PHY 712 Electrodynamics
10-10:50 AM MWF Online
Plan for Lecture 4:

Reading: Chapter 1 - 3 in JDJ

Electrostatic potentials

1. **One, two, and three dimensions (Cartesian coordinates)**
2. **Mean value theorem for the electrostatic potential**

In this lecture, we will continue to develop solution methods for solving electrostatic problems.

Online Colloquium: "ALIX in Wonderland: Multivalency, Phosphorylation-mediated Amyloids, Autoinhibition, and Endosomal Membrane Interactions" — February 4, 2021 at 4 PM

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Thursday, February 4, 2021 4:00 PM EST
Via Video Conference (contact wfuphys@wfu.edu for link information)

Tomorrow's colloquium

PHY 712 Electrodynamics

MWF 10-10:50 PM	Online	http://www.wfu.edu/~natalie/s21phy712/

Instructor: Natalie Holzwarth	Office:300 OPL	e-mail:natalie@wfu.edu

Course schedule for Spring 2021

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Wed: 01/27/2021	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/29/2021
2	Fri: 01/29/2021	Chap. 1	Electrostatic energy calculations	#2	02/01/2021
3	Mon: 02/01/2021	Chap. 1 & 2	Electrostatic potentials and fields	#3	02/03/2021
4	Wed: 02/03/2021	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#4	02/05/2021
5	Fri: 02/05/2021	Chap. 1 - 3	Brief introduction to numerical methods		
6	Mon: 02/08/2021	Chap. 2 & 3	Image charge constructions		

Updated schedule

Poisson Equation

$$\nabla^2 \Phi_P(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}$$

Solution to Poisson equation using Green's function $G(\mathbf{r},\mathbf{r}')$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r' \rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') +$$

$$\frac{1}{4\pi}\int_S d^2r' \left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'.$$

Review of the general methods for solving the Poisson equation in various dimensions and geometries.

Poisson equation for one-dimensional system

$$\frac{d^2\Phi_P(x)}{dx^2} = -\frac{\rho(x)}{\varepsilon_0}$$

Example solution:

$$\Phi_P(x) = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} G(x,x')\rho(x')dx' + C_1 + C_2 x$$

where $G(x,x') = 4\pi x_<$ where $x_<$ is the smaller of x and x';
C_1 and C_2 are constants.

Check:

$$\Phi_P(x) = \frac{1}{\varepsilon_0}\left\{\int_{-\infty}^{x} x'\rho(x')dx' + x\int_{x}^{\infty}\rho(x')dx'\right\} + C_1 + C_2 x$$

$$\frac{d\Phi_P(x)}{dx} = \frac{1}{\varepsilon_0}\int_{x}^{\infty}\rho(x')dx' + C_2 \quad \Rightarrow \frac{d^2\Phi_P(x)}{dx^2} = -\frac{\rho(x)}{\varepsilon_0}$$

For one Cartesian dimension – reviewing previously discussed results.

General procedure for constructing Green's function for one-dimensional system using 2 independent solutions of the homogeneous equations

Consider two independent solutions to the homogeneous equation

$$\nabla^2 \phi_i(x) = 0$$

where $i = 1$ or 2. Let

$$G(x,x') = \frac{4\pi}{W}\phi_1(x_<)\phi_2(x_>).$$

This notation means that $x_<$ should be taken as the smaller of x and x' and $x_>$ should be taken as the larger.

"Wronskian": $W \equiv \frac{d\phi_1(x)}{dx}\phi_2(x) - \phi_1(x)\frac{d\phi_2(x)}{dx}.$

Beautiful method; but only works in one dimension.

Some details

Orthogonal function expansions and Green's functions

Suppose we have a "complete" set of orthogonal functions $\{u_n(x)\}$ defined in the interval $x_1 \leq x \leq x_2$ such that

$$\int_{x_1}^{x_2} u_n(x) u_m(x)\, dx = \delta_{nm}.$$

We can show that the completeness of this functions implies that

$$\sum_{n=1}^{\infty} u_n(x) u_n(x') = \delta(x - x').$$

This relation allows us to use these functions to represent a Green's function for our system. For the 1-dimensional Poisson equation, the Green's function satisfies

$$\frac{\partial^2}{\partial x^2} G(x, x') = -4\pi\delta(x - x').$$

Review of orthogonal function expansions.

