

PHY 712 Electrodynamics

10-10:50 AM MWF Online

Plan for Lecture 9:

Continue reading Chapter 4

Dipolar fields and dielectrics

A. Electric field due to a dipole

B. Electric polarization $\mathbf{P}$

C. Electric displacement $\mathbf{D}$ and dielectric functions

Course schedule for Spring 2021

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Wed: 01/27/2021	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/29/2021
2	Fri: 01/29/2021	Chap. 1	Electrostatic energy calculations	#2	02/01/2021
3	Mon: 02/01/2021	Chap. 1 & 2	Electrostatic potentials and fields	#3	02/03/2021
4	Wed: 02/03/2021	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#4	02/05/2021
5	Fri: 02/05/2021	Chap. 1 - 3	Brief introduction to numerical methods	#5	02/08/2021
6	Mon: 02/08/2021	Chap. 2 & 3	Image charge constructions	#6	02/10/2021
7	Wed: 02/10/2021	Chap. 2 & 3	Cylindrical and spherical geometries		
8	Fri: 02/12/2021	Chap. 3 & 4	Spherical geometry and multipole moments	#7	02/15/2021
9	Mon: 02/15/2021	Chap. 4	Dipoles and Dielectrics	#8	02/17/2021
10	Wed: 02/17/2021	Chap. 4	Polarization and Dielectrics		
11	Fri: 02/19/2021	Chap. 5	Magnetostatics		
12	Mon: 02/22/2021	Chap. 5	Magnetic dipoles and hyperfine interaction		
13	Wed: 02/24/2021	Chap. 5	Magnetic dipoles and hyperfine interaction		

Review: General results for a multipole analysis of the electrostatic potential due to an isolated charge distribution:

General form of electrostatic potential with boundary value $\Phi(r \rightarrow \infty) = 0$ for confined charge density $\rho(\mathbf{r})$:

$$\begin{aligned}\Phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \\ &= \frac{1}{4\pi\epsilon_0} \int d^3r' \rho(\mathbf{r}') \left(\sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi') \right)\end{aligned}$$

Suppose that $\rho(\mathbf{r}) = \sum_{lm} \rho_{lm}(r) Y_{lm}(\theta, \varphi)$

$$\Rightarrow \Phi(\mathbf{r}) = \frac{1}{\epsilon_0} \sum_{lm} \frac{1}{2l+1} Y_{lm}(\theta, \varphi) \left(\frac{1}{r^{l+1}} \int_0^r r'^{l+2} dr' \rho_{lm}(r') + r^l \int_r^\infty r'^{l-1} dr' \rho_{lm}(r') \right)$$

For $r \rightarrow \infty$:
$$\Phi(\mathbf{r}) = \frac{1}{\epsilon_0} \sum_{lm} \frac{1}{2l+1} Y_{lm}(\theta, \varphi) \underbrace{\frac{1}{r^{l+1}} \int_0^\infty r'^{l+2} dr' \rho_{lm}(r')}_{q_{lm}}$$

The the multipole analysis has the following general behavior for $r \rightarrow \text{infinity}$:

For r outside the extent of $\rho(\mathbf{r})$:

$$\begin{aligned}\Phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \sum_{lm} \frac{4\pi}{2l+1} \frac{Y_{lm}(\theta, \varphi)}{r^{l+1}} \left(\int d^3r' r'^l Y_{lm}^*(\theta', \varphi') \rho(\mathbf{r}') \right) \\ &= \frac{1}{4\pi\epsilon_0} \sum_{lm} \frac{4\pi}{2l+1} \frac{q_{lm} Y_{lm}(\theta, \varphi)}{r^{l+1}}\end{aligned}\quad q_{lm} \equiv \int_0^\infty r'^{2+l} dr' \rho_{lm}(r')$$

In terms of Cartesian expansion :

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r} + \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} + \frac{1}{2} \sum_{i,j} Q_{ij} \frac{r_i r_j}{r^5} \dots \right)$$

Here q , p_i , and Q_{ij} are linearly proportional to the q_{lm} multipole values.

