PHY 712 Electrodynamics
10-10:50 AM MWF Online

Class notes for Lecture 7:

Start reading Chapter 3

Solution of Poisson/Laplace equation for special geometries –

A. Cylindrical

B. Spherical

Course schedule for Spring 2021

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Wed: 01/27/2021	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/29/2021
2	Fri: 01/29/2021	Chap. 1	Electrostatic energy calculations	#2	02/01/2021
3	Mon: 02/01/2021	Chap. 1 & 2	Electrostatic potentials and fields	#3	02/03/2021
4	Wed: 02/03/2021	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#4	02/05/2021
5	Fri: 02/05/2021	Chap. 1 - 3	Brief introduction to numerical methods	#5	02/08/2021
6	Mon: 02/08/2021	Chap. 2 & 3	Image charge constructions	#6	02/10/2021
7	Wed: 02/10/2021	Chap. 2 & 3	Cylindrical and spherical geometries		
8	Fri: 02/12/2021	Chap. 3 & 4	Spherical geometry and multipole moments		
9	Mon: 02/15/2021	Chap. 4	Dipoles and Dielectrics		
10	Wed: 02/17/2021	Chap. 4	Polarization and Dielectrics		
11	Fri: 02/19/2021	Chap. 5	Magnetostatics		
12	Mon: 02/22/2021	Chap. 5	Magnetic dipoles and hyperfine interaction		
13	Wed: 02/24/2021	Chap. 5	Magnetic dipoles and dipolar fields		

Tomorrow's colloquium

Online Colloquium: "Therapeutic Opportunities in Glycoscience" — February 11, 2021 at 4 PM

Dr. Carolyn Bertozzi

Baker Family Co-Director, Stanford ChEM-H

Anne T. and Robert M. Bass Professor of Chemistry

Professor of Chemical & Systems Biology and Radiology (by courtesy)

Department of Chemistry and Howard Hughes Medical Institute

Stanford University

Thursday, February 11, 2021, 4 PM EST

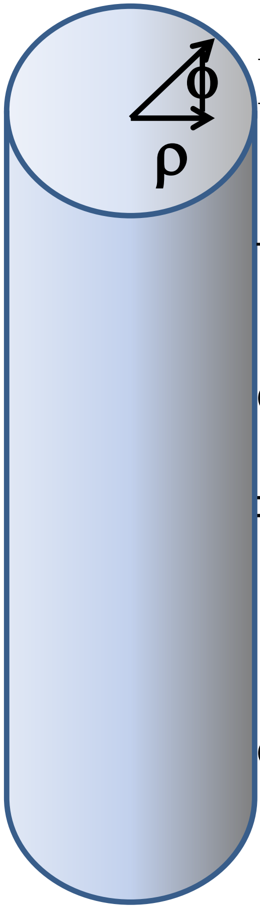
Via Video Conference (contact wfuphys@wfu.edu for link information)

Your questions –

From Tim -- Could you go over how we came to the equation G(p,p',phi, phi') on slide 6?

From Gao – After solving Laplace equations in various geometries and with different boundary conditions, how can we get related Green functions? You have written them out, but according to what you did it? In page 10, 11, Why does only m=0 exist or you just wrote the expression for m=0? In some symmetric systems, or only in spherical symmetry systems, if rho vanishes for r is infinite, then the green function can be chosen as 1/(r-r')?

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no *z*-dependence (infinitely long wire, for example):



Corresponding orthogonal functions from solution of

Laplace equation: $\nabla^2\Phi = 0$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

$$\Phi(\rho,\phi)=\Phi(\rho,\phi+m2\pi)$$

$\Rightarrow$ General solution of the Laplace equation

in these coordinates:

$$\Phi(\rho,\phi)=\mathrm{A}_0+\mathrm{B}_0\ln(\rho)+\sum_{m=1}^{\infty}\left(A_m\rho^{m}+B_m\rho^{-m}\right)\sin(m\phi+\alpha_m)$$

Cylindrical coordinates with trivial *z*-dependence – some details:

$$\Phi(\rho,\phi) = f(\rho)g(\phi)$$

Suppose $\frac{d^2 g(\phi)}{d\phi^2} = -m^2 g(\phi)$

$$g(\phi) = \sin(m\phi + \alpha_m)$$

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{df_m(\rho)}{d\rho}\right) - \frac{m^2}{\rho^2} f_m(\rho) = 0$$

$$f_0(\rho) = \begin{cases} 1 \\ \ln\rho \end{cases} \qquad f_{m>0} = \rho^{\pm m}$$

$\Rightarrow$ General solution of the Laplace equation in these coordinates:

$$\Phi(\rho,\varphi) = \mathrm{A}_0 + \mathrm{B}_0 \ln(\rho) + \sum_{m=1}^{\infty}\left(A_m\rho^m + B_m\rho^{-m}\right)\sin(m\varphi + \alpha_m)$$

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no *z*-dependence (infinitely long wire, for example):

Green's function appropriate for this geometry with boundary conditions at $\rho = 0$ and $\rho = \infty$:

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)G(\rho,\rho',\phi,\phi') = -4\pi\frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi')$$

$$G(\rho,\rho',\phi,\phi') = -\ln\left(\rho_>{}^2\right)+2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m\cos\left(m(\phi-\phi')\right)$$

Questions: How did we get this Green's function

Comment: Jackson very cleverly used the ideas we discussed for the two dimensional Cartesian case --

Green's function construction -- continued

For the two dimensional case, for example, we can assume that the Green's function can be written in the form:

$$G(x,x',y,y') = \sum_n u_n(x)u_n(x')g_n(y,y') \quad \text{where} \quad \frac{d^2}{dx^2}u_n(x) = -\alpha_n u_n(x)$$