Orthogonal function expansion -- continued

Suppose the orthogonal functions satisfy an eigenvalue equation:

$$\frac{d^2}{dx^2}u_n(x) = -\alpha_n u_n(x)$$

where the functions $u_n(x)$ also satisfy the appropriate boundary conditions, then we can construct the Green's function:

$$G(x,x') = 4\pi \sum_n \frac{u_n(x)u_n(x')}{\alpha_n}.$$

Check:

$$\frac{d^2}{dx^2}G(x,x') = 4\pi \sum_n \frac{\left(-\alpha_n u_n(x)\right)u_n(x')}{\alpha_n} = -4\pi \sum_n u_n(x)u_n(x')$$

$$= -4\pi\delta(x-x')$$

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Construction of Green's function for one dimensional case.

Example

For example, consider the previous example in the interval

$-a \le x \le a$:

$$\rho(x) = \begin{cases} 0 & \text{for } x < -a \\ -\rho_0 & \text{for } -a < x < 0 \\ +\rho_0 & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases}$$

We want to solve the Poisson equation with boundary condition $d\Phi(-a)/dx = 0$ and $d\Phi(a)/dx = 0$. We may choose

$u_n(x) = \sqrt{\frac{1}{a}} \sin\left(\frac{[2n+1]\pi x}{2a}\right)$ and the corresponding Green's function

$$G(x,x') = \frac{4\pi}{a} \sum_{n=0}^{\infty} \frac{\sin\left(\frac{[2n+1]\pi x}{2a}\right) \sin\left(\frac{[2n+1]\pi x'}{2a}\right)}{\left(\frac{[2n+1]\pi}{2a}\right)^2}.$$

Reviewing our “favorite” example.

Example -- continued

This form of the one-dimensional Green's function only allows us to find a solution to the Poisson equation within the interval $-a \le x \le a$ from

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\int_{-a}^{a} dx'\, G(x,x')\rho(x') + C_1$$

$$\Rightarrow \Phi(x) = \frac{\rho_0 a^2}{\epsilon_0}\left(16\sum_{n=0}^{\infty}\frac{\sin\left(\frac{[2n+1]\pi x}{2a}\right)}{([2n+1]\pi)^3} + \frac{1}{2}\right),$$

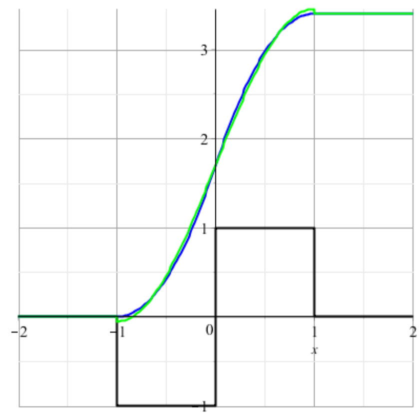
choosing C_1 so that $\Phi(-a) = 0$.

Exact result: $$\Phi(x) = \begin{cases} 0 & \text{for } x < -a \\ \frac{\rho_0}{2\varepsilon_0}(x+a)^2 & \text{for } -a < x < 0 \\ -\frac{\rho_0}{2\varepsilon_0}(x-a)^2 + \frac{\rho_0 a^2}{\varepsilon_0} & \text{for } 0 < x < a \\ \frac{\rho_0}{\varepsilon_0}a^2 & \text{for } x > a \end{cases}$$

Some details.

Example -- continued

$$\Phi(x)=\frac{\rho_0 a^2}{\epsilon_0}\left(16\sum_{n=0}^{\infty}\frac{\sin\left(\frac{[2n+1]\pi x}{2a}\right)}{([2n+1]\pi)^3}+\frac{1}{2}\right)$$



Summary.

Orthogonal function expansions in 2 and 3 dimensions

$$\nabla^2\Phi(\mathbf{r}) \equiv \frac{\partial^2\Phi(\mathbf{r})}{\partial x^2} + \frac{\partial^2\Phi(\mathbf{r})}{\partial y^2} + \frac{\partial^2\Phi(\mathbf{r})}{\partial z^2} = -\rho(\mathbf{r})/\epsilon_0.$$