The multipole analysis also can be used to analyze the the electrostatic fields for $r \rightarrow 0$ as needed in the following example involving a very localized charge density $\rho(\mathbf{r})$ in a electrostatic field $\Phi(\mathbf{r})$ (such as a nucleus in the field produced by electrons in an atom).

charge density
within nucleus



electrostatic potential
due to electrons near
the nucleus.



$$W = \int d^3r \rho(\mathbf{r})\Phi(\mathbf{r})$$

$$\approx \int d^3r \rho(\mathbf{r}) \left(\Phi(0) + \mathbf{r} \cdot \nabla \Phi(\mathbf{r}) \Big|_{r=0} + \frac{1}{2} (\mathbf{r} \cdot \nabla)^2 \Phi(\mathbf{r}) \Big|_{r=0} + \dots \right)$$

$$= q\Phi(0) - \mathbf{p} \cdot \mathbf{E}(0) + \frac{1}{6} \sum_{i,j} Q_{ij} \frac{\partial^2 \Phi(0)}{\partial r_i \partial r_j} + \dots$$

Consider the following example of a charge distribution

$$\rho(\mathbf{r}) = \frac{q}{64\pi a^3} \left(\frac{r}{a}\right)^2 e^{-r/a} \sin^2 \theta$$

Note that : $\sqrt{\frac{4\pi}{5}} Y_{20}(\theta, \phi) = \frac{3}{2} \cos^2 \theta - \frac{1}{2} = 1 - \frac{3}{2} \sin^2 \theta$

$$\sin^2 \theta = \frac{2}{3} - \frac{2}{3} \sqrt{\frac{4\pi}{5}} Y_{20}(\theta, \phi) = \frac{2}{3} \sqrt{\frac{4\pi}{1}} Y_{00}(\theta, \phi) - \frac{2}{3} \sqrt{\frac{4\pi}{5}} Y_{20}(\theta, \phi)$$

$$\Rightarrow \rho(\mathbf{r}) = \rho_{00}(r) Y_{00}(\theta, \phi) + \rho_{20}(r) Y_{20}(\theta, \phi)$$

$$\Phi(\mathbf{r}) = \Phi_{00}(r) Y_{00}(\theta, \phi) + \Phi_{20}(r) Y_{20}(\theta, \phi)$$

$$\Phi_{lm} = \frac{1}{4\pi\epsilon_0} \frac{4\pi}{2l+1} \left(\frac{1}{r^{l+1}} \int_0^r r'^{2+l} dr' \rho_{lm}(r') + r^l \int_r^\infty r'^{1-l} dr' \rho_{lm}(r') \right)$$

$$\rho_{00}(r) = \frac{2}{3} \sqrt{4\pi} \frac{q}{64\pi a^3} \left(\frac{r}{a}\right)^2 e^{-r/a} \quad \rho_{20}(r) = -\frac{2}{3} \sqrt{\frac{4\pi}{5}} \frac{q}{64\pi a^3} \left(\frac{r}{a}\right)^2 e^{-r/a}$$

Writing out the details of the potential from evaluating integrals

$$\Phi_{00}(r) = \frac{1}{4\pi\epsilon_0} \sqrt{4\pi} \frac{q}{r} \left(1 - e^{-r/a} \left(1 + \frac{3r}{4a} + \frac{r^2}{4a^2} + \frac{r^3}{24a^3} \right) \right)$$

$$\Phi_{20}(r) = -\frac{6}{4\pi\epsilon_0} \sqrt{\frac{4\pi}{5}} \frac{qa^2}{r^3} \left(1 - e^{-r/a} \left(1 + \frac{r}{a} + \frac{r^2}{2a^2} + \frac{r^3}{6a^3} + \frac{r^4}{24a^3} + \frac{r^5}{144a^5} \right) \right)$$

For $r \rightarrow \infty$; in terms for Legendre polynomials:

$$\Phi(\mathbf{r}) \rightarrow \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r} - \frac{6a^2}{r^3} P_2(\cos \theta) \right) \qquad Y_{l0}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos \theta)$$

For $r \rightarrow 0$; in terms for Legendre polynomials :

$$\Phi(\mathbf{r}) \rightarrow \frac{q}{4\pi\epsilon_0} \left(\frac{1}{4a} - \frac{r^2}{120a^3} P_2(\cos \theta) \right)$$