The y dependence of this equation will have the required behavior, if we choose:

$$\left[-\alpha_n + \frac{\partial^2}{\partial y^2}\right]g_n(y,y') = -4\pi\delta(y-y'),$$

which in turn can be expressed in terms of the two independent solutions $v_{n_1}(y)$ and $v_{n_2}(y)$ of the homogeneous equation:

$$\left[\frac{d^2}{dy^2} - \alpha_n\right]v_{n_i}(y) = 0,$$

and the Wronskian constant: $K_n \equiv \frac{dv_{n_1}}{dy}v_{n_2} - v_{n_1}\frac{dv_{n_2}}{dy}$

$$\left[-\alpha_n + \frac{\partial^2}{\partial y^2}\right] g_n(y, y') = -4\pi\delta(y - y'),$$

$$g_n(y, y') = \frac{4\pi}{K_n} v_{n_1}(y_<) v_{n_2}(y_>)$$

where: $$\left[\frac{d^2}{dy^2} - \alpha_n\right] v_{n_i}(y) = 0,$$

and $$K_n \equiv \frac{dv_{n_1}}{dy} v_{n_2} - v_{n_1} \frac{dv_{n_2}}{dy}$$

For example, choose $v_{n_1}(y) = \sinh(\sqrt{\alpha_n}\, y)$ and $v_{n_2}(y) = \sinh(\sqrt{\alpha_n}(b - y))$

where $K_n = \sqrt{\alpha_n} \sinh(\sqrt{\alpha_n} b)$

using the identity: $\cosh(r)\sinh(s) + \sinh(r)\cosh(s) = \sinh(r + s)$

$$G(x, x', y, y') = \sum_n u_n(x) u_n(x') \frac{4\pi}{K_n} v_{n_1}(y_<) v_{n_2}(y_>).$$

In the cylindrical geometry case,

$$u_n(x) \rightarrow \{\sin(m\varphi), \cos(m\varphi)\}$$

$$v_{n_{1,2}} \rightarrow \{1, \ln(\rho), \rho^m, \rho^{-m}\}$$

$$G(\rho,\rho',\varphi,\varphi') = -\ln\left(\rho_>{}^2\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m \cos\left(m\left(\varphi-\varphi'\right)\right)$$

Comments and details

Change notation

$\rho \Rightarrow r$

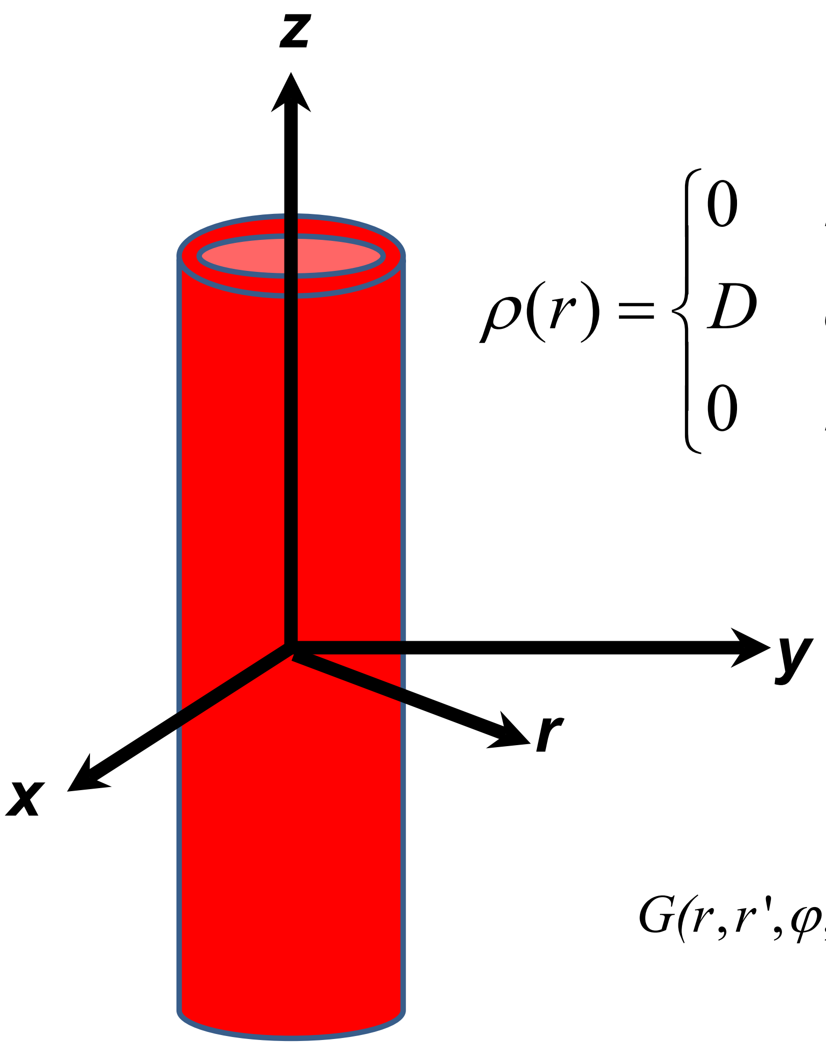
$$G(r,r',\varphi,\varphi') = -\ln\left(r_>^{\,2}\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r_<}{r_>}\right)^m \cos\left(m\left(\varphi-\varphi'\right)\right)$$

$$\Phi(r,\varphi) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\varphi' \int_0^{\infty} r'dr' G(r,r',\varphi,\varphi')\rho(r',\varphi')$$

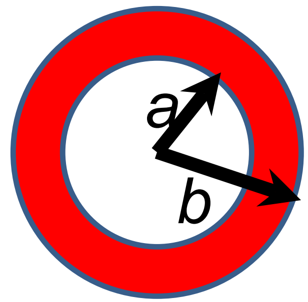
Note that: For this extended charge distribution, Coulomb's law in its original form diverges:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Example – uniform cylindrical shell:



Top view:



$$\rho(r) = \begin{cases} 0 & r < a \\ D & a \le r \le b \\ 0 & r > b \end{cases}$$

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2 \Phi(r,\phi) = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi(r,\phi)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi(r,\phi)}{\partial^2 \phi}$$

$$G(r,r',\varphi,\varphi') = -\ln\left(r_>{}^2\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r_<}{r_>}\right)^m \cos\left(m\left(\varphi-\varphi'\right)\right)$$

$$\Phi(r,\varphi) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\varphi' \int_0^{\infty} r' dr' G(r,r',\varphi,\varphi')\rho(r',\varphi')$$

Question – Why only m=0 for this case?

$$G(r,r',\varphi,\varphi') = -\ln\left(r_>^{\ 2}\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r_<}{r_>}\right)^m \cos\left(m\left(\varphi-\varphi'\right)\right)$$

$$\Phi(r,\varphi) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\varphi' \int_0^{\infty} r'dr' G(r,r',\varphi,\varphi')\rho(r',\varphi')$$

Note that $\int_0^{2\pi} d\varphi' \cos(m(\varphi-\varphi')) = 0$ for $m > 0$

Some details

$$G(r,r',\varphi,\varphi') = -\ln\left(r_>{}^2\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r_<}{r_>}\right)^m \cos\left(m\left(\varphi-\varphi'\right)\right)$$

$$\Phi(r,\varphi) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\varphi' \int_0^{\infty} r'dr' G(r,r',\varphi,\varphi')\rho(r',\varphi')$$

