known condition? If not, how can we use this theorem to get the potential as we have to know potential first to calculate the integral?

One-on-one meetings

Tues   9 PM – Gao

Office hours

Tues 9 AM ????

PHY 712 Electrodynamics

MWF 10-10:50 PM	**Online**	http://www.wfu.edu/~natalie/s21phy712/

Instructor: Natalie Holzwarth	**Office:**300 OPL	**e-mail:**natalie@wfu.edu

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Course schedule for Spring 2021

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Wed: 01/27/2021	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/29/2021
2	Fri: 01/29/2021	Chap. 1	Electrostatic energy calculations	#2	02/01/2021
3	Mon: 02/01/2021	Chap. 1 & 2	Electrostatic potentials and fields	#3	02/03/2021
4	Wed: 02/03/2021	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions		
5	Fri: 02/05/2021	Chap. 1 - 3	Brief introduction to numerical methods		
6	Mon: 02/08/2021	Chap. 2 & 3	Image charge constructions		

Februrary 1, 2021

PHY 712 – Problem Set #3

Continue reading Chaper 1 & 2 in **Jackson**

1. Consider a one-dimensional charge distribution of the form:

$$\rho(x) = \begin{cases} 0 & \text{for} \quad x < -a/2 \\ \rho_0 x/a & \text{for} \quad -a/2 \leq x \leq a/2 \\ 0 & \text{for} \quad x > a/2, \end{cases}$$

   where ρ_0 and a are constants.

   (a) Solve the Poisson equation for the electrostatic potential $\Phi(x)$ with the boundary conditions $\Phi(-a/2) = 0$ and $\frac{d\Phi}{dx}(-a/2) = 0$.

   (b) Find the corresponding electrostatic field $E(x)$.

   (c) Plot $\Phi(x)$ and $E(x)$.

   (d) Discuss your results in terms of elementary application of Gauss's Law arguments.

Poisson and Laplace Equations

We are concerned with finding solutions to the Poisson equation:

$$\nabla^2 \Phi_P(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}$$

and the Laplace equation:

$$\nabla^2 \Phi_L(\mathbf{r}) = 0$$

The Laplace equation is the "homogeneous" version of the Poisson equation. The Green's theorem allows us to determine the electrostatic potential from volume and surface integrals:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r' \rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') +$$

$$\frac{1}{4\pi}\int_S d^2r' \left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'.$$

Poisson equation -- continued

Note that we have previously shown that the differential and integral forms of Coulomb's law is given by:

$$\nabla^2\Phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0} \quad \text{and} \quad \Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Generalization of analysis for non-trivial boundary conditions:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r' \rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') +$$

$$\frac{1}{4\pi}\int_S d^2r' \left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'.$$

General comments on Green's theorem

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r' \rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') +$$

$$\frac{1}{4\pi}\int_S d^2r' \left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}') \right]\cdot \hat{\mathbf{r}}'.$$

This general form can be used in 1, 2, or 3 dimensions. In general, the Green's function must be constructed to satisfy the appropriate (Dirichlet or Neumann) boundary conditions. Alternatively or in addition, boundary conditions can be adjusted using the fact that for any solution to the Poisson equation, $\Phi_P(\mathbf{r})$ other solutions may be generated by use of solutions of the Laplace equation

$$\Phi(\mathbf{r}) = \Phi_P(\mathbf{r}) + C\Phi_L(\mathbf{r}), \text{for any constant } C.$$

Question -- We use the Green theorem to get the potential. In this formula, is phi (r prime) a known condition? If not, how can we use this theorem to get the potential as we have to know potential first to calculate the integral?

Comment – You are correct;   it is assumed that we know the potential or its derivative on the boundary.   As mentioned previously,   in general we cannot know both or in other ways over specify the problem.

"Derivation" of Green's Theorem

Poisson equation: $\nabla^2\Phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}$

Green's relation: $\nabla'^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$.