Let $\{u_n(x)\}$, $\{v_n(y)\}$, $\{w_n(z)\}$ denote complete orthogonal function sets in the x, y, and z dimensions, respectively. The Green's function construction becomes:

$$G(x,x',y,y',z,z') = 4\pi\sum_{lmn}\frac{u_l(x)u_l(x')v_m(y)v_m(y')w_n(z)w_n(z')}{\alpha_l+\beta_m+\gamma_n},$$

where

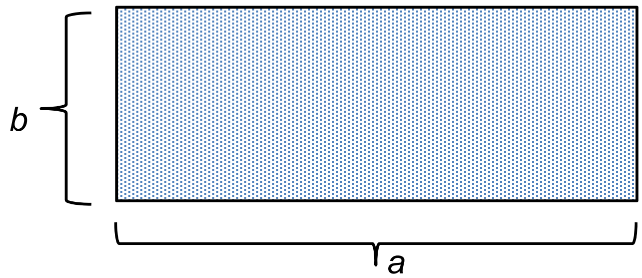
$$\frac{d^2}{dx^2}u_l(x) = -\alpha_l u_l(x), \quad \frac{d^2}{dy^2}v_m(y) = -\beta_m v_m(y), \text{ and } \frac{d^2}{dz^2}w_n(z) = -\gamma_n w_n(z).$$

(See Eq. 3.167 in Jackson for example.)

Extension of the ideas to multiple dimensions.

Details of a two-dimensional example --

Example:



Two dimensional box with sides a and b with boundary conditions: $\Phi(0,y)=\Phi(a,y)=\Phi(x,0)=\Phi(x,b)=0$

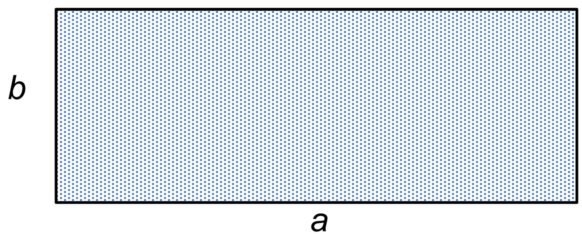
$$\nabla^2\Phi(\mathbf{r}) \equiv \frac{\partial^2\Phi(\mathbf{r})}{\partial x^2} + \frac{\partial^2\Phi(\mathbf{r})}{\partial y^2} = -\rho(\mathbf{r})/\epsilon_0.$$

$$G(x,x',y,y') = 4\pi\sum_{lm}\frac{u_l(x)u_l(x')v_m(y)v_m(y')}{\alpha_l+\beta_m},$$

$$\text{where } \frac{d^2}{dx^2}u_l(x) = -\alpha_l u_l(x), \; \frac{d^2}{dy^2}v_m(x) = -\beta_m v_m(y)$$

Analyzing in detail the two dimensional case.

Two dimensional example continued --



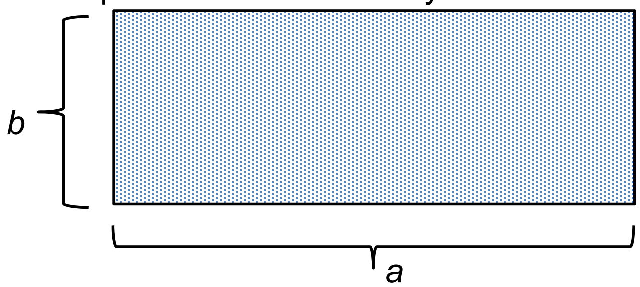
$$u_l(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{l\pi x}{a}\right) \qquad v_m(y) = \sqrt{\frac{2}{b}} \sin\left(\frac{m\pi y}{b}\right) \quad \text{with} \quad \alpha_n = \left(\frac{l\pi}{a}\right)^2 \quad \beta_n = \left(\frac{m\pi}{b}\right)^2$$

$$G(x,x',y,y') = 4\pi \sum_{lm} \frac{u_l(x)u_l(x')v_m(y)v_m(y')}{\alpha_l + \beta_m}$$

$$= \frac{16}{\pi ab} \sum_{lm} \frac{\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)}{\left(\frac{l}{a}\right)^2 + \left(\frac{m}{b}\right)^2}$$

Two dimensional case, using orthogonal functions in both x and y dimensions.

Example two-dimensional system continued -- Two dimensional box with sides a and b with boundary conditions:
$\Phi(0,y)=\Phi(a,y)=$
$\Phi(x,0)=\Phi(x,b)=0$



$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r'\,\rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') +$$

Don't know this term

Know this term=0

$$\frac{1}{4\pi}\int_S d^2r'\left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla'G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'.$$

➔By design $G(\mathbf{r},\mathbf{r}')$ vanishes on boundary.