In our case: $\Phi(r,\varphi) = \frac{2\pi D}{4\pi\epsilon_0}\int_a^b r'dr'\left(-\ln(r_>^2)\right) = \frac{D}{\epsilon_0}\int_a^b r'dr'\left(-\ln(r_>)\right)$

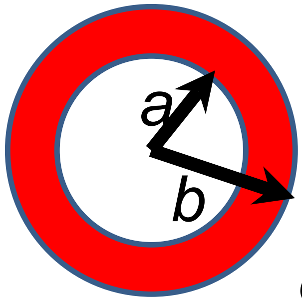
For $0 \le r < a$: $\Phi(r,\varphi) = \frac{D}{\epsilon_0}\int_a^b r'dr'\left(-\ln(r')\right)$

For $a \le r < b$: $\Phi(r,\varphi) = \frac{D}{\epsilon_0}\left(\int_a^r r'dr'\left(-\ln(r)\right) + \int_r^b r'dr'\left(-\ln(r')\right)\right)$

For $r > b$: $\Phi(r,\varphi) = \frac{D}{\epsilon_0}\int_a^b r'dr'\left(-\ln(r)\right)$

Example continued -- *m=0* only --

Top view:



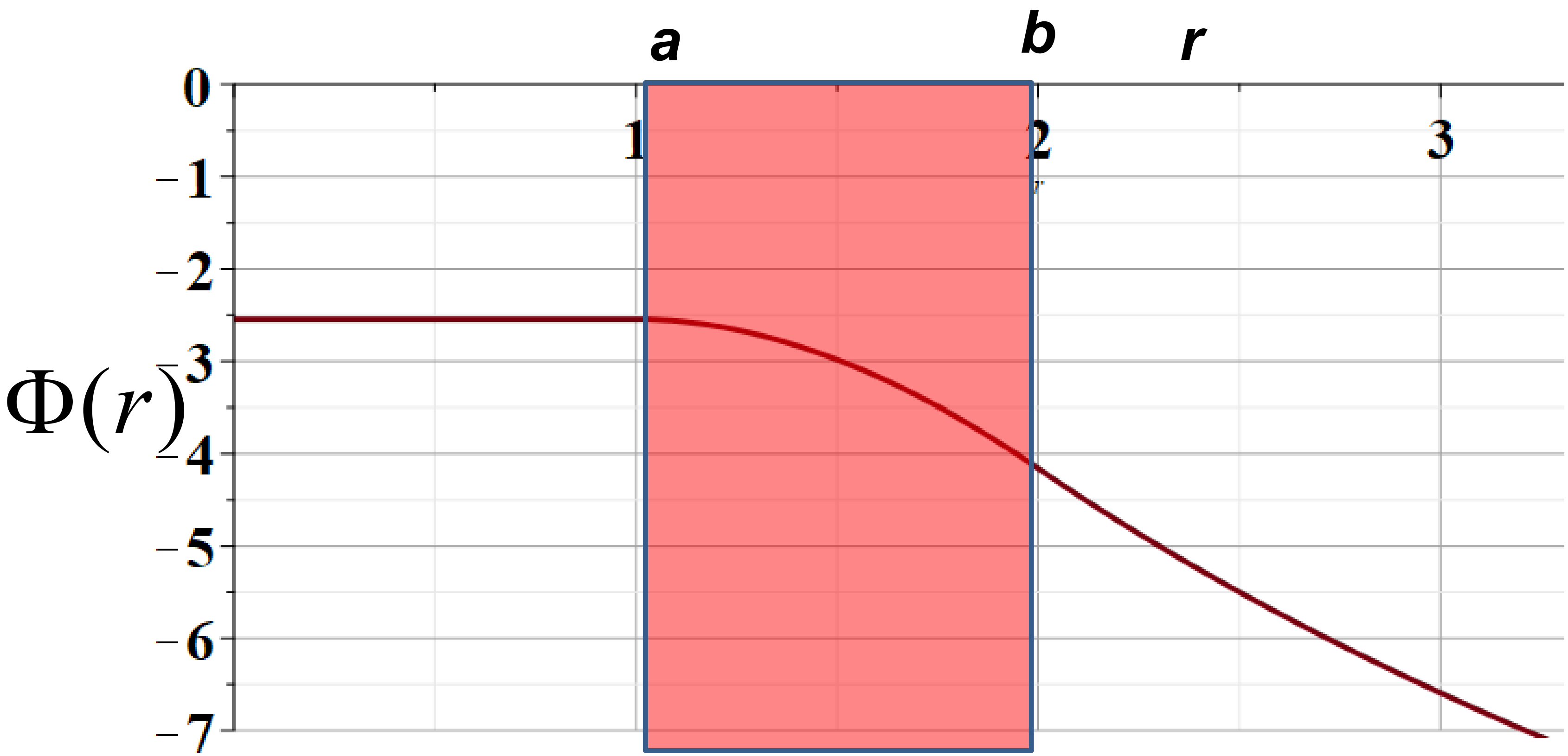
$$\rho(r)=\begin{cases}0 & 0<r<a\\ D & a\leq r\leq b\\ 0 & r>b\end{cases}$$

$$G(r,r',\varphi,\varphi')=-\ln\left(r_>^{\,2}\right)+2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r_<}{r_>}\right)^m\cos\left(m\left(\varphi-\varphi'\right)\right)$$

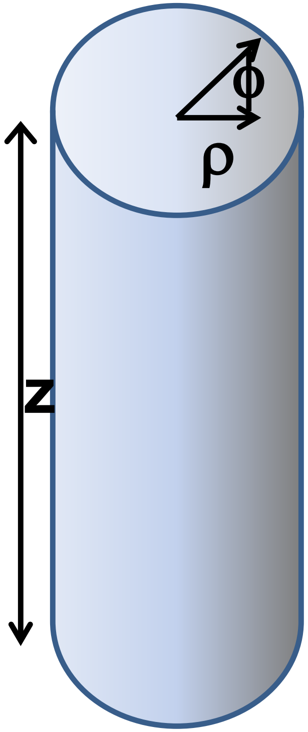
$$\Phi(r,\varphi)=\frac{1}{4\pi\epsilon_0}\int_0^{2\pi}d\varphi'\int_0^{\infty}r'dr'G(r,r',\varphi,\varphi')\rho(r',\varphi')$$

$$\Phi(r)=\begin{cases}\frac{D}{4\epsilon_0}\left(b^2-a^2-b^2\ln(b^2)+a^2\ln(a^2)\right) & 0<r<a\\ \frac{D}{4\epsilon_0}\left(b^2-r^2-b^2\ln(b^2)+a^2\ln(r^2)\right) & a\leq r\leq b\\ \frac{D}{4\epsilon_0}\left(a^2-b^2\right)\ln(r^2) & r>b\end{cases}$$

