

PHY 712 Electrodynamics

10-10:50 AM MWF Online

Plan for Lecture 6:

Continue reading Chapters 2 & 3

- 1. Methods of images -- planes, spheres**
- 2. Solution of Poisson equation in for other geometries -- cylindrical**

PHY 712 Electrodynamics

MWF 10-10:50 PM Online <http://www.wfu.edu/~natalie/s21phy712/>

Instructor: [Natalie Holzwarth](#) Office: 300 OPL e-mail: natalie@wfu.edu

Course schedule for Spring 2021

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Wed: 01/27/2021	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/29/2021
2	Fri: 01/29/2021	Chap. 1	Electrostatic energy calculations	#2	02/01/2021
3	Mon: 02/01/2021	Chap. 1 & 2	Electrostatic potentials and fields	#3	02/03/2021
4	Wed: 02/03/2021	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#4	02/05/2021
5	Fri: 02/05/2021	Chap. 1 - 3	Brief introduction to numerical methods	#5	02/08/2021
6	Mon: 02/08/2021	Chap. 2 & 3	Image charge constructions	#6	02/10/2021
7	Wed: 02/10/2021	Chap. 2 & 3	Cylindrical and spherical geometries		
8	Fri: 02/12/2021	Chap. 3 & 4	Spherical geometry and multipole moments		
9	Mon: 02/15/2021	Chap. 4	Dipoles and Dielectrics		
10	Wed: 02/17/2021	Chap. 4	Polarization and Dielectrics		
11	Fri: 02/19/2021	Chap. 5	Magnetostatics		

Survey of mathematical techniques for analyzing electrostatics – the Poisson equation

$$\nabla^2 \Phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}$$

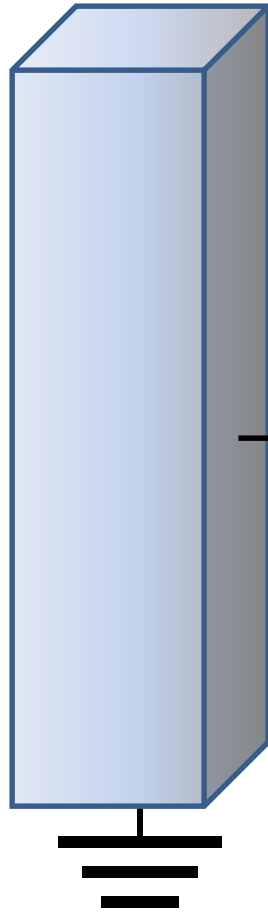
1. Direct solution of differential equation
2. Solution by means of an integral equation;
Green's function techniques
3. Orthogonal function expansions
4. Numerical methods (finite differences and
finite element methods)
5. Method of images

Method of images

Clever trick for specialized geometries:

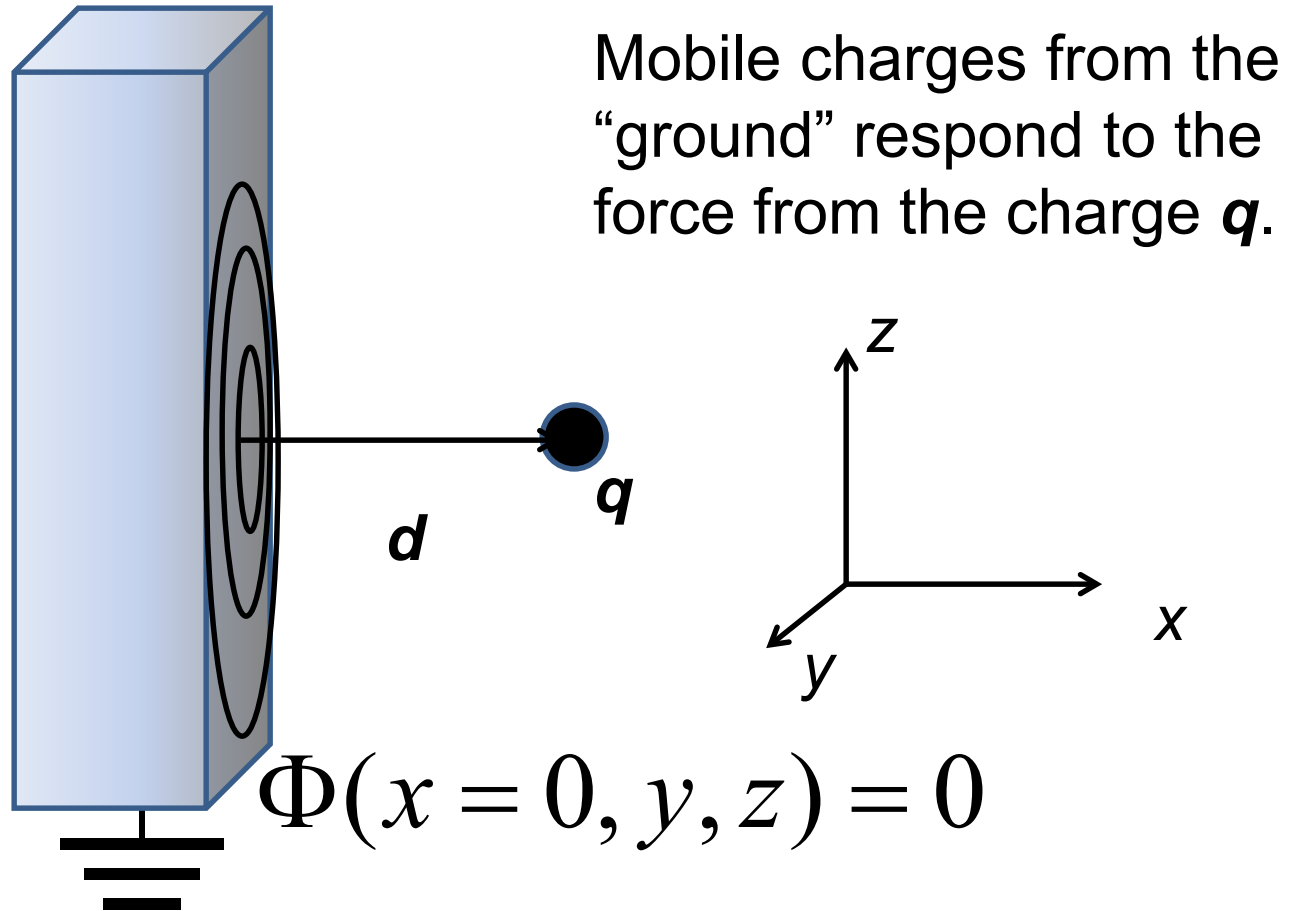
- Flat plane (surface)
- Sphere

Planar case:



Consider a grounded metal sheet, a distance d from a point charge q .

A grounded metal sheet, a distance d from a point charge q .



A grounded metal sheet, a distance d from a point charge q .

$$\nabla^2 \Phi = -\frac{q}{\epsilon_0} \delta^3(\mathbf{r} - d\hat{\mathbf{x}})$$

$$\Phi(x = 0, y, z) = 0$$

Trick for $x \geq 0$:

$$\Phi(x \geq 0, y, z) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{|\mathbf{r} - d\hat{\mathbf{x}}|} - \frac{q}{|\mathbf{r} + d\hat{\mathbf{x}}|} \right)$$

