# PHY 712 Electrodynamics
# 10-10:50 AM MWF Online

## Plan for Lecture 3:

**Reading: Chapter 1 in JDJ**

1. **Review of electrostatics with one-dimensional examples**
2. **Poisson and Laplace Equations**
3. **Green's Theorem and its use in electrostatics**



In this lecture, we will return to the materials presented in our textbook. Some of the ideas were presented in PHY 711.

# PHY 712 Electrodynamics

| MWF 10-10:50 PM | Online | http://www.wfu.edu/~natalie/s21phy712/ |
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## Course schedule for Spring 2021

(Preliminary schedule -- subject to frequent adjustment.)

| | Lecture date | JDJ Reading | Topic | HW | Due date |
|---|---|---|---|---|---|
| 1 | Wed: 01/27/2021 | Chap. 1 & Appen. | Introduction, units and Poisson equation | #1 | 01/29/2021 |
| 2 | Fri: 01/29/2021 | Chap. 1 | Electrostatic energy calculations | #2 | 02/01/2021 |
| 3 | Mon: 02/01/2021 | Chap. 1 & 2 | Electrostatic potentials and fields | #3 | 02/03/2021 |
| 4 | Wed: 02/03/2021 | Chap. 1 - 3 | Poisson's equation in 2 and 3 dimensions | | |
| 5 | Fri: 02/05/2021 | Chap. 1 - 3 | Brief introduction to numerical methods | | |
| 6 | Mon: 02/08/2021 | Chap. 2 & 3 | Image charge constructions | | |



Updated schedule. Note new homework assignment which follows from today's lecture.

Februrary 1, 2021

**PHY 712 – Problem Set #3**

Continue reading Chaper 1 & 2 in **Jackson**

1. Consider a one-dimensional charge distribution of the form:

$$\rho(x) = \begin{cases} 0 & \text{for} \quad x < -a/2 \\ \rho_0 x/a & \text{for} \quad -a/2 \leq x \leq a/2 \\ 0 & \text{for} \quad x > a/2, \end{cases}$$

where $\rho_0$ and $a$ are constants.

(a) Solve the Poisson equation for the electrostatic potential $\Phi(x)$ with the boundary conditions $\Phi(-a/2) = 0$ and $\frac{d\Phi}{dx}(-a/2) = 0$.

(b) Find the corresponding electrostatic field $E(x)$.

(c) Plot $\Phi(x)$ and $E(x)$.

(d) Discuss your results in terms of elementary application of Gauss's Law arguments.



Content of lHW 3.

## Poisson and Laplace Equations

We are concerned with finding solutions to the Poisson equation:

$$\nabla^2 \Phi_P(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}$$

and the Laplace equation:

$$\nabla^2 \Phi_L(\mathbf{r}) = 0$$

The Laplace equation is the "homogeneous" version of the Poisson equation. The Green's theorem allows us to determine the electrostatic potential from volume and surface integrals:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r' \, \rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') +$$

$$\frac{1}{4\pi}\int_S d^2r' \left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\right]\cdot \hat{\mathbf{r}}'.$$



Here we start our systematic derivations of solution of the electrostatic equation for a potential with a given charge source and the associated homogeneous equation.

**Poisson equation -- continued**

Note that we have previously shown that the differential and integral forms of Coulomb's law is given by:

$$\nabla^2\Phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0} \quad \text{and} \quad \Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Generalization of analysis for non-trivial boundary conditions:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r' \rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') +$$

$$\frac{1}{4\pi}\int_S d^2r' \left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'.$$



What we discussed last week is still true for isolated charges. Now we consider the case where the charges are within a volume V whose surface may have some imposed restrictions (boundary condisions).