Divergence theorm: $\int_V d^3r\ \nabla\cdot\mathbf{A} = \oint_S d^2r\ \mathbf{A}\cdot\hat{\mathbf{r}}$

Let $\mathbf{A} = f(\mathbf{r})\nabla g(\mathbf{r}) - g(\mathbf{r})\nabla f(\mathbf{r})$

$$\int_V d^3r\ \nabla\cdot\left(f(\mathbf{r})\nabla g(\mathbf{r}) - g(\mathbf{r})\nabla f(\mathbf{r})\right) = \oint_S d^2r\ \left(f(\mathbf{r})\nabla g(\mathbf{r}) - g(\mathbf{r})\nabla f(\mathbf{r})\right)\cdot\hat{\mathbf{r}}$$

$$\int_V d^3r\ \left(f(\mathbf{r})\nabla^2 g(\mathbf{r}) - g(\mathbf{r})\nabla^2 f(\mathbf{r})\right)$$

Your question -- When we choose f and g to be Phi and G, where does that intuition come from in plugging into divergence theorem?

Comment – I assume this was the genius of Green (perhaps with the help of Jackson).

Another comment – Ideally the Green's function will be designed to take into account the boundary conditions for your given problem.   In some cases, further adjustments may be needed.

“Derivation” of Green’s Theorem

Poisson equation: $\nabla^2\Phi(\mathbf{r}) = -\dfrac{\rho(\mathbf{r})}{\varepsilon_0}$

Green's relation: $\nabla'^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$.

$$\int_V d^3r\, \left(f(\mathbf{r})\nabla^2 g(\mathbf{r}) - g(\mathbf{r})\nabla^2 f(\mathbf{r})\right) = \oint_S d^2r\, \left(f(\mathbf{r})\nabla g(\mathbf{r}) - g(\mathbf{r})\nabla f(\mathbf{r})\right)\cdot\hat{\mathbf{r}}$$

$$f(\mathbf{r}) \leftrightarrow \Phi(\mathbf{r}) \qquad\qquad g(\mathbf{r}) = G(\mathbf{r},\mathbf{r}')$$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r'\,\rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') +$$

$$\frac{1}{4\pi}\int_S d^2r'\left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'.$$

Example of charge density and potential varying in one dimension

Consider the following one dimensional charge distribution:

$$\rho(x) = \begin{cases} 0 & \text{for } x < -a \\ -\rho_0 & \text{for } -a < x < 0 \\ +\rho_0 & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases}$$

We want to find the electrostatic potential such that

$$\frac{d^2\Phi(x)}{dx^2} = -\frac{\rho(x)}{\varepsilon_0},$$

with the boundary condition $\Phi(-\infty) = 0$ and $\frac{d\Phi}{dx}(-\infty) = 0$