More details continued --

For $r \rightarrow 0$; in terms for Legendre polynomials :

$$\Phi(\mathbf{r}) \rightarrow \frac{q}{4\pi\epsilon_0} \left(\frac{1}{4a} - \frac{r^2}{120a^3} P_2(\cos \theta) \right)$$

Implications for electric quadrupole interaction :

$$W = \frac{1}{6} \sum_{i,j} Q_{ij} \frac{\partial^2 \Phi(0)}{\partial r_i \partial r_j} + \dots \quad P_2(\cos \theta) = \frac{3}{2} \cos^2 \theta - \frac{1}{2} = \frac{1}{2r^2} (3z^2 - r^2) \\ = \frac{1}{2r^2} (2z^2 - x^2 - y^2)$$

For $r \rightarrow 0$; in terms of Cartesian coordinates

$$\Phi(\mathbf{r}) \rightarrow \frac{q}{4\pi\epsilon_0} \left(\frac{1}{4a} - \frac{2z^2 - x^2 - y^2}{240a^3} \right)$$

$$\frac{\partial^2 \Phi(0)}{\partial x^2} = \frac{\partial^2 \Phi(0)}{\partial y^2} = -\frac{1}{2} \frac{\partial^2 \Phi(0)}{\partial z^2} = \frac{q}{4\pi\epsilon_0} \frac{1}{120a^3}$$

Another example of multipole distribution -- continued

Electric quadrupole interaction:

$$W = \frac{1}{6} \sum_{i,j} Q_{ij} \frac{\partial^2 \Phi(0)}{\partial r_i \partial r_j} = \frac{1}{6} \left(Q_{xx} \frac{\partial^2 \Phi(0)}{\partial x^2} + Q_{yy} \frac{\partial^2 \Phi(0)}{\partial y^2} + Q_{zz} \frac{\partial^2 \Phi(0)}{\partial z^2} \right)$$

For symmetric nuclei, $Q_{zz} \equiv Q_q = -\frac{1}{2} Q_{xx} = -\frac{1}{2} Q_{yy}$

$$W \approx -\frac{q^2}{4\pi\epsilon_0} \frac{Q}{240a^3}$$

Here q stands for the elementary charge
 $q=e=1.602176634 \times 10^{-19} \text{C}$

Notion of multipole moment:

In the spherical harmonic representation --

define the moment q_{lm} of the (confined) charge distribution $\rho(\mathbf{r})$:

$$q_{lm} \equiv \int d^3 r' r'^l Y_{lm}^*(\theta', \phi') \rho(\mathbf{r}')$$

In the Cartesian representation --

define the monopole moment q :

$$q \equiv \int d^3 r' \rho(\mathbf{r}')$$

define the dipole moment $\mathbf{p}$:

$$\mathbf{p} \equiv \int d^3 r' \mathbf{r}' \rho(\mathbf{r}')$$

define the quadrupole moment components Q_{ij} ($i, j \rightarrow x, y, z$):

$$Q_{ij} \equiv \int d^3 r' (3r'_i r'_j - r'^2 \delta_{ij}) \rho(\mathbf{r}')$$

General form of electrostatic potential in terms of multipole moments:

For r outside the extent of $\rho(\mathbf{r})$:

$$\begin{aligned}\Phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \sum_{lm} \frac{4\pi}{2l+1} \frac{Y_{lm}(\theta, \varphi)}{r^{l+1}} \left(\int d^3r' r'^l Y_{lm}^*(\theta', \varphi') \rho(\mathbf{r}') \right) \\ &= \frac{1}{4\pi\epsilon_0} \sum_{lm} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta, \varphi)}{r^{l+1}}\end{aligned}$$

In terms of Cartesian expansion :

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r} + \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} + \frac{1}{2} \sum_{i,j} Q_{ij} \frac{r_i r_j}{r^5} \dots \right)$$

Focus on dipolar contributions:

For r outside the extent of $\rho(\mathbf{r})$:

Electrostatic potential:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{\mathbf{p} \cdot \mathbf{r}}{r^3} \right)$$

