

PHY 712 Electrodynamics

10-10:50 AM MWF Online

Plan for Lecture 8:

Finish reading Chap. 3 and start Chap. 4

**Multipole moment expansion of
electrostatic potential –**

A. Spherical coordinates

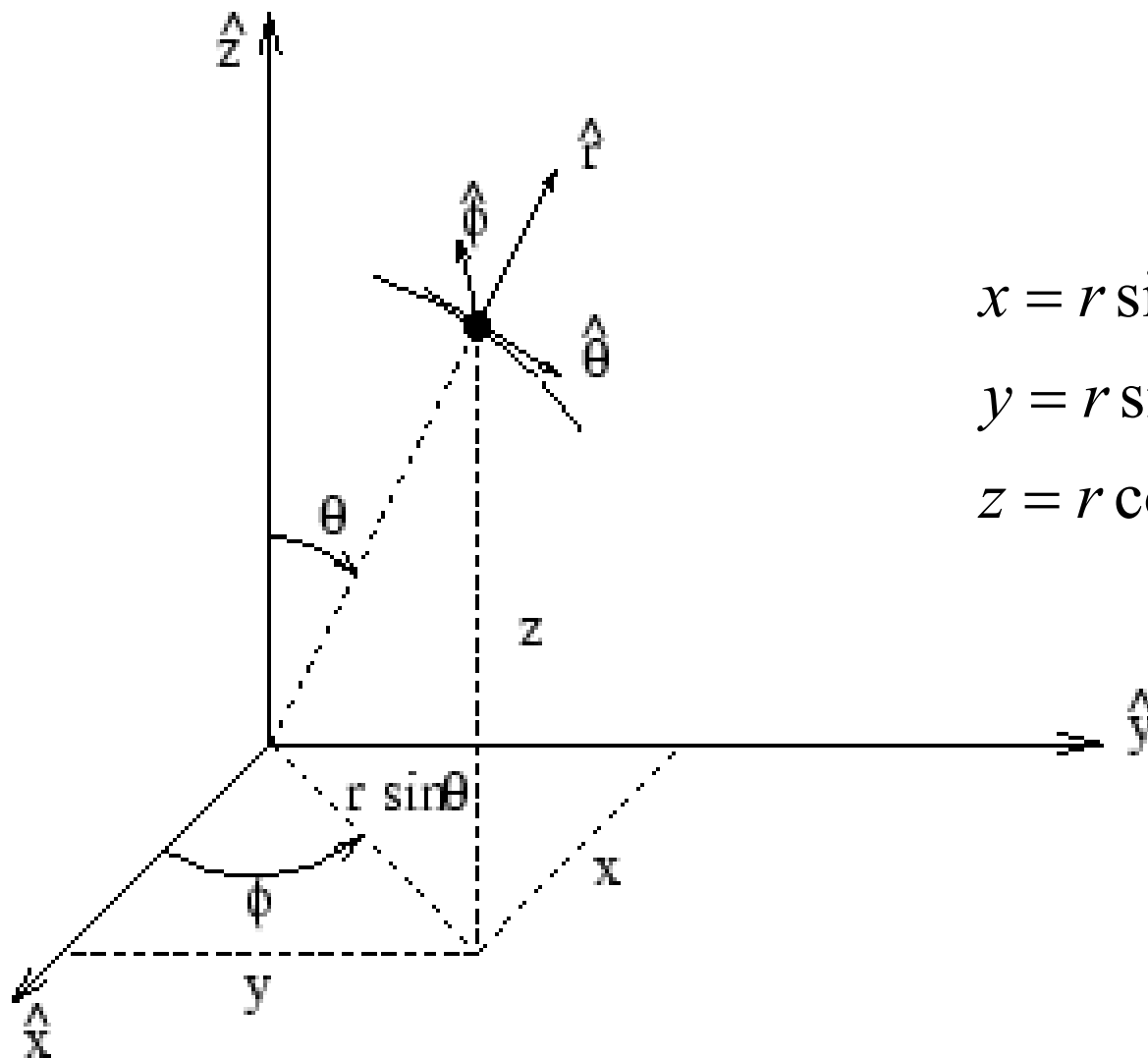
B. Cartesian coordinates

Course schedule for Spring 2021

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Wed: 01/27/2021	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/29/2021
2	Fri: 01/29/2021	Chap. 1	Electrostatic energy calculations	#2	02/01/2021
3	Mon: 02/01/2021	Chap. 1 & 2	Electrostatic potentials and fields	#3	02/03/2021
4	Wed: 02/03/2021	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#4	02/05/2021
5	Fri: 02/05/2021	Chap. 1 - 3	Brief introduction to numerical methods	#5	02/08/2021
6	Mon: 02/08/2021	Chap. 2 & 3	Image charge constructions	#6	02/10/2021
7	Wed: 02/10/2021	Chap. 2 & 3	Cylindrical and spherical geometries		
8	Fri: 02/12/2021	Chap. 3 & 4	Spherical geometry and multipole moments	#7	02/15/2021
9	Mon: 02/15/2021	Chap. 4	Dipoles and Dielectrics		
10	Wed: 02/17/2021	Chap. 4	Polarization and Dielectrics		
11	Fri: 02/19/2021	Chap. 5	Magnetostatics		
12	Mon: 02/22/2021	Chap. 5	Magnetic dipoles and hyperfine interaction		
13	Wed: 02/24/2021	Chap. 5	Magnetic dipoles and dipolar fields		

Poisson and Laplace equation in spherical polar coordinates



$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

<http://www.uic.edu/classes/eecs/eecs520/textbook/node32.html>

Poisson and Laplace equation in spherical polar coordinates -- continued

Laplace equation for electrostatic potential $\Phi(r, \theta, \varphi)$:

$$\frac{1}{r} \frac{\partial^2}{\partial r^2} (r\Phi) + \frac{1}{r^2} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right) \Phi = 0$$

$$\Phi(r, \theta, \varphi) = \sum_{lm} R_{lm}(r) Y_{lm}(\theta, \varphi)$$

Spherical harmonic functions:

$$\left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right) Y_{lm}(\theta, \varphi) = -l(l+1) Y_{lm}(\theta, \varphi)$$

Properties of spherical harmonic functions

$$Y_{lm}(\theta, \varphi) = (-1)^m Y_{l(-m)}^*(\theta, \varphi) \quad (\text{standard Condon-Shortley convention})$$