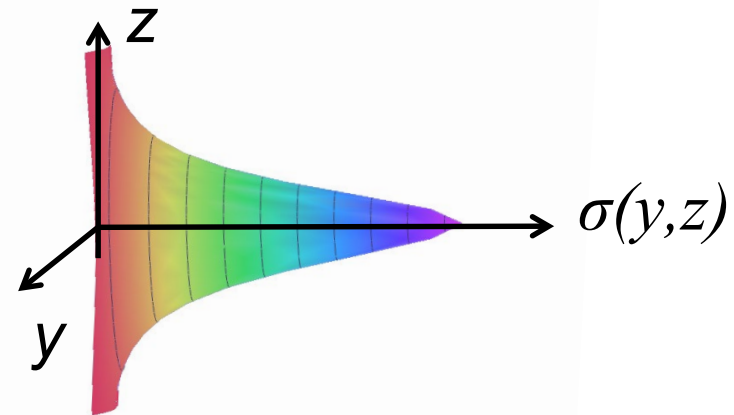
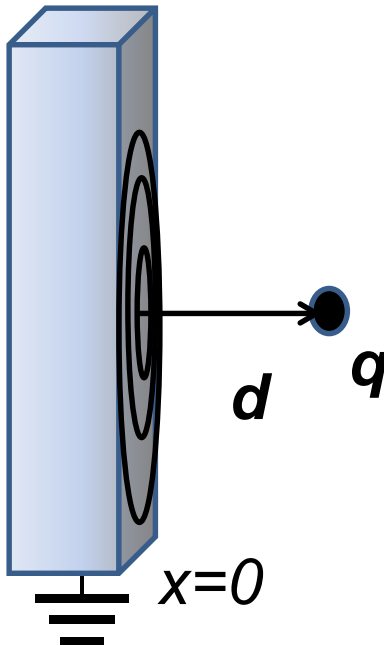
Surface charge density:

$$\sigma(y, z) = \epsilon_0 E(0, y, z) = -\epsilon_0 \frac{d\Phi(0, y, z)}{dx} = -\frac{q}{4\pi} \left(\frac{2d}{(d^2 + y^2 + z^2)^{3/2}} \right)$$

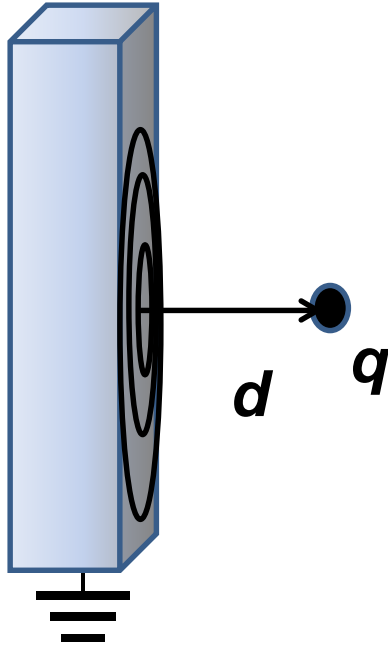
A grounded metal sheet, a distance d from a point charge q .

Surface charge density :
$$\sigma(y,z) = -\frac{q}{4\pi} \left(\frac{2d}{(d^2 + y^2 + z^2)^{3/2}} \right)$$

Note :
$$\iint dydz \sigma(y,z) = -\frac{q2d}{4\pi} 2\pi \int_0^\infty \frac{u du}{(d^2 + u^2)^{3/2}} = -q$$



A grounded metal sheet, a distance d from a point charge q .



Surface charge density :

$$\sigma(y,z) = -\frac{q}{4\pi} \left(\frac{2d}{(d^2 + y^2 + z^2)^{3/2}} \right)$$