Electrostatic field:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\underbrace{\frac{3\mathbf{r} (\mathbf{p} \cdot \mathbf{r}) - r^2 \mathbf{p}}{r^5}}_{\text{Poorly defined for } r \rightarrow 0} - \underbrace{\frac{4\pi}{3} \mathbf{p} \delta^3(\mathbf{r})}_{\text{Correct value for } r \rightarrow 0} \right)$$

Poorly defined for $r \rightarrow 0$

Correct value for $r \rightarrow 0$

“Justification” of surprising δ -function term in dipole electric field -- Assuming dipole is located at $r=0$, we need to need to evaluate the electrostatic field near $r=0$:

We will use the approximation:

$$\mathbf{E}(\mathbf{r} \approx \mathbf{0}) \approx \left(\int_{\text{sphere}} \mathbf{E}(\mathbf{r}) d^3 r \right) \delta^3(\mathbf{r}).$$

First we note that:

$$\int_{r \leq R} \mathbf{E}(\mathbf{r}) d^3 r = -R^2 \int_{r=R} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\Omega.$$

Some details:

$$\int_{r \leq R} \mathbf{E}(\mathbf{r}) d^3 r = -R^2 \int_{r=R} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\Omega.$$

This result follows from the divergence theorem:

$$\int_{\text{vol}} \nabla \cdot \mathcal{V} d^3 r = \int_{\text{surface}} \mathcal{V} \cdot d\mathbf{A}.$$

In our case, this theorem can be used for each cartesian coordinate if we choose $\mathcal{V} \equiv \hat{\mathbf{x}}\Phi(\mathbf{r})$ for the x component, etc.

$$\int_{r \leq R} \nabla \Phi(\mathbf{r}) d^3 r = \hat{\mathbf{x}} \int_{r \leq R} \nabla \cdot (\hat{\mathbf{x}}\Phi) d^3 r + \hat{\mathbf{y}} \int_{r \leq R} \nabla \cdot (\hat{\mathbf{y}}\Phi) d^3 r + \hat{\mathbf{z}} \int_{r \leq R} \nabla \cdot (\hat{\mathbf{z}}\Phi) d^3 r,$$

which is equal to:

$$\int_{r=R} \Phi(\mathbf{r}) R^2 d\Omega ((\hat{\mathbf{x}} \cdot \hat{\mathbf{r}})\hat{\mathbf{x}} + (\hat{\mathbf{y}} \cdot \hat{\mathbf{r}})\hat{\mathbf{y}} + (\hat{\mathbf{z}} \cdot \hat{\mathbf{r}})\hat{\mathbf{z}}) = \int_{r=R} \Phi(\mathbf{r}) R^2 d\Omega \hat{\mathbf{r}}.$$

Therefore --

$$\int_{r \leq R} \mathbf{E}(\mathbf{r}) d^3 r = - \int_{r \leq R} \nabla \Phi(\mathbf{r}) d^3 r = -R^2 \int_{r=R} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\Omega.$$

More details

$$\int_{r \leq R} \mathbf{E}(\mathbf{r}) d^3 r = -R^2 \int_{r=R} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\Omega.$$

Now, we notice that the electrostatic potential can be determined from the charge density $\rho(\mathbf{r})$ according to:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3 r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{4\pi\epsilon_0} \sum_{lm} \frac{4\pi}{2l+1} \int d^3 r' \rho(\mathbf{r}') \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}^*(\hat{\mathbf{r}}) Y_{lm}(\hat{\mathbf{r}}').$$

We also note that the unit vector can be written in terms of spherical harmonic functions:

$$\hat{\mathbf{r}} = \begin{cases} \sin(\theta) \cos(\phi) \hat{\mathbf{x}} + \sin(\theta) \sin(\phi) \hat{\mathbf{y}} + \cos(\theta) \hat{\mathbf{z}} \\ \sqrt{\frac{4\pi}{3}} \left(Y_{1-1}(\hat{\mathbf{r}}) \frac{\hat{\mathbf{x}} + i\hat{\mathbf{y}}}{\sqrt{2}} + Y_{11}(\hat{\mathbf{r}}) \frac{-\hat{\mathbf{x}} + i\hat{\mathbf{y}}}{\sqrt{2}} + Y_{10}(\hat{\mathbf{r}}) \hat{\mathbf{z}} \right) \end{cases}$$