Electrostatic field solution

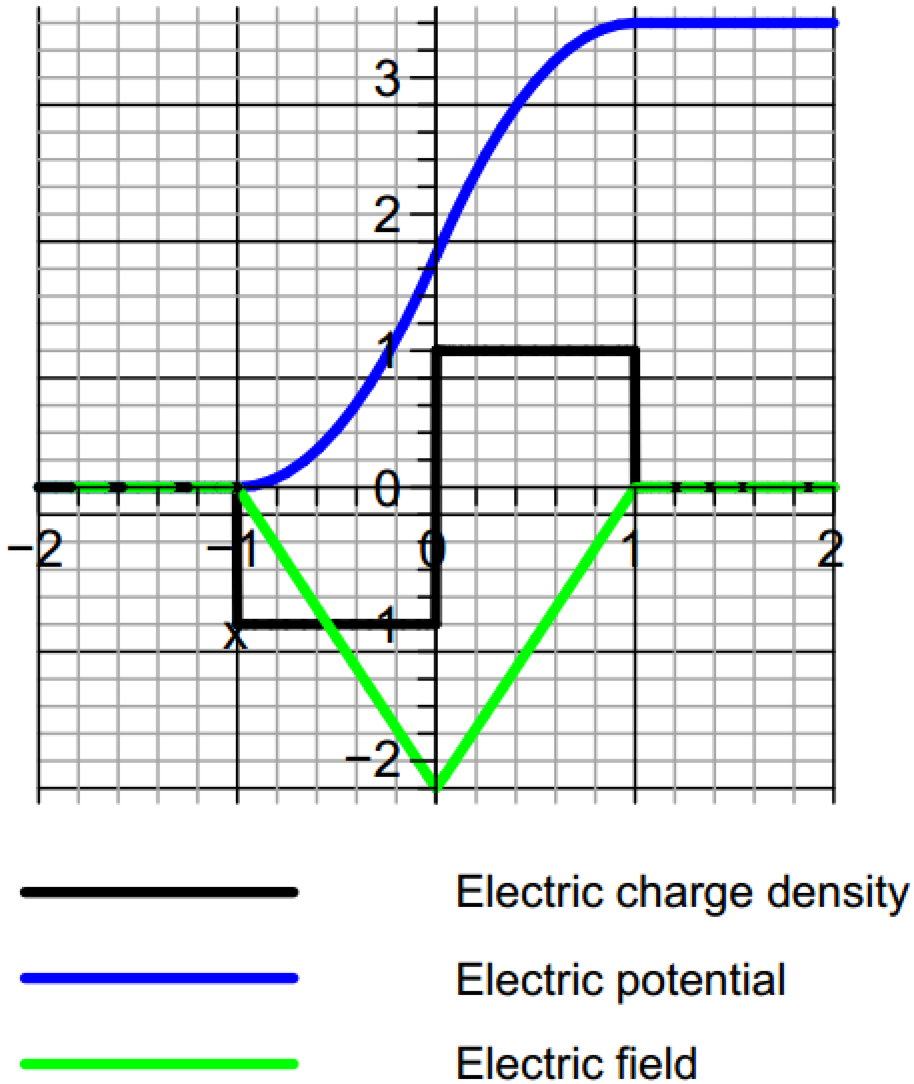
The solution to the Poisson equation is given by:

$$\Phi(x) = \begin{cases} 0 & \text{for } x < -a \\ \frac{\rho_0}{2\varepsilon_0}(x+a)^2 & \text{for } -a < x < 0 \\ -\frac{\rho_0}{2\varepsilon_0}(x-a)^2 + \frac{\rho_0 a^2}{\varepsilon_0} & \text{for } 0 < x < a \\ \frac{\rho_0}{\varepsilon_0} a^2 & \text{for } x > a \end{cases} .$$

Laplace
Poisson
Poisson
Laplace

The electrostatic field is given by:

$$E(x) = \begin{cases} 0 & \text{for } x < -a \\ -\frac{\rho_0}{\varepsilon_0}(x+a) & \text{for } -a < x < 0 \\ \frac{\rho_0}{\varepsilon_0}(x-a) & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases} .$$


3
2
1
0
−2
−1
0
1
2
x
−1
−2
Electric charge density
Electric potential
Electric field

Comment about the example and solution

This particular example is one that is used to model semiconductor junctions where the charge density is controlled by introducing charged impurities near the junction.

The solution of the Poisson equation for this case can be determined by piecewise solution within each of the four regions. Alternatively, from Green's theorem in one-dimension, one can use the Green's function

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} G(x,x')\rho(x')dx' \quad \text{where} \quad G(x,x') = 4\pi x_<$$

$x_<$ should be take as the smaller of x and x'.

Notes on the one-dimensional Green's function

The Green's function for the one-dimensional Poisson equation can be defined as a solution to the equation: $\nabla^2 G(x,x') = -4\pi\delta(x-x')$

Here the factor of 4π is not really necessary, but ensures consistency with your text's treatment of the 3-dimensional case. The meaning of this expression is that x' is held fixed while taking the derivative with respect to x.

Construction of a Green's function in one dimension

Consider two independent solutions to the homogeneous equation

$$\nabla^2 \phi_i(x) = 0$$

where $i = 1$ or 2. Let

$$G(x, x') = \frac{4\pi}{W} \phi_1(x_<)\phi_2(x_>).$$

This notation means that $x_<$ should be taken as the smaller of x and x' and $x_>$ should be taken as the larger.