Example continued -- *m=0* only --



Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with *z*-dependence

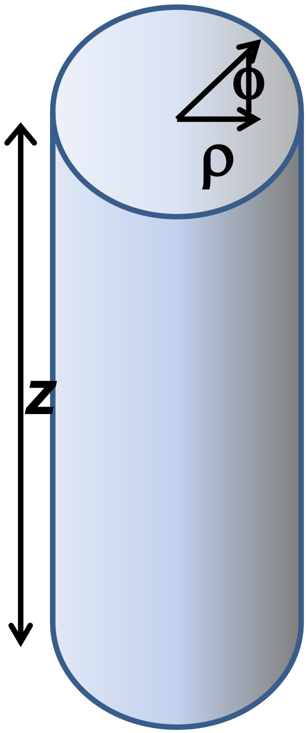


Laplace equation: $\nabla^2 \Phi = 0$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\Phi(\rho, \phi, z) = R(\rho) Q(\phi) Z(z)$$

Cylindrical geometry continued:



Laplace equation : $\nabla^2\Phi = 0$

$\Phi(\rho,\phi,z) = R(\rho)Q(\phi)Z(z)$

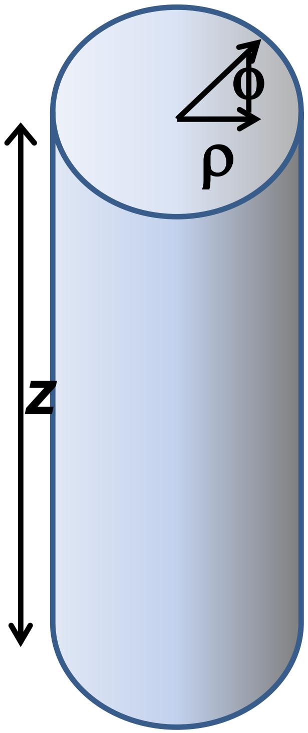
One possibility :

$$\frac{d^2Z}{dz^2} - k^2 Z = 0 \qquad \Rightarrow Z(z) = \sinh(kz), \cosh(kz), e^{\pm kz}$$

$$\frac{d^2Q}{d\phi^2} + m^2 Q = 0 \qquad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2}\right)R = 0 \qquad \Rightarrow J_m(k\rho), N_m(k\rho)$$

Cylindrical geometry continued:



Laplace equation : $\nabla^2\Phi = 0$

$$\Phi(\rho,\phi,z) = R(\rho)Q(\phi)Z(z)$$

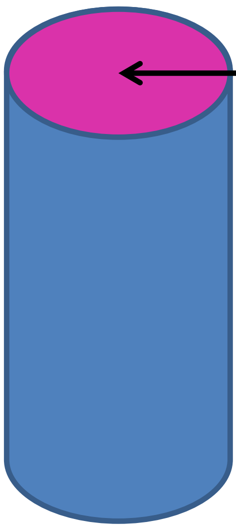
Another possibility :

$$\frac{d^2Z}{dz^2} + k^2 Z = 0 \qquad \Rightarrow Z(z) = \sin(kz), \cos(kz), e^{\pm ikz}$$

$$\frac{d^2Q}{d\phi^2} + m^2 Q = 0 \qquad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(-k^2 - \frac{m^2}{\rho^2}\right)R = 0 \qquad \Rightarrow I_m(k\rho), K_m(k\rho)$$

Solutions of Laplace equation inside cylindrical shape
Example with non-trivial boundary value at *z=L*

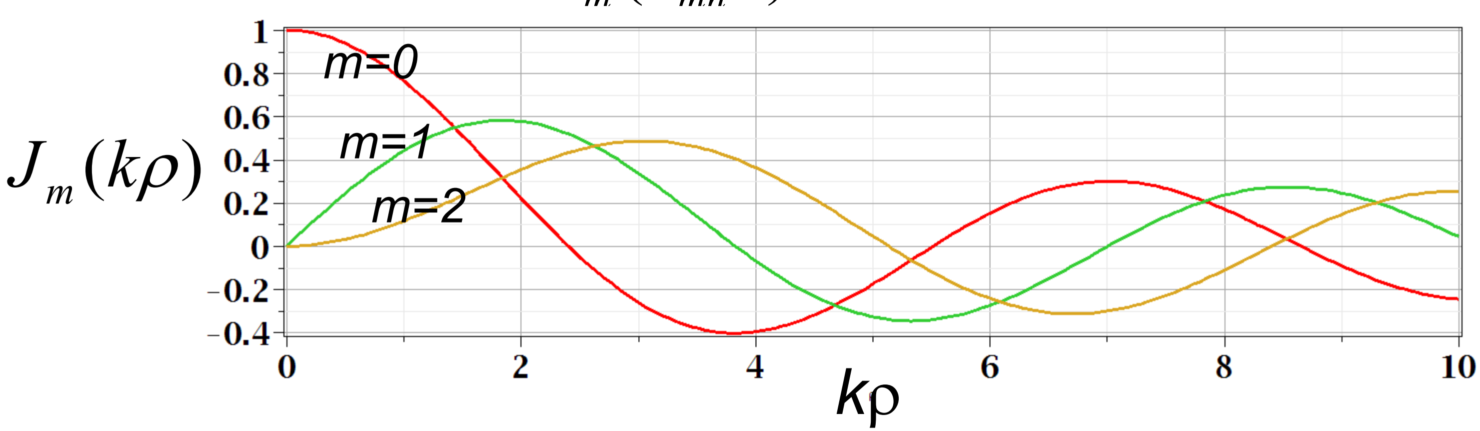


$$\Phi(\rho,\phi,z=L)=V(\rho,\phi)$$

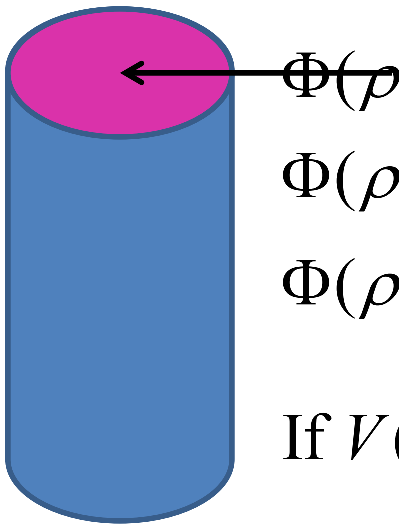
$$\Phi(\rho,\phi,z)=0 \quad \text{on all other boundaries}$$