General comments on Green's theorem

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r' \rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') +$$

$$\frac{1}{4\pi}\int_S d^2r' \left[ G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}') \right] \cdot \hat{\mathbf{r}}'.$$

This general form can be used in 1, 2, or 3 dimensions. In general, the Green's function must be constructed to satisfy the appropriate (Dirichlet or Neumann) boundary conditions. Alternatively or in addition, boundary conditions can be adjusted using the fact that for any solution to the Poisson equation, $\Phi_P(\mathbf{r})$ other solutions may be generated by use of solutions of the Laplace equation

$$\Phi(\mathbf{r}) = \Phi_P(\mathbf{r}) + C\Phi_L(\mathbf{r}), \text{for any constant } C.$$



Comment about how the boundary conditions may or may not work.

## "Derivation" of Green's Theorem

Poisson equation: $\nabla^2\Phi(\mathbf{r}) = -\dfrac{\rho(\mathbf{r})}{\varepsilon_0}$

Green's relation: $\nabla'^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}').$

Divergence theorm: $\int_V d^3r\ \nabla\cdot\mathbf{A} = \oint_S d^2r\ \mathbf{A}\cdot\hat{\mathbf{r}}$

Let $\quad \mathbf{A} = f(\mathbf{r})\nabla g(\mathbf{r}) - g(\mathbf{r})\nabla f(\mathbf{r})$

$$\int_V d^3r\ \nabla\cdot\big(f(\mathbf{r})\nabla g(\mathbf{r}) - g(\mathbf{r})\nabla f(\mathbf{r})\big) = \oint_S d^2r\ \big(f(\mathbf{r})\nabla g(\mathbf{r}) - g(\mathbf{r})\nabla f(\mathbf{r})\big)\cdot\hat{\mathbf{r}}$$

$$\int_V d^3r\ \big(f(\mathbf{r})\nabla^2 g(\mathbf{r}) - g(\mathbf{r})\nabla^2 f(\mathbf{r})\big)$$



Here we derive the equations stated on the previous slides.

"Derivation" of Green's Theorem

Poisson equation: $\nabla^2\Phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}$

Green's relation: $\nabla'^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$.

$$\int_V d^3r \left( f(\mathbf{r})\nabla^2 g(\mathbf{r}) - g(\mathbf{r})\nabla^2 f(\mathbf{r})\right) = \oint_S d^2r \left( f(\mathbf{r})\nabla g(\mathbf{r}) - g(\mathbf{r})\nabla f(\mathbf{r})\right)\cdot\hat{\mathbf{r}}$$

$$f(\mathbf{r}) \leftrightarrow \Phi(\mathbf{r}) \qquad\qquad g(\mathbf{r}) = G(\mathbf{r},\mathbf{r}')$$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r'\,\rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') +$$

$$\frac{1}{4\pi}\int_S d^2r'\left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'.$$



Derivation continued.

## Example of charge density and potential varying in one dimension

Consider the following one dimensional charge distribution:

$$\rho(x) = \begin{cases} 0 & \text{for } x < -a \\ -\rho_0 & \text{for } -a < x < 0 \\ +\rho_0 & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases}$$

We want to find the electrostatic potential such that

$$\frac{d^2\Phi(x)}{dx^2} = -\frac{\rho(x)}{\varepsilon_0},$$

with the boundary condition $\Phi(-\infty) = 0$ and $\frac{d\Phi}{dx}(-\infty) = 0$



Simple one-dimensional example of a particular charge distribution.

## Electrostatic field solution

The solution to the Poisson equation is given by:

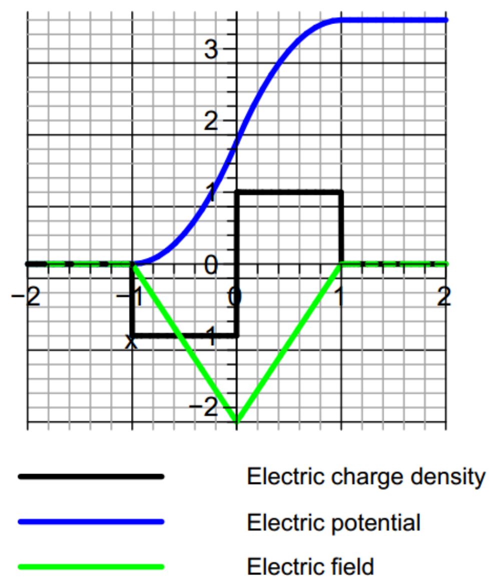
$$\Phi(x) = \begin{cases} 0 & \text{for } x < -a \\ \frac{\rho_0}{2\varepsilon_0}(x+a)^2 & \text{for } -a < x < 0 \\ -\frac{\rho_0}{2\varepsilon_0}(x-a)^2 + \frac{\rho_0 a^2}{\varepsilon_0} & \text{for } 0 < x < a \\ \frac{\rho_0}{\varepsilon_0}a^2 & \text{for } x > a \end{cases} .$$