$$\begin{aligned} \int_{r=R} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\Omega &= \frac{1}{3\epsilon_0} \int d^3 r' \rho(\mathbf{r}') \frac{r_{<}}{r_{>}^2} \sqrt{\frac{4\pi}{3}} \left(Y_{1-1}(\hat{\mathbf{r}}') \frac{\hat{\mathbf{x}} + i\hat{\mathbf{y}}}{\sqrt{2}} + Y_{11}(\hat{\mathbf{r}}') \frac{-\hat{\mathbf{x}} + i\hat{\mathbf{y}}}{\sqrt{2}} + Y_{10}(\hat{\mathbf{r}}') \hat{\mathbf{z}} \right) \\ &= \frac{1}{3\epsilon_0} \int d^3 r' \rho(\mathbf{r}') \frac{r_{<}}{r_{>}^2} \hat{\mathbf{r}}' \end{aligned}$$

More details continued --

When we evaluate the integral over solid angle $d\Omega$, only the $l = 1$ terms contribute, and the result of the integration reduces to:

$$-R^2 \int_{r=R} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\Omega = -\frac{1}{4\pi\epsilon_0} \frac{4\pi R^2}{3} \int d^3 r' \rho(\mathbf{r}') \frac{r_{<}}{r_{>}^2} \hat{\mathbf{r}}'.$$

The choice of $r_{<}$ and $r_{>}$ is a choice between the integration variables r' and the sphere radius R . If the sphere encloses the charge distribution, $\rho(\mathbf{r}')$, then $r_{<} = r'$ and $r_{>} = R$ so that the result is:

$$-R^2 \int_{r=R} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\Omega = -\frac{1}{4\pi\epsilon_0} \frac{4\pi R^2}{3} \frac{1}{R^2} \int d^3 r' \rho(\mathbf{r}') r' \hat{\mathbf{r}}' \equiv -\frac{\mathbf{p}}{3\epsilon_0}.$$

Otherwise, if the charge distribution $\rho(\mathbf{r}')$ lies outside of the sphere, then $r_{<} = R$ and $r_{>} = r'$ and the result is:

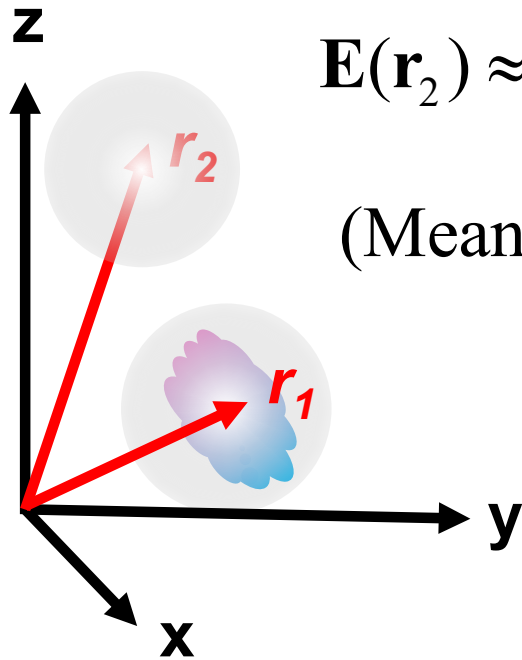
$$-R^2 \int_{r=R} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\Omega = -\frac{1}{4\pi\epsilon_0} \frac{4\pi R^2}{3} R \int d^3 r' \frac{\rho(\mathbf{r}')}{r'^2} \hat{\mathbf{r}}' \equiv \frac{4\pi R^3}{3} \mathbf{E}(0).$$

In summary --

Electrostatic dipolar field for dipole moment $\mathbf{p}$ at $\mathbf{r}=0$:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{3\mathbf{r} (\mathbf{p} \cdot \mathbf{r}) - r^2 \mathbf{p}}{r^5} - \frac{4\pi}{3} \mathbf{p} \delta^3(\mathbf{r}) \right)$$

Summary of key argument:



$$\mathbf{E}(\mathbf{r}_2) \approx \frac{3}{4\pi R^3} \int_{r \leq R} d^3 r \mathbf{E}(\mathbf{r}_2 + \mathbf{r}) = \mathbf{E}(\mathbf{r}_2)$$