$$\Phi(\rho,\phi,z)=\sum_{n,m} A_{mn} J_m(k_{mn}\rho)\sinh(k_{mn}z)\sin\left(m\phi+\alpha_{mn}\right)$$

$$\text{where} \quad J_m(k_{mn}a)=0$$



Solutions of Laplace equation inside cylindrical shape
Example with non-trivial boundary value at *z=L*



$$\Phi(\rho,\varphi,z=L)=V(\rho,\varphi)$$

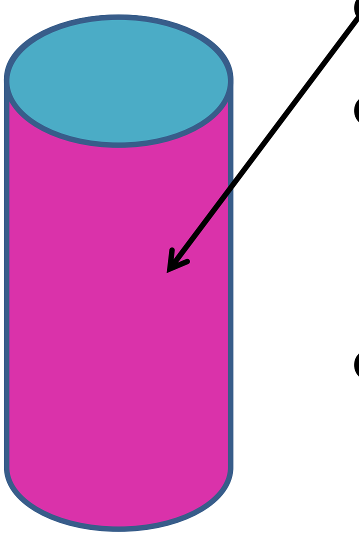
$$\Phi(\rho,\varphi,z)=0 \quad \text{on all other boundaries}$$

$$\Phi(\rho,\varphi,z)=\sum_{n,m} A_{mn} J_m(k_{mn}\rho)\sinh(k_{mn}z)\sin\left(m\varphi+\alpha_{mn}\right)$$

If $V(\rho,\varphi)$ is an even function of φ so that $\alpha_{mn}=\pi/2$:

$$A_{mn}=\frac{\int_0^{2\pi} d\varphi \cos\left(m\varphi\right)\int_0^a \rho d\rho J_m(k_{mn}\rho)V(\rho,\varphi)}{\sinh(k_{mn}L)\int_0^{2\pi} d\varphi \cos^2\left(m\varphi\right)\int_0^a \rho d\rho J_m{}^2(k_{mn}\rho)}$$

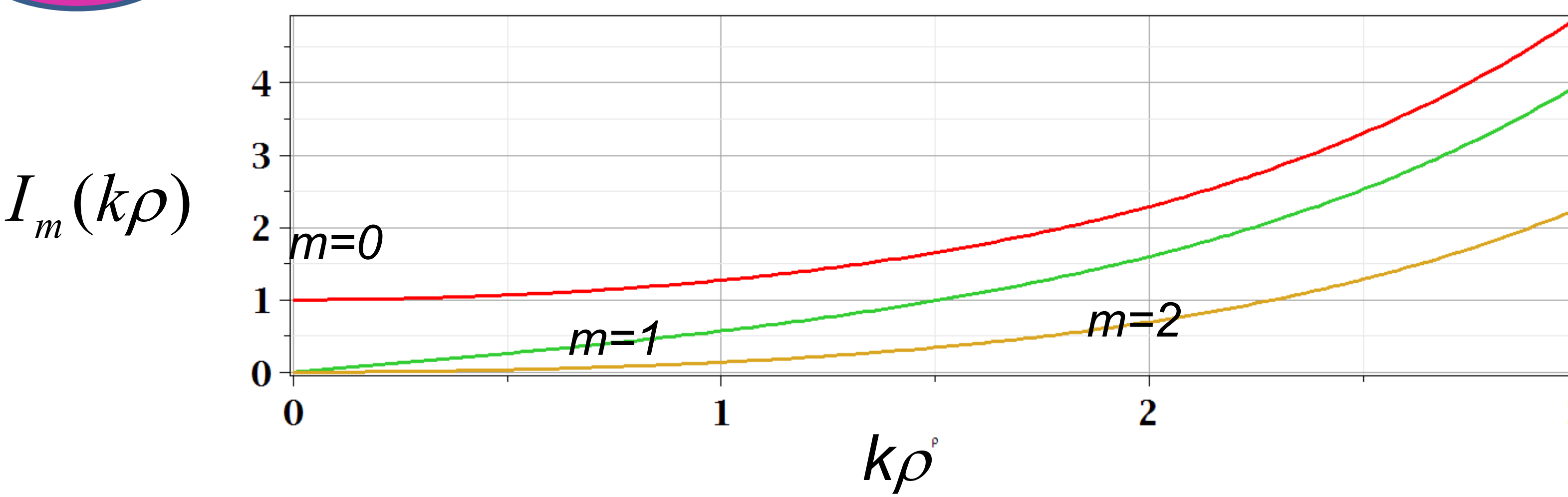
Solutions of Laplace equation inside cylindrical shape
Example with non-trivial boundary value at $\rho=a$



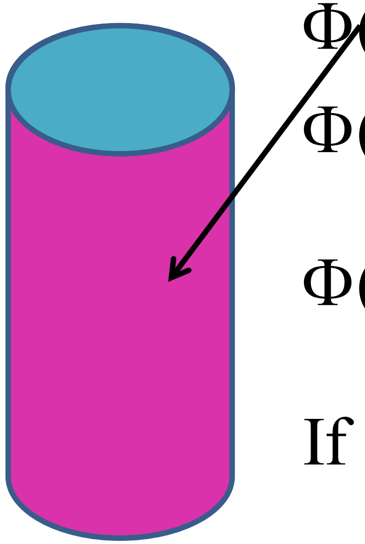
$$\Phi(\rho=a,\phi,z)=V(\phi,z)$$

$$\Phi(\rho,\phi,z)=0 \quad \text{on all other boundaries}$$

$$\Phi(\rho,\phi,z)=\sum_{n,m} A_{mn} I_m\left(\frac{n\pi\rho}{L}\right)\sin\left(\frac{n\pi z}{L}\right)\sin\left(m\phi+\alpha_{mn}\right)$$