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Specific example.

Example #1: $\rho(x,y) = \rho_0 \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)$

Example #2: $\rho(x,y) = \rho_0$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r'\,\rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}')$$

For this case:

$$G(x,x',y,y') = \frac{16}{\pi ab}\sum_{lm}\frac{\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)}{\left(\frac{l}{a}\right)^2+\left(\frac{m}{b}\right)^2}$$

For example #1: $\Phi(x,y) = \frac{\rho_0 a^2 b^2}{\epsilon_0 \pi^2 (a^2+b^2)}\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)$

Two examples.

Combined orthogonal function expansion and homogeneous solution construction of Green's function in 2 and 3 dimensions.

An alternative method of finding Green's functions for a second order ordinary differential equations (in 1 dimension) is based on a product of two independent solutions of the homogeneous equation, $\phi_1(x)$ and $\phi_2(x)$:

$$G(x,x') = K\phi_1(x_<)\phi_2(x_>), \text{ where } K \equiv \frac{4\pi}{\dfrac{d\phi_1}{dx}\phi_2 - \phi_1\dfrac{d\phi_2}{dx}},$$

where $x_<$ denotes the smaller of x and x'.

For the two and three dimensional cases, we can use this technique in one of the dimensions in order to reduce the number of summation terms. These ideas are discussed in Section 3.11 of Jackson.

Now consider using orthogonal function expansion in the x dimension and the homogeneous solution construction in the y dimension.

Green’s function construction -- continued

For the two dimensional case, for example, we can assume that the Green's function can be written in the form:

$$G(x,x',y,y') = \sum_n u_n(x)u_n(x')g_n(y,y') \text{ where } \frac{d^2}{dx^2}u_n(x) = -\alpha_n u_n(x)$$

The y dependence of this equation will have the required behavior, if we choose: $\left[-\alpha_n + \frac{\partial^2}{\partial y^2}\right]g_n(y,y') = -4\pi\delta(y-y'),$

which in turn can be expressed in terms of the two independent solutions $v_{n_1}(y)$ and $v_{n_2}(y)$ of the homogeneous equation:

$$\left[\frac{d^2}{dy^2} - \alpha_n\right]v_{n_i}(y) = 0,$$

and the Wronskian constant: $K_n \equiv \frac{dv_{n_1}}{dy}v_{n_2} - v_{n_1}\frac{dv_{n_2}}{dy}$

Some details.

$$\left[-\alpha_n + \frac{\partial^2}{\partial y^2}\right] g_n(y,y') = -4\pi\delta(y-y'),$$

$$g_n(y,y') = \frac{4\pi}{K_n} v_{n_1}(y_<) v_{n_2}(y_>)$$

where: $\left[\frac{d^2}{dy^2} - \alpha_n\right] v_{n_i}(y) = 0,$

and $K_n \equiv \frac{dv_{n_1}}{dy} v_{n_2} - v_{n_1} \frac{dv_{n_2}}{dy}$

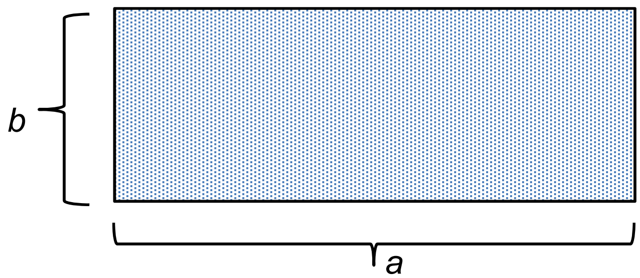
For example, choose $v_{n_1}(y) = \sinh(\sqrt{\alpha_n}\, y)$ and $v_{n_2}(y) = \sinh(\sqrt{\alpha_n}\,(b-y))$

where $K_n = \sqrt{\alpha_n} \sinh(\sqrt{\alpha_n}\, b)$

using the identity: $\cosh(r)\sinh(s) + \sinh(r)\cosh(s) = \sinh(r+s)$

More details.