(Mean value theorem for Laplace equation)

$$\begin{aligned} \mathbf{E}(\mathbf{r}_1) &\approx \frac{3}{4\pi R^3} \int_{r \leq R} d^3 r \mathbf{E}(\mathbf{r}_1 + \mathbf{r}) \\ &\approx \frac{3}{4\pi R^3} \left(-\frac{\mathbf{p}}{3\epsilon_0} \right) \Rightarrow -\frac{\mathbf{p}}{3\epsilon_0} \delta^3(\mathbf{r} - \mathbf{r}_1) \end{aligned}$$

Summary:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{3\mathbf{r} (\mathbf{p} \cdot \mathbf{r}) - r^2 \mathbf{p}}{r^5} - \frac{4\pi}{3} \mathbf{p} \delta^3(\mathbf{r}) \right)$$

Coarse grain representation of macroscopic distribution of dipoles:

Electric polarization $\mathbf{P}(\mathbf{r})$ due to collection of dipoles :

$$\mathbf{P}(\mathbf{r}) \equiv \sum_i \mathbf{p}_i \delta^3(\mathbf{r} - \mathbf{r}_i)$$

Monopole electric charge density $\rho_{\text{mono}}(\mathbf{r})$:

$$\rho_{\text{mono}}(\mathbf{r}) \equiv \sum_i q_i \delta^3(\mathbf{r} - \mathbf{r}_i)$$

Electrostatic potential for a single monopole charge q
and a single dipole $\mathbf{p}$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r} + \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} \right)$$

Coarse grain representation of macroscopic distribution of dipoles -- continued:

Electrostatic potential for a single monopole charge q
and a single dipole $\mathbf{p}$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r} + \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} \right)$$

Electrostatic potential for collections of monopole charges q_i
and dipoles $\mathbf{p}_i$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\int d^3r' \frac{\rho_{mono}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \int d^3r' \frac{\mathbf{P}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \right)$$

Note:
$$\int d^3r' \frac{\mathbf{P}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} = \int d^3r' \mathbf{P}(\mathbf{r}') \cdot \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} = - \int d^3r' \frac{\nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

Coarse grain representation of macroscopic distribution of dipoles -- continued:

Electrostatic potential for collections of monopole charges q_i and dipoles $\mathbf{p}_i$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\int d^3r' \frac{\rho_{mono}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} - \int d^3r' \frac{\nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right)$$

$$-\nabla^2 \Phi(\mathbf{r}) = \nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{1}{\epsilon_0} (\rho_{mono}(\mathbf{r}) - \nabla \cdot \mathbf{P}(\mathbf{r}))$$

$$\Rightarrow \nabla \cdot (\epsilon_0 \mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r})) = \rho_{mono}(\mathbf{r})$$

Define Displacement field: $\mathbf{D}(\mathbf{r}) \equiv \epsilon_0 \mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r})$

Macroscopic form of Gauss's law: $\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho_{mono}(\mathbf{r})$

Coarse grain representation of macroscopic distribution of dipoles -- continued:

Many materials are polarizable and produce a polarization field in the presence of an electric field with a proportionality constant χ_e :

$$\mathbf{P}(\mathbf{r}) = \varepsilon_0 \chi_e \mathbf{E}(\mathbf{r})$$

$$\mathbf{D}(\mathbf{r}) \equiv \varepsilon_0 \mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r}) = \varepsilon_0 (1 + \chi_e) \mathbf{E}(\mathbf{r}) \equiv \varepsilon \mathbf{E}(\mathbf{r})$$