Solutions of Laplace equation inside cylindrical shape
Example with non-trivial boundary value at ρ=a



$$\Phi(\rho = a, \varphi, z) = V(\varphi, z)$$

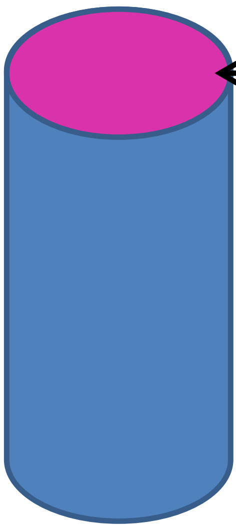
$$\Phi(\rho, \varphi, z) = 0 \quad \text{on all other boundaries}$$

$$\Phi(\rho, \varphi, z) = \sum_{n,m} A_{mn} I_m\left(\frac{n\pi\rho}{L}\right)\sin\left(\frac{n\pi z}{L}\right)\sin\left(m\varphi + \alpha_{mn}\right)$$

If $V(z,\varphi)$ is an even function of φ so that $\alpha_{mn} = \pi/2$:

$$A_{mn} = \frac{\int_0^{2\pi} d\varphi \cos\left(m\varphi\right) \int_0^L dz \sin\left(\frac{n\pi z}{L}\right) V(z,\varphi)}{I_m\left(\frac{n\pi a}{L}\right) \int_0^{2\pi} d\varphi \cos^2\left(m\varphi\right) \int_0^L dz \sin^2\left(\frac{n\pi z}{L}\right)}$$

Green's function for Dirchelet boundary value inside cylindar:



$$\Phi(\rho,\phi,z=L)=V(\rho,\phi)$$

$$\Phi(\rho=a,\phi,z)=0, \ \ \Phi(\rho,\phi,z=0)=0$$

$$\text{Expansion in terms of Bessel function zeros: } J_m(k_{mn}a)=0$$

$$G(\rho,\rho',\phi,\phi',z,z')=$$

$$\frac{8\pi}{\pi a^2}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{im(\phi-\phi')}J_m(k_{mn}\rho)J_m(k_{mn}\rho')\sinh(k_{mn}z_<)\sinh(k_{mn}(L-z_>))}{k_{mn}(J_{m+1}(k_{mn}a))^2\sinh(k_{mn}L)}$$

$$\Phi(\rho,\phi,z)=\frac{1}{4\pi\varepsilon_0}\int_V d\phi'\,\rho'\,d\rho'\,dz'\,G(\rho,\rho',\phi,\phi',z,z')\rho(\rho',\phi',z')$$

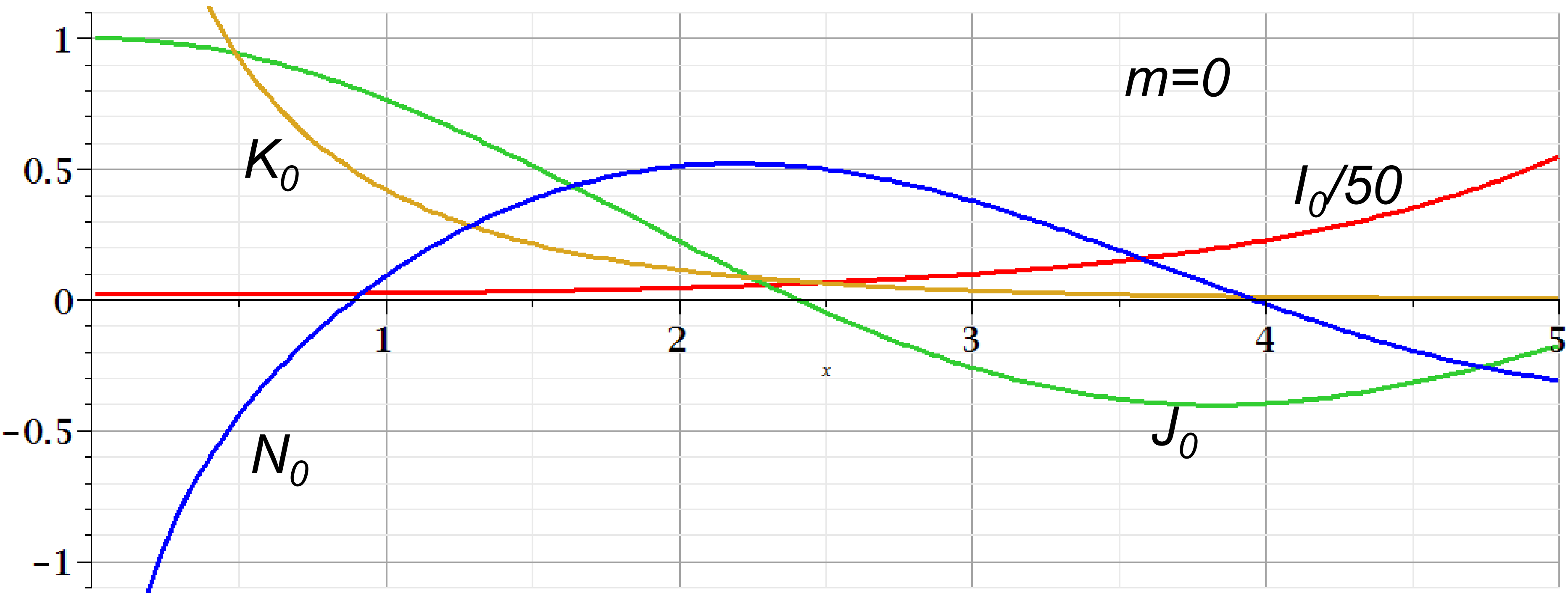
$$+\frac{1}{4\pi}\int_{S;z'=L} d\phi'\,\rho'\,d\rho'\left.\frac{\partial G(\rho,\rho',\phi,\phi',z,z')}{\partial z'}\right|_{z'=L} V(\rho',\phi')$$