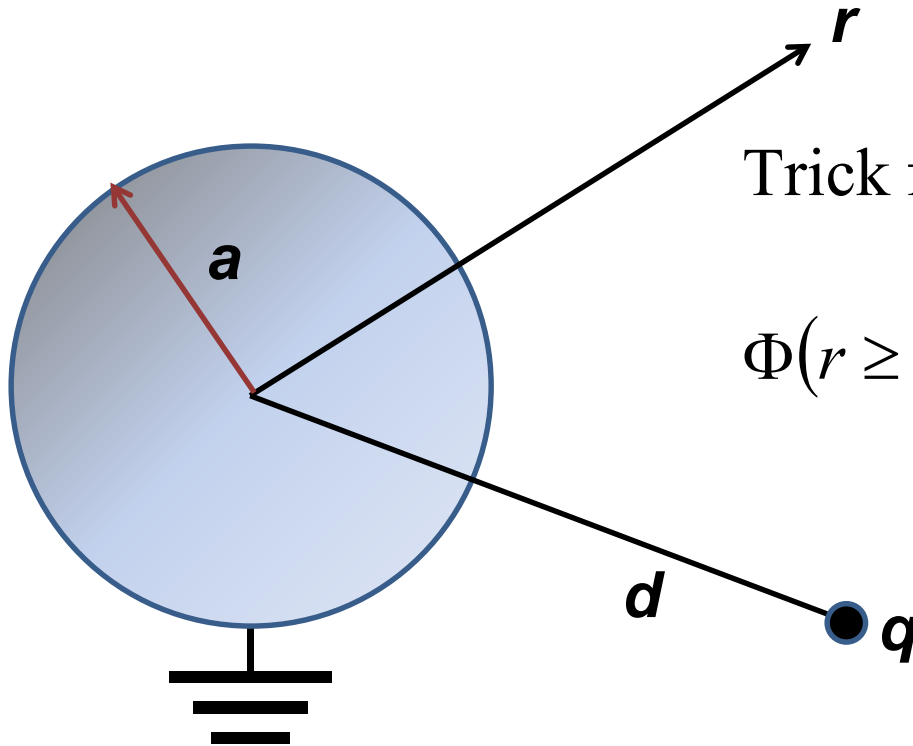
Force between charge and sheet :

$$\mathbf{F} = \frac{-q^2 \hat{\mathbf{x}}}{4\pi\epsilon_0 (2d)^2}$$

Image potential between charge and sheet at distance x :

$$V(x) = \frac{-q^2}{4\pi\epsilon_0 (4x)}$$

A grounded metal sphere of radius a , in the presence of