Example:



Two dimensional box with sides a and b with boundary conditions: $\Phi(0,y)=\Phi(a,y)=\Phi(x,0)=\Phi(x,b)=0$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r'\,\rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') +$$

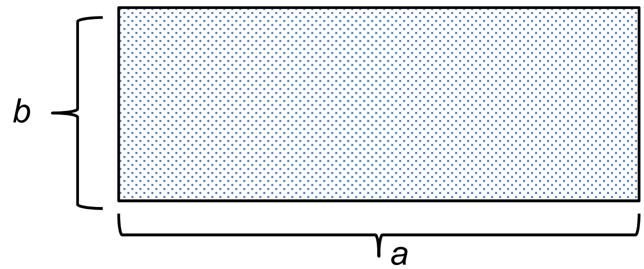
Don't know this term — Know this term

$$\frac{1}{4\pi}\int_S d^2r'\left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla'G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'.$$

$$G(x,x',y,y') = \sum_n u_n(x)u_n(x')\frac{4\pi}{K_n}v_{n_1}(y_<)v_{n_2}(y_>).$$

More details.

Example:



Two dimensional box with sides a and b with boundary conditions: $\Phi(0,y)=\Phi(a,y)=\Phi(x,0)=\Phi(x,b)=0$

For this type of problem, it is necessary to construct $G(x,x',y,y')$ so that it vanishes on the boundary:

$$G(x,x',y,0)=G(x,x',y,b)=G(x,0,y,y')=G(x,a,y,y')=0$$

Checking boundary values.

$$G(x,x',y,y') = \sum_n u_n(x) u_n(x') \frac{4\pi}{K_n} v_{n_1}(y_<) v_{n_2}(y_>).$$

$$\frac{d^2}{dx^2} u_n(x) = -\alpha_n u_n(x) \quad \text{where} \quad u_n(0) = u_n(a) = 0$$

$$\Rightarrow u_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \qquad \alpha_n = \left(\frac{n\pi}{a}\right)^2$$

$$\left[\frac{d^2}{dy^2} - \left(\frac{n\pi}{a}\right)^2\right] v_{n_i}(y) = 0$$

$$v_{n_1}(y) = \sinh\left(\frac{n\pi}{a} y\right) \qquad v_{n_2}(y) = \sinh\left(\frac{n\pi}{a}(b-y)\right)$$

$$K_n = \frac{n\pi}{a} \sinh\left(\frac{n\pi b}{a}\right)$$

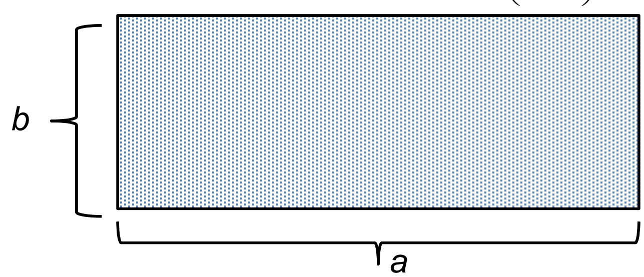
More details.

Green’s function construction -- continued

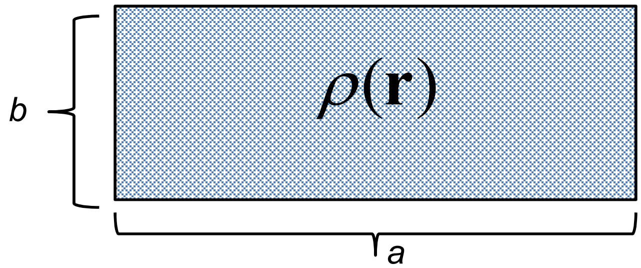
$$G(x,x',y,y') = \sum_n u_n(x)u_n(x')K_n v_{n_1}(y_<)v_{n_2}(y_>).$$

For example, a Green's function for a two-dimensional rectangular system with $0 \le x \le a$ and $0 \le y \le b$, which vanishes on the rectangular boundaries:

$$G(x,x',y,y') = 8\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi x'}{a}\right)\sinh\left(\frac{n\pi y_<}{a}\right)\sinh\left(\frac{n\pi}{a}(b-y_>)\right)}{n\sinh\left(\frac{n\pi b}{a}\right)}.$$



Resultant effective Green’s function for this case.