$$\int d\Omega Y_{lm}(\theta, \varphi) Y_{l'm'}^*(\theta, \varphi) \equiv \int \sin \theta d\theta d\varphi Y_{lm}(\theta, \varphi) Y_{l'm'}^*(\theta, \varphi) = \delta_{ll'} \delta_{mm'}$$

Completeness:

$$\sum_{lm} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi') = \delta(\hat{\mathbf{r}} - \hat{\mathbf{r}}') \equiv \delta(\cos \theta - \cos \theta') \delta(\varphi - \varphi')$$

Relationship to Legendre polynomials:

$$Y_{l0}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos \theta)$$

Relationship to Associated Legendre polynomials:

$$Y_{lm}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos \theta) e^{im\varphi}$$

Legendre and Associated Legendre functions

Legendre differential equation :

$$\left(\frac{d}{dx} \left((1-x^2) \frac{d}{dx} \right) + l(l+1) \right) P_l(x) = 0$$

Associated Legendre differential equation :

$$\left(\frac{d}{dx} \left((1-x^2) \frac{d}{dx} \right) + l(l+1) - \frac{m^2}{1-x^2} \right) P_l^m(x) = 0$$

For $m \geq 0$

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \left(\frac{d^m}{dx^m} P_l(x) \right)$$

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x)$$

Useful identity:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi')$$

Example for isolated charge density $\rho(\mathbf{r})$ with electrostatic potential vanishing for $r \rightarrow \infty$:

$$\begin{aligned} \Phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \\ &= \frac{1}{4\pi\epsilon_0} \int d^3r' \rho(\mathbf{r}') \left(\sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi') \right) \end{aligned}$$

Some spherical harmonic functions:

$$Y_{00}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1(\pm 1)}(\hat{\mathbf{r}}) = \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi}$$

$$Y_{10}(\hat{\mathbf{r}}) = \sqrt{\frac{3}{4\pi}} \cos \theta$$

$$Y_{2(\pm 2)}(\hat{\mathbf{r}}) = \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{\pm 2i\phi}$$

$$Y_{2(\pm 1)}(\hat{\mathbf{r}}) = \mp \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{\pm i\phi}$$

$$Y_{20}(\hat{\mathbf{r}}) = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$$