a point charge q at a distance d from its center.



Trick for $r \geq a$:

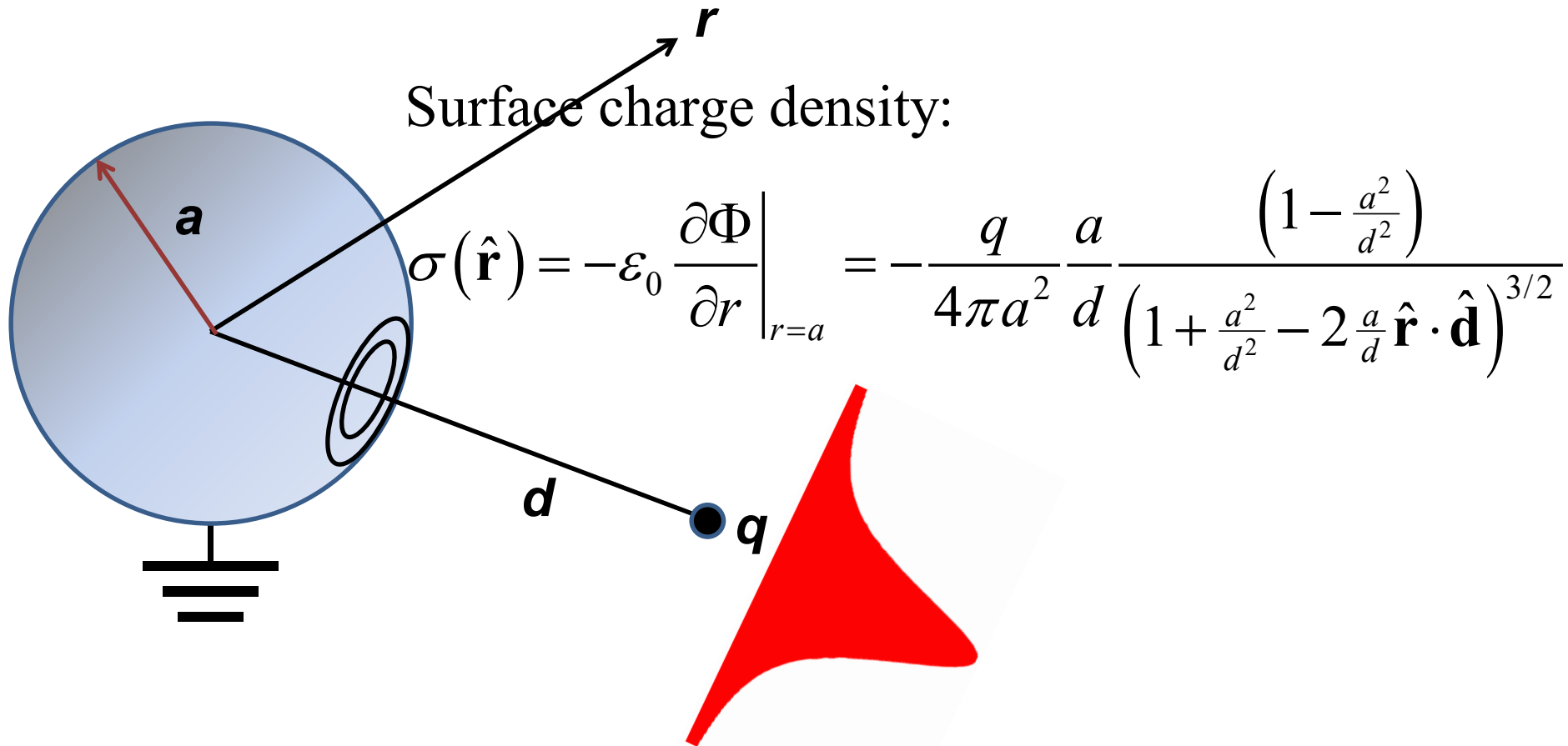
$$\Phi(r \geq a) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{|\mathbf{r} - \mathbf{d}|} - \frac{q}{\frac{d}{a} \left| \mathbf{r} - \mathbf{d} \frac{a^2}{d^2} \right|} \right)$$