W is defined as the "Wronskian":

$$W \equiv \frac{d\phi_1(x)}{dx}\phi_2(x) - \phi_1(x)\frac{d\phi_2(x)}{dx}.$$

Summary

$$\nabla^2 G(x,x') = -4\pi\delta(x-x')$$

$$G(x,x') = \frac{4\pi}{W}\phi_1(x_<)\phi_2(x_>)$$

$$W \equiv \frac{d\phi_1(x)}{dx}\phi_2(x) - \phi_1(x)\frac{d\phi_2(x)}{dx}$$

$$\left.\frac{dG(x,x')}{dx}\right|_{x=x'+\epsilon} - \left.\frac{dG(x,x')}{dx}\right|_{x=x'-\epsilon} = -4\pi$$

One dimensional Green's function in practice

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} G(x,x')\rho(x')dx'$$

$$= \frac{1}{4\pi\varepsilon_0}\left\{\int_{-\infty}^{x} G(x,x')\rho(x')dx' + \int_{x}^{\infty} G(x,x')\rho(x')dx'\right\}$$

For the one-dimensional Poisson equation, we can construct the Green's function by choosing $\phi_1(x) = x$ and $\phi_2(x) = 1; W = 1$:

$$\Phi(x) = \frac{1}{\varepsilon_0}\left\{\int_{-\infty}^{x} x'\rho(x')dx' + x\int_{x}^{\infty} \rho(x')dx'\right\}.$$

$$G(x,x') = 4\pi x_<$$

This expression gives the same result as previously obtained for the example $\rho(x)$ and more generally is appropriate for any neutral charge distribution.

Question -- On slide 16, how do we know which one of x and x' is the x_< term?

$$G(x,x') = 4\pi x_<$$

$$\Phi(x) = \frac{1}{\varepsilon_0}\left\{\int_{-\infty}^{x} x'\rho(x')dx' + x\int_{x}^{\infty} \rho(x')dx'\right\}.$$

$x'<x$ $\qquad\qquad$ $x'>x$

Orthogonal function expansions and Green's functions

Suppose we have a "complete" set of orthogonal functions $\{u_n(x)\}$ defined in the interval $x_1 \leq x \leq x_2$ such that

$$\int_{x_1}^{x_2} u_n(x) u_m(x)\, dx = \delta_{nm}.$$

We can show that the completeness of this functions implies that

$$\sum_{n=1}^{\infty} u_n(x) u_n(x') = \delta(x - x').$$

This relation allows us to use these functions to represent a Green's function for our system. For the 1-dimensional Poisson equation, the Green's function satisfies

$$\frac{\partial^2}{\partial x^2} G(x, x') = -4\pi\delta(x - x').$$

Orthogonal function expansions –continued

Therefore, if

$$\frac{d^2}{dx^2} u_n(x) = -\alpha_n u_n(x),$$

where $\{u_n(x)\}$ also satisfy the appropriate boundary conditions, then we can write t
Green's functions as

$$G(x, x') = 4\pi \sum_n \frac{u_n(x) u_n(x')}{\alpha_n}.$$

Example

For example, consider the example discussed earlier in the interval $-a \leq x \leq a$ with

$$\rho(x) = \begin{cases} 0 & \text{for } x < -a \\ -\rho_0 & \text{for } -a < x < 0 \\ +\rho_0 & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases} \tag{24}$$

We want to solve the Poisson equation with boundary condition $d\Phi(-a)/dx = 0$ and $d\Phi(a)/dx = 0$. For this purpose, we may choose

$$u_n(x) = \sqrt{\frac{1}{a}} \sin\left(\frac{[2n+1]\pi x}{2a}\right). \tag{25}$$

The Green's function for this case as:

$$G(x, x') = \frac{4\pi}{a} \sum_{n=0}^{\infty} \frac{\sin\left(\frac{[2n+1]\pi x}{2a}\right) \sin\left(\frac{[2n+1]\pi x'}{2a}\right)}{\left(\frac{[2n+1]\pi}{2a}\right)^2}. \tag{26}$$

Example – continued

$$\Phi(x) = \frac{\rho_0 a^2}{\epsilon_0}\left(16\sum_{n=0}^{\infty}\frac{\sin\left(\frac{[2n+1]\pi x}{2a}\right)}{([2n+1]\pi)^3} + \frac{1}{2}\right).$$

Constant shift to allow $\Phi(0)=0$.