The electrostatic field is given by:

$$E(x) = \begin{cases} 0 & \text{for } x < -a \\ -\frac{\rho_0}{\varepsilon_0}(x+a) & \text{for } -a < x < 0 \\ \frac{\rho_0}{\varepsilon_0}(x-a) & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases} .$$



Resultant potential and electric field.






Graph of results for this example

## Comment about the example and solution

This particular example is one that is used to model semiconductor junctions where the charge density is controlled by introducing charged impurities near the junction.

The solution of the Poisson equation for this case can be determined by piecewise solution within each of the four regions. Alternatively, from Green's theorem in one-dimension, one can use the Green's function

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} G(x,x')\rho(x')dx' \quad \text{where} \quad G(x,x') = 4\pi x_<$$

$x_<$ should be take as the smaller of $x$ and $x'$.



Comment and generalization.

## Notes on the one-dimensional Green's function

The Green's function for the one-dimensional Poisson equation can be defined as a solution to the equation: $\nabla^2 G(x,x') = -4\pi\delta(x-x')$

Here the factor of $4\pi$ is not really necessary, but ensures consistency with your text's treatment of the 3-dimensional case. The meaning of this expression is that $x'$ is held fixed while taking the derivative with respect to $x$.



Some details.

## Construction of a Green's function in one dimension

Consider two independent solutions to the homogeneous equation

$$\nabla^2 \phi_i(x) = 0$$

where $i = 1$ or 2. Let

$$G(x, x') = \frac{4\pi}{W} \phi_1(x_<) \phi_2(x_>).$$

This notation means that $x_<$ should be taken as the smaller of $x$ and $x'$ and $x_>$ should be taken as the larger.

$W$ is defined as the "Wronskian":

$$W \equiv \frac{d\phi_1(x)}{dx} \phi_2(x) - \phi_1(x) \frac{d\phi_2(x)}{dx}.$$



Details continued for one dimensional Poisson equation.

## Summary

$$\nabla^2 G(x,x') = -4\pi\delta(x-x')$$

$$G(x,x') = \frac{4\pi}{W}\phi_1(x_<)\phi_2(x_>)$$

$$W \equiv \frac{d\phi_1(x)}{dx}\phi_2(x) - \phi_1(x)\frac{d\phi_2(x)}{dx}$$

$$\left.\frac{dG(x,x')}{dx}\right|_{x=x'+\epsilon} - \left.\frac{dG(x,x')}{dx}\right|_{x=x'-\epsilon} = -4\pi$$



Summary for one dimensional Poisson equation.

## One dimensional Green's function in practice

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} G(x,x')\rho(x')dx'$$

$$= \frac{1}{4\pi\varepsilon_0}\left\{\int_{-\infty}^{x} G(x,x')\rho(x')dx' + \int_{x}^{\infty} G(x,x')\rho(x')dx'\right\}$$

For the one-dimensional Poisson equation, we can construct the Green's function by choosing $\phi_1(x) = x$ and $\phi_2(x) = 1; W = 1$:

$$\Phi(x) = \frac{1}{\varepsilon_0}\left\{\int_{-\infty}^{x} x'\rho(x')dx' + x\int_{x}^{\infty} \rho(x')dx'\right\}.$$

$$G(x,x') = 4\pi x_<$$

This expression gives the same result as previously obtained for the example $\rho(x)$ and more generally is appropriate for any neutral charge distribution.