Interpreted as

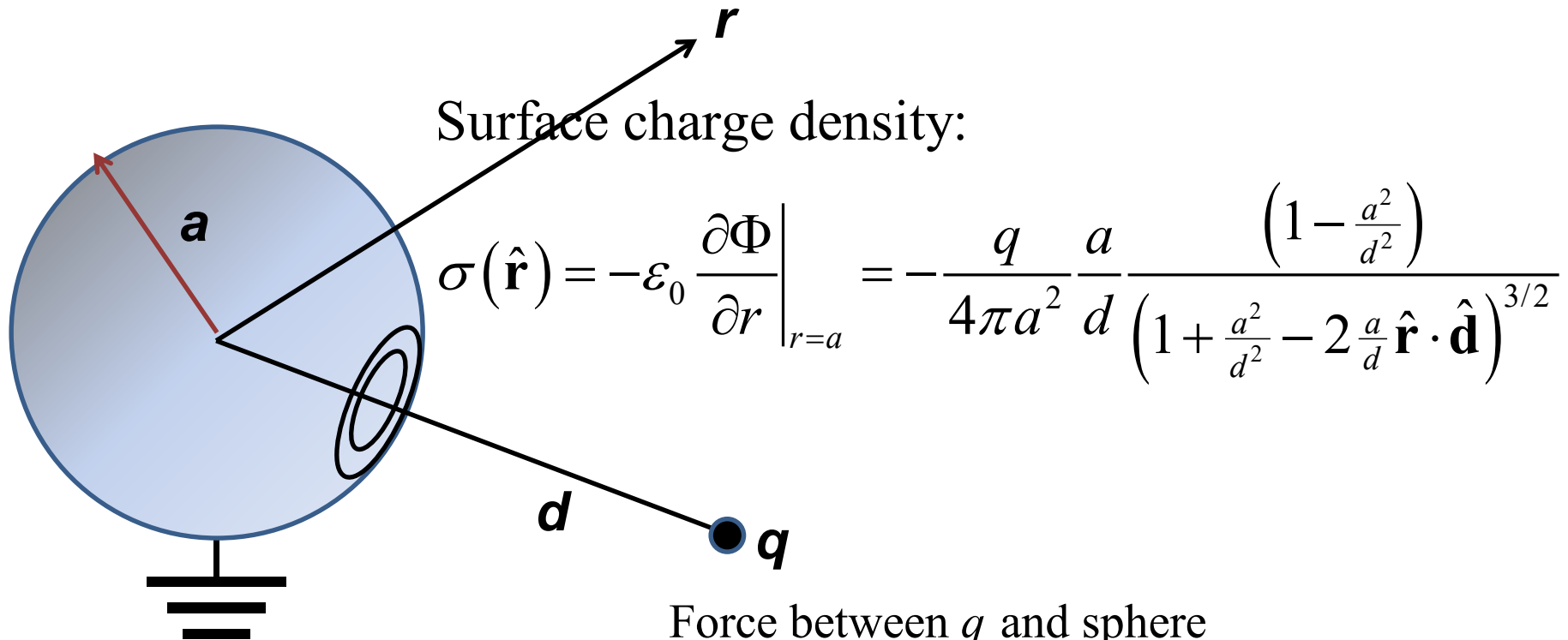
Image charge of $q' = -q \frac{a}{d}$

Located along $\hat{\mathbf{d}}$ at $\hat{\mathbf{d}} a \frac{a}{d}$

A grounded metal sphere of radius a , in the presence of a point charge q at a distance d from its center.



A grounded metal sphere of radius a , in the presence of a point charge q at a distance d from its center.



$$|\mathbf{F}| = \frac{1}{4\pi\epsilon_0} \frac{q^2 (a/d)}{\left(d - a^2/d\right)^2} = \frac{q^2}{4\pi\epsilon_0} \frac{ad}{\left(d^2 - a^2\right)^2}$$

Use of image charge formalism to construct Green's function

Example:

Suppose we have a Dirichlet boundary value problem
on a sphere of radius a :

$$\nabla^2 \Phi = -\frac{\rho(\mathbf{r})}{\epsilon_0} \quad \Phi(r = a) = 0$$

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = -4\pi\delta^3(\mathbf{r} - \mathbf{r}')$$

$$\Rightarrow \text{For } r, r' > a : \quad G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{\frac{r'}{a} \left| \mathbf{r} - \frac{a^2}{r'^2} \mathbf{r}' \right|}$$

Analysis of Poisson/Laplace equation in various regular geometries

1. Rectangular geometries
2. Cylindrical geometries
3. Spherical geometries

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no z-dependence (infinitely long wire, for example):

Corresponding orthogonal functions from solution of

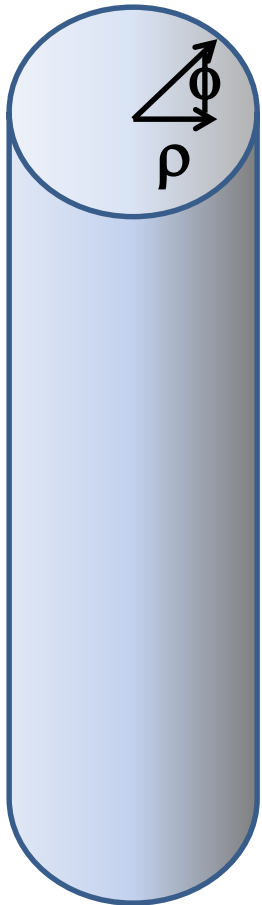
Laplace equation : $\nabla^2 \Phi = 0$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

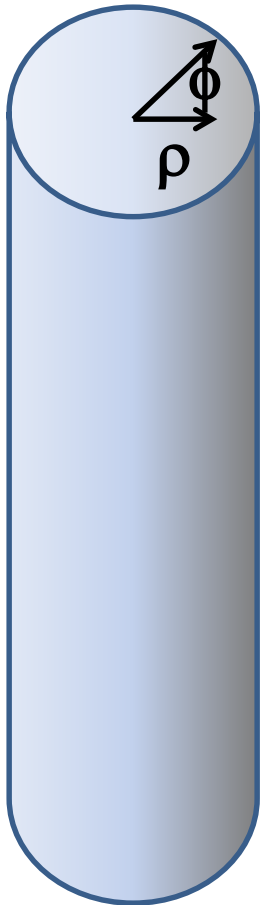
$$\Phi(\rho, \phi) = \Phi(\rho, \phi + m2\pi)$$