Comments on cylindrical Bessel functions

$$\left(\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}+\left(\pm 1-\frac{m^2}{u^2}\right)\right)F_m^{\pm}(u)=0$$

$$F_m^{+}(u)=J_m(u),N_m(u),H_m(u)\equiv J_m(u)\pm iN_m(u)$$

$$F_m^{-}(u)=I_m(u),K_m(u)$$

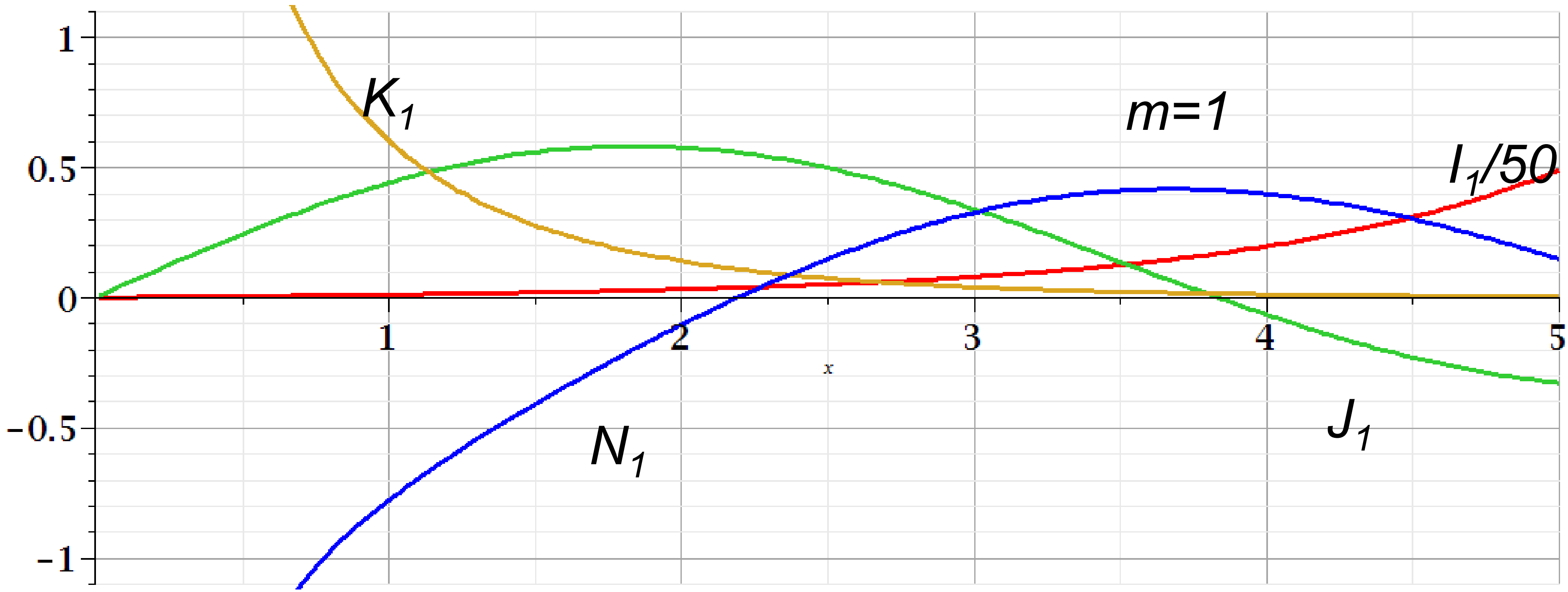


Comments on cylindrical Bessel functions

$$\left(\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}+\left(\pm 1-\frac{m^2}{u^2}\right)\right)F_m^{\pm}(u)=0$$

$$F_m^{+}(u)=J_m(u),N_m(u),H_m(u)\equiv J_m(u)\pm iN_m(u)$$

$$F_m^{-}(u)=I_m(u),K_m(u)$$



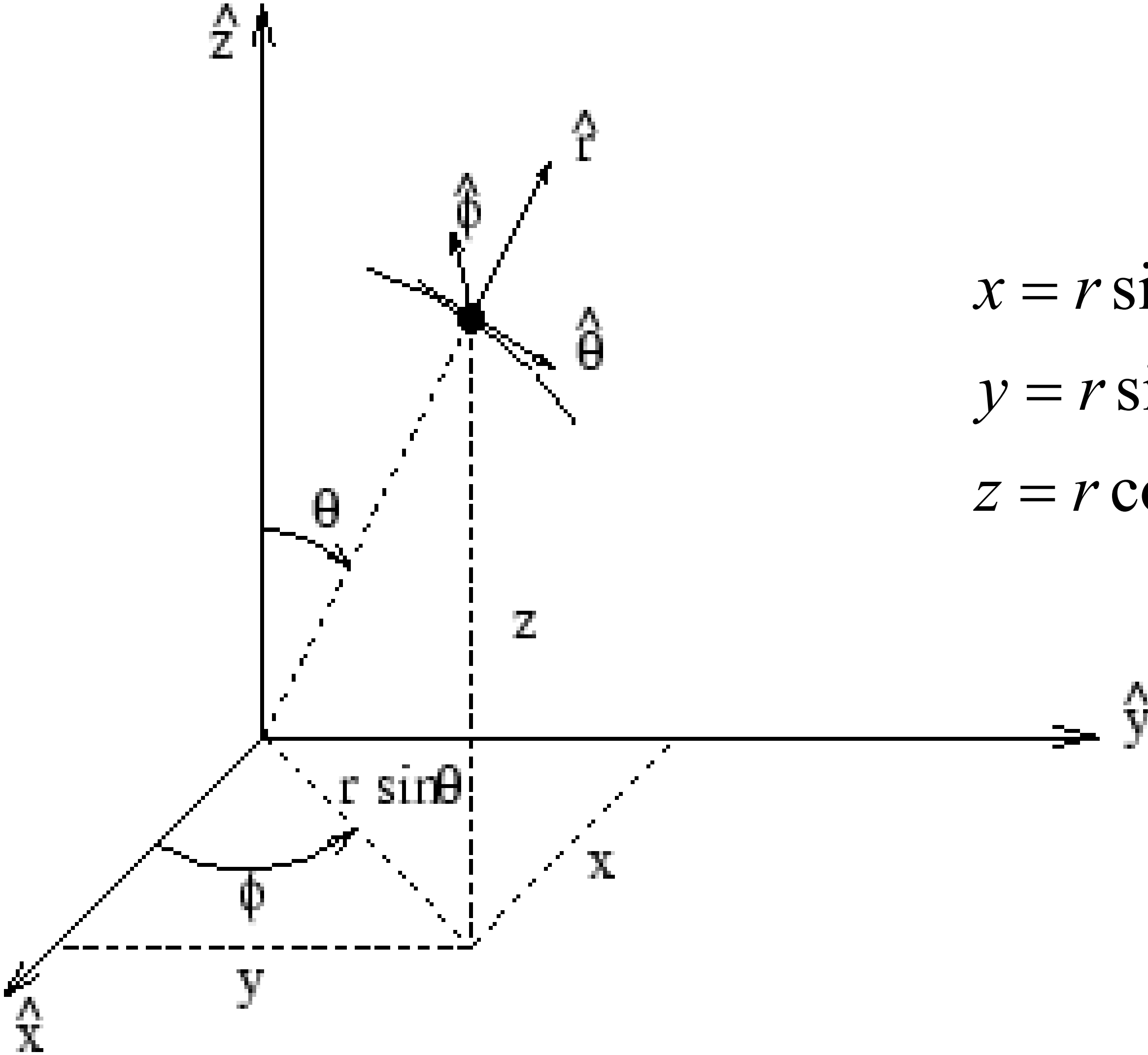
Some useful identities involving cylindrical Bessel functions

$$\left(\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}+\left(1-\frac{m^2}{u^2}\right)\right)J_m(u)=0 \qquad \text{for integer } m$$