General form of electrostatic potential with boundary value

$r \rightarrow \infty$, for isolated charge density $\rho(\mathbf{r})$:

$$\begin{aligned}\Phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \\ &= \frac{1}{4\pi\epsilon_0} \int d^3r' \rho(\mathbf{r}') \left(\sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi') \right)\end{aligned}$$

Suppose that $\rho(\mathbf{r}) = \sum_{lm} \rho_{lm}(r) Y_{lm}(\theta, \varphi)$

$$\Rightarrow \Phi(\mathbf{r}) = \sum_{lm} F_{lm}(r) Y_{lm}(\theta, \varphi) \quad \text{where}$$

$$F_{lm}(r) = \frac{1}{\epsilon_0} \frac{1}{2l+1} \left(\frac{1}{r^{l+1}} \int_0^r r'^{2+l} dr' \rho_{lm}(r') + r^l \int_r^\infty r'^{1-l} dr' \rho_{lm}(r') \right)$$

In summary:

$$\Phi(\mathbf{r}) = \frac{1}{\epsilon_0} \sum_{lm} \frac{1}{2l+1} Y_{lm}(\theta, \varphi) \left(\frac{1}{r^{l+1}} \int_0^r r'^{2+l} dr' \rho_{lm}(r') + r^l \int_r^\infty r'^{1-l} dr' \rho_{lm}(r') \right)$$

Example:

$$\text{Suppose } \rho(\mathbf{r}) = \begin{cases} \frac{qr \sin \theta \cos \varphi}{Va} = \frac{qr}{Va} \left(\frac{1}{2} \sqrt{\frac{8\pi}{3}} (Y_{1-1}(\theta, \varphi) - Y_{11}(\theta, \varphi)) \right) & r \leq a \\ 0 & r > a \end{cases}$$

$$\Phi(\mathbf{r}) = \frac{1}{\epsilon_0} \sum_{lm} \frac{1}{2l+1} Y_{lm}(\theta, \varphi) \left(\frac{1}{r^{l+1}} \int_0^r r'^{2+l} dr' \rho_{lm}(r') + r^l \int_r^\infty r'^{1-l} dr' \rho_{lm}(r') \right)$$

For $r \leq a$

$$\Phi(\mathbf{r}) = \frac{q}{Va\epsilon_0} \left(\frac{1}{6} \sqrt{\frac{8\pi}{3}} (Y_{1-1}(\theta, \varphi) - Y_{11}(\theta, \varphi)) \right) \left(\frac{1}{r^2} \int_0^r r'^4 dr' + r \int_r^a r' dr' \right)$$

For $r > a$

$$\Phi(\mathbf{r}) = \frac{q}{Va\epsilon_0} \left(\frac{1}{6} \sqrt{\frac{8\pi}{3}} (Y_{1-1}(\theta, \varphi) - Y_{11}(\theta, \varphi)) \right) \left(\frac{1}{r^2} \int_0^a r'^4 dr' \right)$$

Example -- continued:

$$\text{Suppose } \rho(\mathbf{r}) = \begin{cases} \frac{qx}{Va} = \frac{qr}{Va} \left(\frac{1}{2} \sqrt{\frac{8\pi}{3}} (Y_{1-1}(\theta, \varphi) - Y_{11}(\theta, \varphi)) \right) & r \leq a \\ 0 & r > a \end{cases}$$

For $r \leq a$

$$\begin{aligned} \Phi(\mathbf{r}) &= \frac{q}{Va\epsilon_0} \left(\frac{1}{6} \sqrt{\frac{8\pi}{3}} (Y_{1-1}(\theta, \varphi) - Y_{11}(\theta, \varphi)) \right) \left(\frac{1}{r^2} \int_0^r r'^4 dr' + r \int_r^a r' dr' \right) \\ &= \frac{q}{6Va\epsilon_0} \sin \theta \cos \varphi \left(r \left(a^2 - \frac{3}{5} r^2 \right) \right) \end{aligned}$$