$\Rightarrow$ General solution of the Laplace equation
in these coordinates :

$$\Phi(\rho, \phi) = A_0 + B_0 \ln(\rho) + \sum_{m=1}^{\infty} (A_m \rho^m + B_m \rho^{-m}) \sin(m\phi + \alpha_m)$$



Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no z-dependence (infinitely long wire, for example):

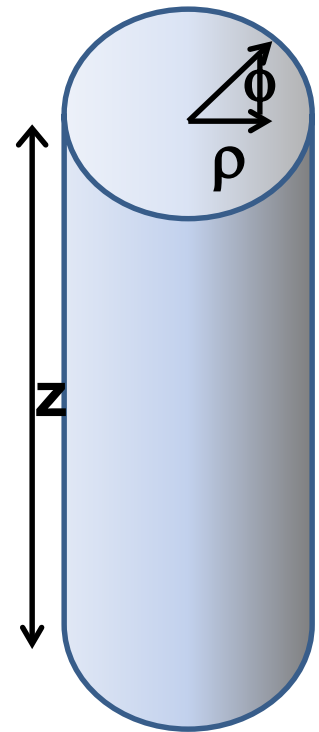


Green's function appropriate for this geometry with boundary conditions at $\rho = 0$ and $\rho = \infty$:

$$\left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \right) G(\rho, \rho', \phi, \phi') = -4\pi \frac{\delta(\rho - \rho')}{\rho} \delta(\phi - \phi')$$

$$G(\rho, \rho', \phi, \phi') = -\ln(\rho_{>}^2) + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho_{<}}{\rho_{>}} \right)^m \cos(m(\phi - \phi'))$$

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with z-dependence



Corresponding orthogonal functions from solution of Laplace equation : $\nabla^2 \Phi = 0$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\Phi(\rho, \phi, z) = \Phi(\rho, \phi + m2\pi, z)$$

$$\Phi(\rho, \phi, z) = R(\rho)Q(\phi)Z(z)$$

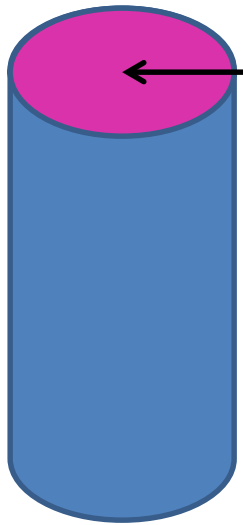
Cylindrical geometry continued:

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0 \quad \Rightarrow Z(z) = \sinh(kz), \cosh(kz), e^{\pm kz}$$

$$\frac{d^2 Q}{d\phi^2} + m^2 Q = 0 \quad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2} \right) R = 0 \quad \Rightarrow J_m(k\rho), N_m(k\rho)$$

Cylindrical geometry example:

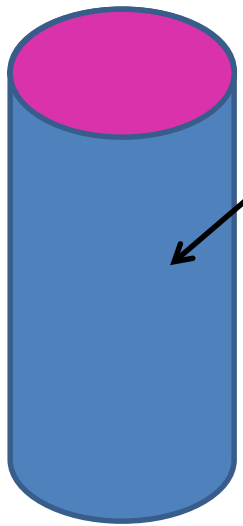


$$\Phi(\rho, \phi, z = L) = V(\rho, \phi)$$

$$\Phi(\rho, \phi, z) = 0 \quad \text{on all other boundaries}$$

$$\Phi(\rho, \phi, z) = \sum_{n,m} A_{mn} J_m(k_{mn}\rho) \sinh(k_{mn}z) \sin(m\phi + \alpha_{mn})$$