Properties of Bessel functions in terms of zeros: x_{mn}; $\quad J_m(x_{mn})=0$

$$\int_0^a \rho d\rho J_m(x_{mn}\rho/a)J_m(x_{mn'}\rho/a)=\frac{a^2}{2}\left(J_{m+1}(x_{mn})\right)^2\delta_{nn'}$$

Poisson and Laplace equation in spherical polar coordinates



$$x = r\sin\theta\cos\varphi$$

$$y = r\sin\theta\sin\varphi$$

$$z = r\cos\theta$$

http://www.uic.edu/classes/eecs/eecs520/textbook/node32.html

Poisson and Laplace equation in spherical polar coordinates -- continued

Laplace equation for electrostatic potential $\Phi\left(r,\theta,\varphi\right)$:

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r\Phi\right)+\frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\Phi=0$$

$$\Phi\left(r,\theta,\varphi\right)=\sum_{lm}R_{lm}(r)Y_{lm}\left(\theta,\varphi\right)$$

Spherical harmonic functions:

$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)Y_{lm}\left(\theta,\varphi\right)=-l\left(l+1\right)Y_{lm}\left(\theta,\varphi\right)$$

Properties of spherical harmonic functions

$$Y_{lm}(\theta,\phi) = (-1)^m Y_{l(-m)}{}^*(\theta,\phi) \qquad \text{(standard Condon - Shortley convention)}$$

$$\int d\Omega Y_{lm}(\theta,\varphi) Y_{l'm'}{}^*(\theta,\varphi) \equiv \int \sin\theta \, d\theta \, d\varphi \, Y_{lm}(\theta,\varphi) Y_{l'm'}{}^*(\theta,\varphi) = \delta_{ll'}\delta_{mm'}$$

Completeness :

$$\sum_{lm} Y_{lm}(\theta,\varphi) Y_{lm}{}^*(\theta',\varphi') = \delta(\hat{\mathbf{r}} - \hat{\mathbf{r}}') \equiv \delta(\cos\theta - \cos\theta')\delta(\phi - \phi')$$

Relationship to Legendre polynomials :

$$Y_{l0}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)$$

Useful identity:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^{\,l}}{r_>^{\,l+1}}Y_{lm}(\theta,\varphi)Y_{lm}^*(\theta',\varphi')$$

Example for isolated charge density $\rho(\mathbf{r})$ with electrostatic potential vanishing for $r\rightarrow\infty$:

$$\Phi(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\int d^3r'\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$=\frac{1}{4\pi\varepsilon_0}\int d^3r'\rho(\mathbf{r}')\left(\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^{\,l}}{r_>^{\,l+1}}Y_{lm}(\theta,\varphi)Y_{lm}^*(\theta',\varphi')\right)$$

Example -- continued

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\rho(\mathbf{r}')\left(\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^{\,l}}{r_>^{\,l+1}}Y_{lm}(\theta,\varphi)Y_{lm}^{\;*}(\theta',\varphi')\right)$$

$$\text{Suppose:}\quad \rho(\mathbf{r}') = \frac{Q}{a^3\pi^{3/2}}e^{-r'^2/a^2}$$

$$\int d\Omega'\,Y_{lm}^{\;*}(\theta',\varphi') = \sqrt{4\pi}\,\delta_{l0}\delta_{m0}$$

$$\Phi(\mathbf{r}) = \frac{4\pi}{4\pi\varepsilon_0}\int_0^\infty r'^2\,dr'\frac{r_<^{\,0}}{r_>^{\,1}}\frac{Q}{a^3\pi^{3/2}}e^{-r'^2/a^2}$$

$$= \frac{Q}{4\pi\varepsilon_0}\frac{\mathrm{erf}(r/a)}{r}$$

Useful identity:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_<^{\,l}}{r_>^{\,l+1}} Y_{lm}(\theta,\varphi) Y_{lm}^{\ *}(\theta',\varphi')$$

Elements of "proof":

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r_>\left(1+\left(\frac{r_<}{r_>}\right)^2 - 2\left(\frac{r_<}{r_>}\right)\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'\right)^{1/2}} =$$

$$= \frac{1}{r_>}\left(1+\left(\frac{r_<}{r_>}\right)\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}' + \left(\frac{r_<}{r_>}\right)^2\left(\frac{3}{2}(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')^2 - \frac{1}{2}\right) + \cdots\right)$$

$$= \frac{1}{r_>}\left(\sum_{l=0}^{\infty}\left(\frac{r_<}{r_>}\right)^l P_l(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')\right)$$

Useful identity:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^{\,l}}{r_>^{\,l+1}}Y_{lm}(\theta,\varphi)Y_{lm}^{\;*}(\theta',\varphi')$$

Elements of "proof" -- continued:

Sum rule for spherical harmonics:

$$P_l(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')=\frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y_{lm}(\hat{\mathbf{r}})\,Y_{lm}^{\;*}(\hat{\mathbf{r}}')$$

Note that for $\hat{\mathbf{r}}=\hat{\mathbf{r}}'$, $P_l(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')=1$

$$\Rightarrow\frac{4\pi}{2l+1}\sum_{m=-l}^{l}|Y_{lm}(\hat{\mathbf{r}})|^2=1$$

Some spherical harmonic functions:

$$Y_{00}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1(\pm 1)}(\hat{\mathbf{r}}) = \mp\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{\pm i\phi}$$

$$Y_{10}(\hat{\mathbf{r}}) = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_{2(\pm 2)}(\hat{\mathbf{r}}) = \sqrt{\frac{15}{32\pi}} \sin^2\theta \, e^{\pm 2i\phi}$$

$$Y_{2(\pm 1)}(\hat{\mathbf{r}}) = \mp\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \, e^{\pm i\phi}$$

$$Y_{20}(\hat{\mathbf{r}}) = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$