Some general comments.

## Orthogonal function expansions and Green's functions

Suppose we have a "complete" set of orthogonal functions $\{u_n(x)\}$ defined in the interval $x_1 \leq x \leq x_2$ such that

$$\int_{x_1}^{x_2} u_n(x) u_m(x)\, dx = \delta_{nm}.$$

We can show that the completeness of this functions implies that

$$\sum_{n=1}^{\infty} u_n(x) u_n(x') = \delta(x - x').$$

This relation allows us to use these functions to represent a Green's function for our system. For the 1-dimensional Poisson equation, the Green's function satisfies

$$\frac{\partial^2}{\partial x^2} G(x, x') = -4\pi\delta(x - x').$$



Now we will discuss another approach to analyzing Green's functions based on expansion in terms of a complete set of orthogonal functions.

## Orthogonal function expansions –continued

Therefore, if

$$\frac{d^2}{dx^2}u_n(x) = -\alpha_n u_n(x),$$

where $\{u_n(x)\}$ also satisfy the appropriate boundary conditions, then we can write t
Green's functions as

$$G(x, x') = 4\pi \sum_n \frac{u_n(x)u_n(x')}{\alpha_n}.$$



Some details for orthogonal function expansion method.

**Example**

For example, consider the example discussed earlier in the interval $-a \leq x \leq a$ with

$$\rho(x) = \begin{cases} 0 & \text{for } x < -a \\ -\rho_0 & \text{for } -a < x < 0 \\ +\rho_0 & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases} \tag{24}$$

We want to solve the Poisson equation with boundary condition $d\Phi(-a)/dx = 0$ and $d\Phi(a)/dx = 0$. For this purpose, we may choose

$$u_n(x) = \sqrt{\frac{1}{a}} \sin\left(\frac{[2n+1]\pi x}{2a}\right). \tag{25}$$

The Green's function for this case as:

$$G(x, x') = \frac{4\pi}{a} \sum_{n=0}^{\infty} \frac{\sin\left(\frac{[2n+1]\pi x}{2a}\right) \sin\left(\frac{[2n+1]\pi x'}{2a}\right)}{\left(\frac{[2n+1]\pi}{2a}\right)^2}. \tag{26}$$

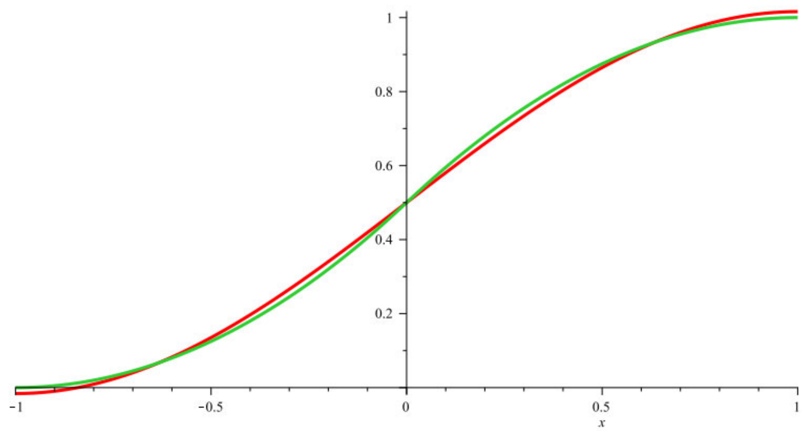


Application to our example.

**Example – continued**

Constant shift to allow $\Phi(0) = 0$.

$$\Phi(x) = \frac{\rho_0 a^2}{\epsilon_0}\left(16\sum_{n=0}^{\infty}\frac{\sin\left(\frac{[2n+1]\pi x}{2a}\right)}{([2n+1]\pi)^3} + \frac{1}{2}\right).$$




2/01/2021 PHY 712 Spring 2021 -- Lecture 3 20

Graph of potential (green) and expansion for a few terms. Note that it was necessary to shift the potential by a constant.