Cylindrical geometry example:



$$\Phi(\rho = a, \phi, z) = V(\phi, z)$$

$$\Phi(\rho, \phi, z) = 0 \quad \text{on all other boundaries}$$

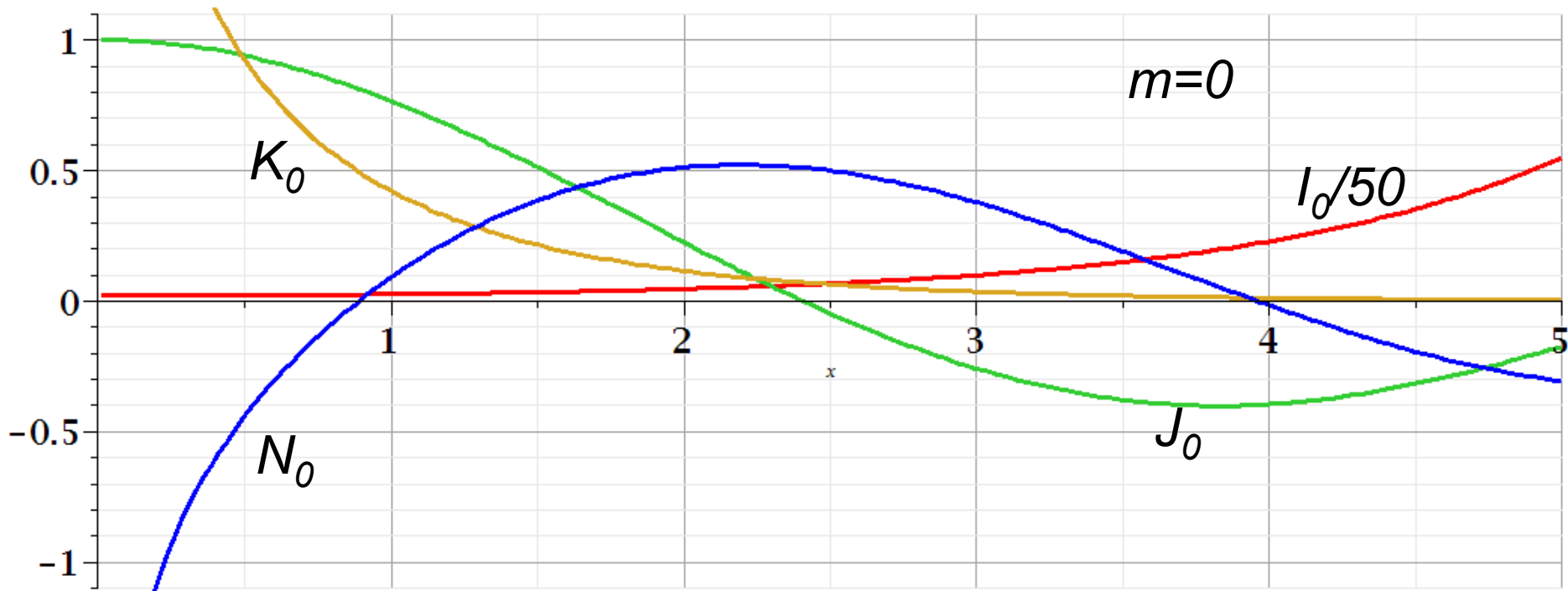
$$\Phi(\rho, \phi, z) = \sum_{n,m} A_{mn} I_m \left(\frac{n\pi\rho}{L} \right) \sin \left(\frac{n\pi z}{L} \right) \sin(m\phi + \alpha_{mn})$$

Comments on cylindrical Bessel functions

$$\left(\frac{d^2}{du^2} + \frac{1}{u} \frac{d}{du} + \left(\pm 1 - \frac{m^2}{u^2} \right) \right) F_m^\pm(u) = 0$$

$$F_m^+(u) = J_m(u), N_m(u), H_m(u) \equiv J_m(u) \pm iN_m(u)$$

$$F_m^-(u) = I_m(u), K_m(u)$$



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