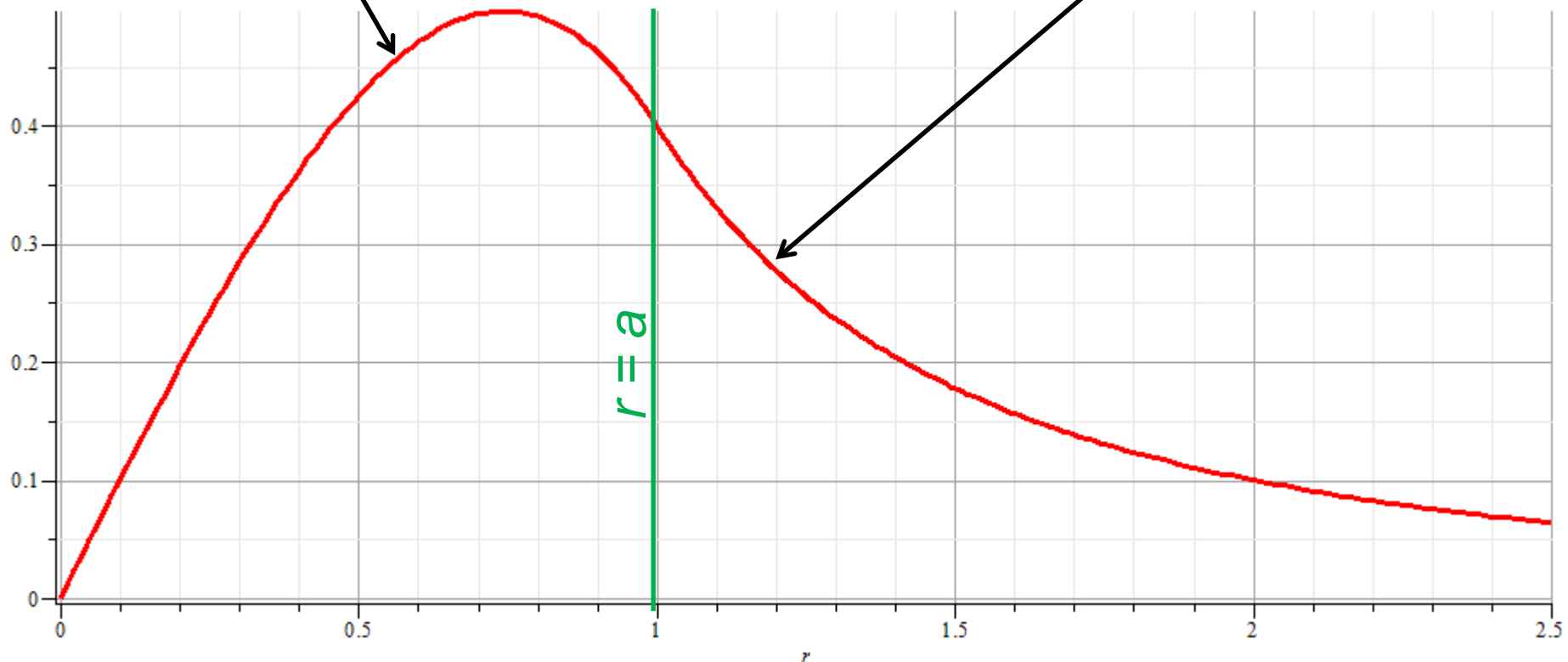
For $r > a$

$$\begin{aligned} \Phi(\mathbf{r}) &= \frac{q}{Va\epsilon_0} \left(\frac{1}{6} \sqrt{\frac{8\pi}{3}} (Y_{1-1}(\theta, \varphi) - Y_{11}(\theta, \varphi)) \right) \left(\frac{1}{r^2} \int_0^a r'^4 dr' \right) \\ &= \frac{q}{6Va\epsilon_0} \sin \theta \cos \varphi \left(\frac{\frac{2}{5} a^5}{r^2} \right) \end{aligned}$$

Example -- continued:

For $r \leq a$: $\Phi(\mathbf{r}) = \frac{q}{6Va\epsilon_0} \sin \theta \cos \varphi \left(r \left(a^2 - \frac{3}{5} r^2 \right) \right)$

For $r > a$: $\Phi(\mathbf{r}) = \frac{q}{6V\epsilon_0} \sin \theta \cos \varphi \left(\frac{\frac{2}{5} a^5}{r^2} \right) = \frac{qa^5}{15V\epsilon_0} \frac{x}{r^3}$



Notion of multipole moment:

In the spherical harmonic representation --

define the moment q_{lm} of the (confined) charge distribution $\rho(\mathbf{r})$:

$$q_{lm} \equiv \int d^3 r' r'^l Y_{lm}^*(\theta', \varphi') \rho(\mathbf{r}')$$

In the Cartesian representation --

define the monopole moment q :

$$q \equiv \int d^3 r' \rho(\mathbf{r}')$$

define the dipole moment $\mathbf{p}$:

$$\mathbf{p} \equiv \int d^3 r' \mathbf{r}' \rho(\mathbf{r}')$$

define the quadrupole moment components Q_{ij} ($i, j \rightarrow x, y, z$):

$$Q_{ij} \equiv \int d^3 r' (3r'_i r'_j - r'^2 \delta_{ij}) \rho(\mathbf{r}')$$

Significance of multipole moments

Recall general form of electrostatic potential with boundary value $r \rightarrow \infty$, for isolated charge density $\rho(\mathbf{r})$:

$$\begin{aligned}\Phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \\ &= \frac{1}{4\pi\epsilon_0} \int d^3r' \rho(\mathbf{r}') \left(\sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi') \right)\end{aligned}$$

For r outside the extent of $\rho(\mathbf{r})$:

$$\begin{aligned}\Phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \sum_{lm} \frac{4\pi}{2l+1} \frac{Y_{lm}(\theta, \varphi)}{r^{l+1}} \underbrace{\left(\int_0^\infty d^3r' r'^l Y_{lm}^*(\theta', \varphi') \rho(\mathbf{r}') \right)}_{q_{lm}} \\ &= \frac{1}{4\pi\epsilon_0} \sum_{lm} \frac{4\pi q_{lm}}{2l+1} \frac{Y_{lm}(\theta, \varphi)}{r^{l+1}}\end{aligned}$$

Multipole moments continued:

For r outside the extent of $\rho(\mathbf{r})$:

$$q_{lm} = \int_0^\infty d^3r' r'^l Y_{lm}^*(\theta', \phi') \rho(\mathbf{r}')$$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{lm} \frac{4\pi q_{lm}}{2l+1} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}}$$

Relationship between spherical harmonic and Cartesian forms of multipole moments:

$$q_{00} = \sqrt{\frac{1}{4\pi}} q$$

$$q_{2\pm 2} = \sqrt{\frac{15}{288\pi}} (Q_{xx} \mp 2iQ_{xy} - Q_{yy})$$

$$q_{1\pm 1} = \mp \sqrt{\frac{3}{8\pi}} (p_x \mp ip_y)$$

$$q_{2\pm 1} = \mp \sqrt{\frac{15}{72\pi}} (Q_{xz} \mp iQ_{yz})$$

$$q_{10} = \sqrt{\frac{3}{4\pi}} p_z$$

$$q_{20} = \sqrt{\frac{5}{16\pi}} Q_{zz}$$

Consider previous example:

$$\rho(\mathbf{r}) = \begin{cases} \frac{qx}{Va} = \frac{qr}{Va} \left(\frac{1}{2} \sqrt{\frac{8\pi}{3}} (Y_{1-1}(\theta, \varphi) - Y_{11}(\theta, \varphi)) \right) & r \leq a \\ 0 & r > a \end{cases}$$

We previously showed that for $r > a$

$$\begin{aligned} \Phi(\mathbf{r}) &= \frac{q}{Va\epsilon_0} \left(\frac{1}{6} \sqrt{\frac{8\pi}{3}} (Y_{1-1}(\theta, \varphi) - Y_{11}(\theta, \varphi)) \right) \left(\frac{1}{r^2} \int_0^a r'^4 dr' \right) \\ &= \frac{q}{Va\epsilon_0} \left(\frac{1}{6} \sqrt{\frac{8\pi}{3}} (Y_{1-1}(\theta, \varphi) - Y_{11}(\theta, \varphi)) \right) \frac{a^5}{5r^2} = \frac{q}{6V\epsilon_0} \sin\theta \cos\varphi \left(\frac{2a^5}{5r^2} \right) \end{aligned}$$

Note that: $q_{1\pm 1} = \mp \frac{q}{Va} \frac{1}{2} \sqrt{\frac{8\pi}{3}} \frac{a^5}{5}$

$$p_x = \frac{1}{2} \sqrt{\frac{8\pi}{3}} (-q_{11} + q_{1-1}) = \frac{4\pi}{3} \frac{q}{Va} \frac{a^5}{5}$$

General form of electrostatic potential in terms of multipole moments:

For r outside the extent of $\rho(\mathbf{r})$:

$$\begin{aligned}\Phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \sum_{lm} \frac{4\pi}{2l+1} \frac{Y_{lm}(\theta, \varphi)}{r^{l+1}} \left(\int d^3r' r'^l Y_{lm}^*(\theta', \varphi') \rho(\mathbf{r}') \right) \\ &= \frac{1}{4\pi\epsilon_0} \sum_{lm} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta, \varphi)}{r^{l+1}}\end{aligned}$$