ε represents the dielectric function of the material

Boundary value problems in dielectric materials

$$\text{For } \rho_{\text{mono}}(\mathbf{r}) = 0$$

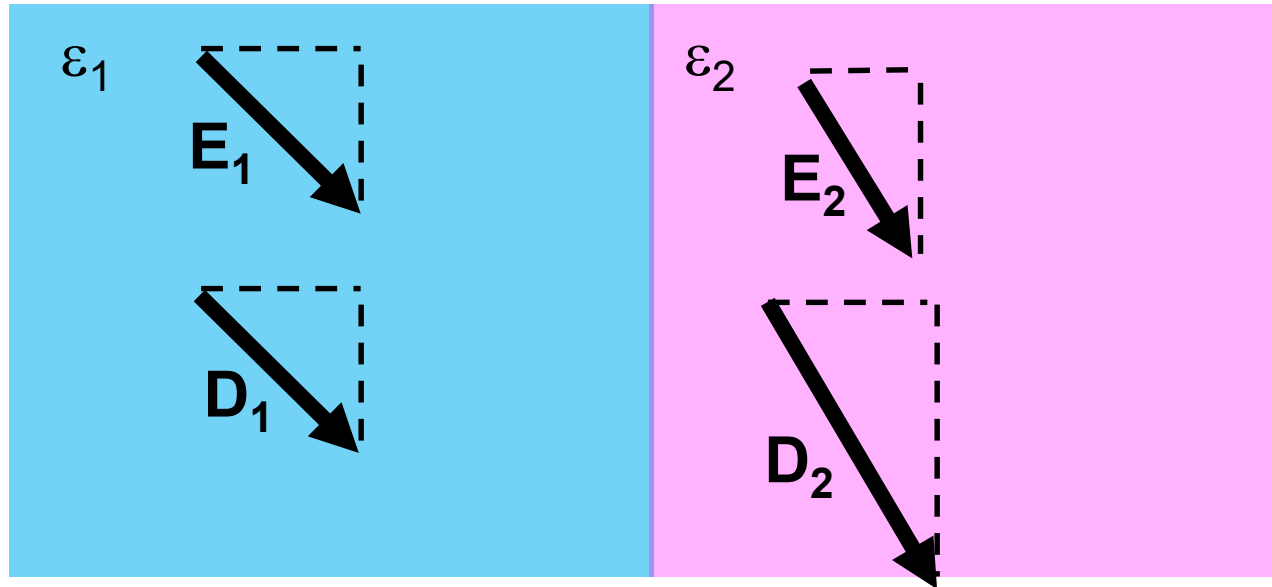
$$\nabla \cdot \mathbf{D}(\mathbf{r}) = 0 \quad \text{and} \quad \nabla \times \mathbf{E}(\mathbf{r}) = 0$$

$\Rightarrow$ At a surface between two dielectrics, in terms of surface normal $\hat{\mathbf{r}}$:

$$\hat{\mathbf{r}} \cdot \mathbf{D}(\mathbf{r}) = \text{continuous} = \hat{\mathbf{r}} \times \mathbf{E}(\mathbf{r})$$

Boundary value problems in the presence of dielectrics

– example:



$$\text{For } \frac{\epsilon_2}{\epsilon_1} = 2$$

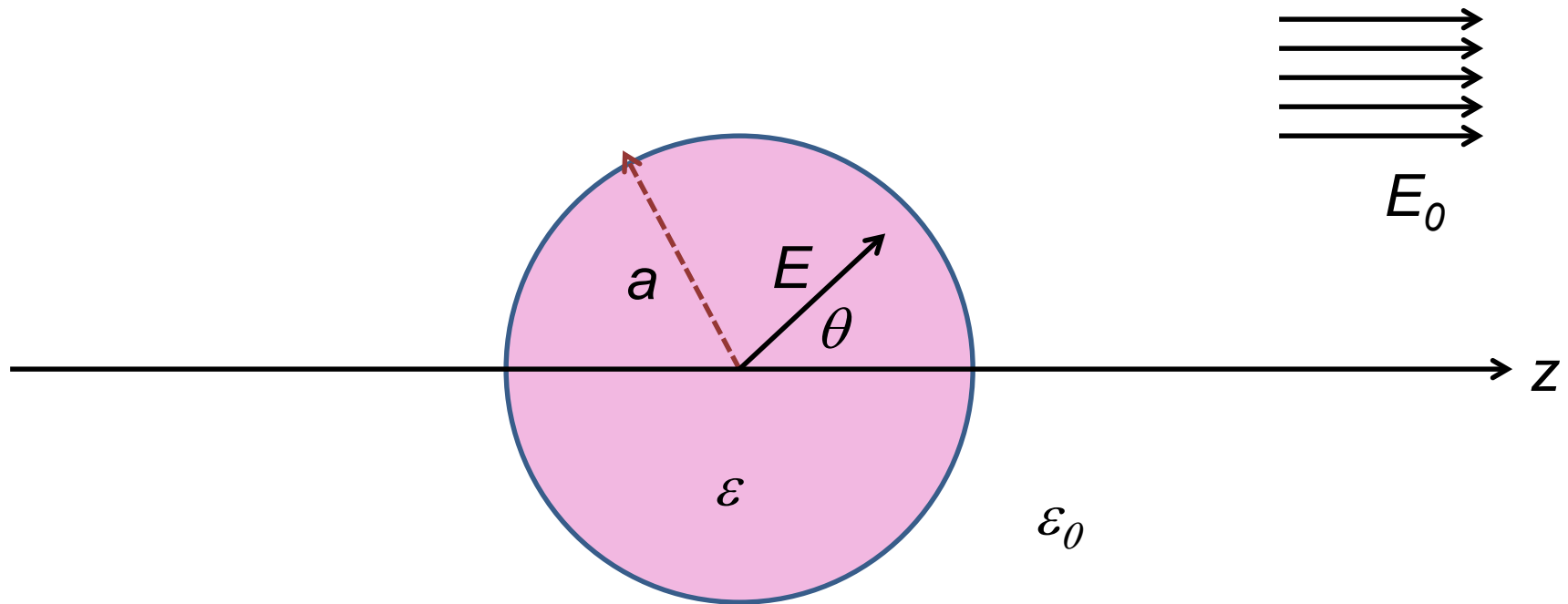
For isotropic dielectrics:

$$D_{1n} = D_{2n} \quad \epsilon_1 E_{1n} = \epsilon_2 E_{2n}$$

$$E_{1t} = E_{2t} \quad \frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2}$$

Boundary value problems in the presence of dielectrics

– example:



$$\nabla \cdot \mathbf{D}(\mathbf{r}) = 0 \quad \text{and} \quad \nabla \times \mathbf{E}(\mathbf{r}) = 0 \quad \text{At } r = a: \quad \epsilon \frac{\partial \Phi_{<}(\mathbf{r})}{\partial r} = \epsilon_0 \frac{\partial \Phi_{>}(\mathbf{r})}{\partial r}$$

$$\text{For } r \leq a \quad \mathbf{D}(\mathbf{r}) = -\epsilon \nabla \Phi(\mathbf{r})$$

$$\text{For } r > a \quad \mathbf{D}(\mathbf{r}) = -\epsilon_0 \nabla \Phi(\mathbf{r})$$

$$\frac{\partial \Phi_{<}(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_{>}(\mathbf{r})}{\partial \theta}$$

Boundary value problems in the presence of dielectrics

– example -- continued:

$$\Phi_{<}(\mathbf{r}) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta)$$

$$\Phi_{>}(\mathbf{r}) = \sum_{l=0}^{\infty} \left(B_l r^l + \frac{C_l}{r^{l+1}} \right) P_l(\cos \theta)$$

$$\text{At } r = a : \quad \varepsilon \frac{\partial \Phi_{<}(\mathbf{r})}{\partial r} = \varepsilon_0 \frac{\partial \Phi_{>}(\mathbf{r})}{\partial r}$$

$$\frac{\partial \Phi_{<}(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_{>}(\mathbf{r})}{\partial \theta}$$

$$\text{For } r \rightarrow \infty \quad \Phi_{>}(\mathbf{r}) = -E_0 r \cos \theta$$

Solution -- only $l = 1$ contributes

$$B_1 = -E_0$$

$$A_1 = -\left(\frac{3}{2 + \varepsilon / \varepsilon_0} \right) E_0$$

$$C_1 = \left(\frac{\varepsilon / \varepsilon_0 - 1}{2 + \varepsilon / \varepsilon_0} \right) a^3 E_0$$

Boundary value problems in the presence of dielectrics

– example -- continued:

$$\Phi_{<}(\mathbf{r}) = -\left(\frac{3}{2 + \epsilon / \epsilon_0}\right) E_0 r \cos \theta$$

$$\Phi_{>}(\mathbf{r}) = -\left(r - \left(\frac{\epsilon / \epsilon_0 - 1}{2 + \epsilon / \epsilon_0}\right) \frac{a^3}{r^2}\right) E_0 \cos \theta$$