$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r'\,\rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') +$$

=0

$$\frac{1}{4\pi}\int_S d^2r'\left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'.$$

$$G(x,x',y,y') = 8\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi x'}{a}\right)\sinh\left(\frac{n\pi y_<}{a}\right)\sinh\left(\frac{n\pi}{a}(b-y_>)\right)}{n\sinh\left(\frac{n\pi b}{a}\right)}.$$

Checking boundary values.

$$G(x,x',y,y') = 8\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi x'}{a}\right)\sinh\left(\frac{n\pi y_<}{a}\right)\sinh\left(\frac{n\pi}{a}(b-y_>)\right)}{n\sinh\left(\frac{n\pi b}{a}\right)}.$$

Example: $\rho(x,y) = \rho_0 \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r'\rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}')$$

In this example, only n=1 contributes because

$$\int_0^a dx'\sin\left(\frac{\pi x'}{a}\right)\sin\left(\frac{n\pi x'}{a}\right) = \frac{a}{2}\delta_{1n}$$

$$\Phi(x,y) = \frac{8\rho_0}{4\pi\epsilon_0}\frac{a}{2\sinh(\pi a/b)}\sin\left(\frac{\pi x}{a}\right)\times$$

$$\left(\sinh\left(\frac{\pi(b-y)}{a}\right)\int_0^y dy'\sin\left(\frac{\pi y'}{b}\right)\sinh\left(\frac{\pi y'}{a}\right) + \sinh\left(\frac{\pi y}{a}\right)\int_y^b dy'\sin\left(\frac{\pi y'}{b}\right)\sinh\left(\frac{\pi(b-y')}{a}\right)\right)$$

When the dust clears: $\Phi(x,y) = \frac{\rho_0}{\epsilon_0}\frac{a^2b^2}{\pi^2(a^2+b^2)}\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)$

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Application to previously discussed examples (also your homework examples).

A useful theorem for electrostatics
The mean value theorem (Problem 1.10 in Jackson)

The "mean value theorem" value theorem (problem 1.10 of your textbook) states that the value of $\Phi(\mathbf{r})$ at the arbitrary (charge-free) point $\mathbf{r}$ is equal to the average of $\Phi(\mathbf{r}')$ over the surface of any sphere centered on the point $\mathbf{r}$ (see Jackson problem #1.10). One way to prove this theorem is the following. Consider a point $\mathbf{r}' = \mathbf{r} + \mathbf{u}$, where $\mathbf{u}$ will describe a sphere of radius R about the fixed point $\mathbf{r}$. We can make a Taylor series expansion of the electrostatic potential $\Phi(\mathbf{r}')$ about the fixed point $\mathbf{r}$:

$$\Phi(\mathbf{r}+\mathbf{u}) = \Phi(\mathbf{r}) + \mathbf{u}\cdot\nabla\Phi(\mathbf{r}) + \frac{1}{2!}(\mathbf{u}\cdot\nabla)^2\Phi(\mathbf{r}) + \frac{1}{3!}(\mathbf{u}\cdot\nabla)^3\Phi(\mathbf{r}) + \frac{1}{4!}(\mathbf{u}\cdot\nabla)^4\Phi(\mathbf{r}) + \cdots . \quad (1)$$

According to the premise of the theorem, we want to integrate both sides of the equation 1 over a sphere of radius R in the variable $\mathbf{u}$:

$$\int_{\text{sphere}} dS_u = R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u). \quad (2)$$

Changing gears slightly -- discussion of the mean value theorem for electrostatics.

Mean value theorem – continued

We note that

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) 1 = 4\pi R^2,$$

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) \mathbf{u} \cdot \nabla = 0,$$

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) (\mathbf{u} \cdot \nabla)^2 = \frac{4\pi R^4}{3} \nabla^2,$$

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) (\mathbf{u} \cdot \nabla)^3 = 0,$$

and

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) (\mathbf{u} \cdot \nabla)^4 = \frac{4\pi R^6}{5} \nabla^4.$$

Since $\nabla^2 \Phi(\mathbf{r}) = 0$, the only non-zero term of the average is thus the first term:

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) \Phi(\mathbf{r} + \mathbf{u}) = 4\pi R^2 \Phi(\mathbf{r}),$$

or

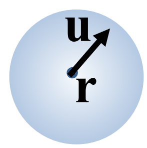
$$\Phi(\mathbf{r}) = \frac{1}{4\pi R^2} \; R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) \Phi(\mathbf{r} + \mathbf{u}) \equiv \frac{1}{4\pi R^2} \int_{\text{sphere}} dS_u \Phi(\mathbf{r} + \mathbf{u}).$$

Since this result is independent of the radius R, we see that we have the theorem.

Some details

Summary: Mean value theorem

$$\Phi(\mathbf{r}) = \frac{1}{4\pi R^2}\int R^2 d\Omega_u \ \Phi(\mathbf{r}+\mathbf{u})$$



Summary of results.