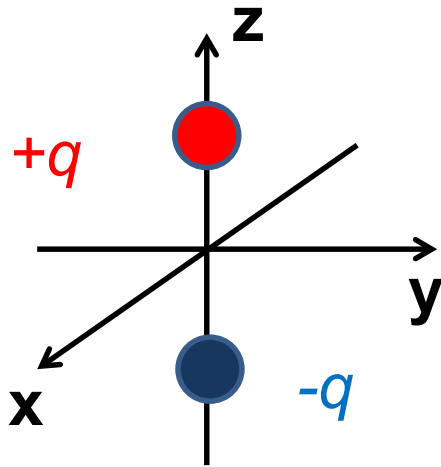
In terms of Cartesian expansion :

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r} + \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} + \frac{1}{2} \sum_{i,j} Q_{ij} \frac{r_i r_j}{r^5} \dots \right)$$

Example of multipole expansion in evaluating energy of a very localized charge density $\rho(\mathbf{r})$ in a electrostatic field $\Phi(\mathbf{r})$ (such as an nucleus in the field produced by electrons in an atom).

$$\begin{aligned} W &= \int d^3r \rho(\mathbf{r})\Phi(\mathbf{r}) \\ &\approx \int d^3r \rho(\mathbf{r}) \left(\Phi(0) + \mathbf{r} \cdot \nabla \Phi(\mathbf{r}) \Big|_{r=0} + \frac{1}{2} (\mathbf{r} \cdot \nabla)^2 \Phi(\mathbf{r}) \Big|_{r=0} + \dots \right) \\ &= q\Phi(0) - \mathbf{p} \cdot \mathbf{E}(0) + \frac{1}{6} \sum_{i,j} Q_{ij} \frac{\partial^2 \Phi(0)}{\partial r_i \partial r_j} + \dots \end{aligned}$$

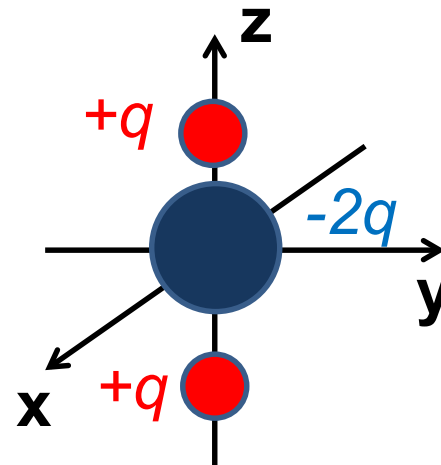
Simple examples of multipole distributions



$$\rho(\mathbf{r}) = q(\delta^3(\mathbf{r} - d\hat{\mathbf{z}}) - \delta^3(\mathbf{r} + d\hat{\mathbf{z}}))$$

$$p_z = 2qd$$

$$p_x = p_y = 0$$



$$\rho(\mathbf{r}) = q(\delta^3(\mathbf{r} - d\hat{\mathbf{z}}) + \delta^3(\mathbf{r} + d\hat{\mathbf{z}}) - 2\delta^3(\mathbf{r}))$$

$$Q_{zz} = 4qd^2 = -2Q_{xx} = -2Q_{yy}$$

Another example of multipole distribution

$$\rho(\mathbf{r}) = \frac{q}{64\pi a^3} \left(\frac{r}{a}\right)^2 e^{-r/a} \sin^2 \theta$$

Note that : $\sqrt{\frac{4\pi}{5}} Y_{20}(\theta, \phi) = \frac{3}{2} \cos^2 \theta - \frac{1}{2} = 1 - \frac{3}{2} \sin^2 \theta$

$$\sin^2 \theta = \frac{2}{3} - \frac{2}{3} \sqrt{\frac{4\pi}{5}} Y_{20}(\theta, \phi) = \frac{2}{3} \sqrt{\frac{4\pi}{1}} Y_{00}(\theta, \phi) - \frac{2}{3} \sqrt{\frac{4\pi}{5}} Y_{20}(\theta, \phi)$$

$$\Rightarrow \rho(\mathbf{r}) = \rho_{00}(r) Y_{00}(\theta, \phi) + \rho_{20}(r) Y_{20}(\theta, \phi)$$

$$\Phi(\mathbf{r}) = \Phi_{00}(r) Y_{00}(\theta, \phi) + \Phi_{20}(r) Y_{20}(\theta, \phi)$$

$$\Phi_{lm} = \frac{1}{4\pi\epsilon_0} \frac{4\pi}{2l+1} \left(\frac{1}{r^{l+1}} \int_0^r r'^{2+l} dr' \rho_{lm}(r') + r^l \int_r^\infty r'^{1-l} dr' \rho_{lm}(r') \right)$$

$$\rho_{00}(r) = \frac{2}{3} \sqrt{4\pi} \frac{q}{64\pi a^3} \left(\frac{r}{a}\right)^2 e^{-r/a} \quad \rho_{20}(r) = -\frac{2}{3} \sqrt{\frac{4\pi}{5}} \frac{q}{64\pi a^3} \left(\frac{r}{a}\right)^2 e^{-r/a}$$

Another example of multipole distribution -- continued

$$\Phi_{00}(r) = \frac{1}{4\pi\epsilon_0} \sqrt{4\pi} \frac{q}{r} \left(1 - e^{-r/a} \left(1 + \frac{3r}{4a} + \frac{r^2}{4a^2} + \frac{r^3}{24a^3} \right) \right)$$

$$\Phi_{20}(r) = -\frac{6}{4\pi\epsilon_0} \sqrt{\frac{4\pi}{5}} \frac{qa^2}{r^3} \left(1 - e^{-r/a} \left(1 + \frac{r}{a} + \frac{r^2}{2a^2} + \frac{r^3}{6a^3} + \frac{r^4}{24a^3} + \frac{r^5}{144a^5} \right) \right)$$

For $r \rightarrow \infty$; in terms for Legendre polynomials :

$$\Phi(\mathbf{r}) \rightarrow \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r} - \frac{6a^2}{r^3} P_2(\cos \theta) \right) \qquad Y_{l0}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos \theta)$$

For $r \rightarrow 0$; in terms for Legendre polynomials :

$$\Phi(\mathbf{r}) \rightarrow \frac{q}{4\pi\epsilon_0} \left(\frac{1}{4a} - \frac{r^2}{120a^3} P_2(\cos \theta) \right)$$

Another example of multipole distribution -- continued

For $r \rightarrow 0$; in terms for Legendre polynomials :

$$\Phi(\mathbf{r}) \rightarrow \frac{q}{4\pi\epsilon_0} \left(\frac{1}{4a} - \frac{r^2}{120a^3} P_2(\cos \theta) \right)$$

Implications for electric quadrupole interaction :

$$W = \frac{1}{6} \sum_{i,j} Q_{ij} \frac{\partial^2 \Phi(0)}{\partial r_i \partial r_j} + \dots \quad P_2(\cos \theta) = \frac{3}{2} \cos^2 \theta - \frac{1}{2} = \frac{1}{2r^2} (3z^2 - r^2) \\ = \frac{1}{2r^2} (2z^2 - x^2 - y^2)$$

For $r \rightarrow 0$; in terms of Cartesian coordinates

$$\Phi(\mathbf{r}) \rightarrow \frac{q}{4\pi\epsilon_0} \left(\frac{1}{4a} - \frac{2z^2 - x^2 - y^2}{240a^3} \right)$$

$$\frac{\partial^2 \Phi(0)}{\partial x^2} = \frac{\partial^2 \Phi(0)}{\partial y^2} = -\frac{1}{2} \frac{\partial^2 \Phi(0)}{\partial z^2} = \frac{q}{4\pi\epsilon_0} \frac{1}{120a^3}$$

Another example of multipole distribution -- continued

Electric quadrupole interaction :

$$W = \frac{1}{6} \sum_{i,j} Q_{ij} \frac{\partial^2 \Phi(0)}{\partial r_i \partial r_j} = \frac{1}{6} \left(Q_{xx} \frac{\partial^2 \Phi(0)}{\partial x^2} + Q_{yy} \frac{\partial^2 \Phi(0)}{\partial y^2} + Q_{zz} \frac{\partial^2 \Phi(0)}{\partial z^2} \right)$$

For symmetric nuclei, $Q_{zz} \equiv Qq = -\frac{1}{2}Q_{xx} = -\frac{1}{2}Q_{yy}$

$$W \approx -\frac{q^2}{4\pi\epsilon_0} \frac{Q}{240a^3}$$