PHY 712 Electrodynamics
10-10:50 AM MWF Online

Class for Lecture 9:

Continue reading Chapter 4

Dipolar fields and dielectrics

A. Electric field due to a dipole

B. Electric polarization P

C. Electric displacement D and dielectric functions

Course schedule for Spring 2021

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Wed: 01/27/2021	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/29/2021
2	Fri: 01/29/2021	Chap. 1	Electrostatic energy calculations	#2	02/01/2021
3	Mon: 02/01/2021	Chap. 1 & 2	Electrostatic potentials and fields	#3	02/03/2021
4	Wed: 02/03/2021	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#4	02/05/2021
5	Fri: 02/05/2021	Chap. 1 - 3	Brief introduction to numerical methods	#5	02/08/2021
6	Mon: 02/08/2021	Chap. 2 & 3	Image charge constructions	#6	02/10/2021
7	Wed: 02/10/2021	Chap. 2 & 3	Cylindrical and spherical geometries		
8	Fri: 02/12/2021	Chap. 3 & 4	Spherical geometry and multipole moments	#7	02/15/2021
9	Mon: 02/15/2021	Chap. 4	Dipoles and Dielectrics	#8	02/17/2021
10	Wed: 02/17/2021	Chap. 4	Polarization and Dielectrics		
11	Fri: 02/19/2021	Chap. 5	Magnetostatics		
12	Mon: 02/22/2021	Chap. 5	Magnetic dipoles and hyperfine interaction		

Your questions –

From Gao -- About boundary value, assuming the screen is the boundary surface, and for E1t=E2t, if E1t is in x direction and E2t is in y direction, of course E1t and E2t are in the same plane, the screen surface. Will the equation E1t=E2t still be correct?

Review: General results for a multipole analysis of the electrostatic potential due to an isolated charge distribution:

General form of electrostatic potential with boundary value $\Phi(r \to \infty) = 0$

for confined charge density $\rho(\mathbf{r})$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$= \frac{1}{4\pi\varepsilon_0}\int d^3r' \rho(\mathbf{r}')\left(\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^{\,l}}{r_>^{\,l+1}}Y_{lm}(\theta,\varphi)Y_{lm}^*(\theta',\varphi')\right)$$

Suppose that $\rho(\mathbf{r}) = \sum_{lm}\rho_{lm}(r)Y_{lm}(\theta,\varphi)$

$$\Rightarrow \Phi(\mathbf{r}) = \frac{1}{\varepsilon_0}\sum_{lm}\frac{1}{2l+1}Y_{lm}(\theta,\varphi)\left(\frac{1}{r^{l+1}}\int_0^r r'^{2+l}\,dr'\rho_{lm}(r') + r^l\int_r^\infty r'^{1-l}\,dr'\rho_{lm}(r')\right)$$

For $r \to \infty$: $\Phi(\mathbf{r}) = \frac{1}{\varepsilon_0}\sum_{lm}\frac{1}{2l+1}Y_{lm}(\theta,\varphi)\frac{1}{r^{l+1}}\underbrace{\int_0^\infty r'^{2+l}\,dr'\rho_{lm}(r')}_{q_{lm}}$

Comment --

Acts like a projection operator

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$= \frac{1}{4\pi\varepsilon_0}\int d^3r' \rho(\mathbf{r}')\left(\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^{\,l}}{r_>^{\,l+1}} Y_{lm}(\theta,\varphi)Y_{lm}{}^*(\theta',\varphi')\right)$$

Suppose that $\rho(\mathbf{r}') = \sum_{lm}\rho_{lm}(r')Y_{lm}(\theta',\varphi')$

$$\Rightarrow \Phi(\mathbf{r}) = \frac{1}{\varepsilon_0}\sum_{lm}\frac{1}{2l+1}Y_{lm}(\theta,\varphi)\left(\frac{1}{r^{l+1}}\int_0^r r'^{2+l}\,dr'\rho_{lm}(r') + r^l\int_r^\infty r'^{1-l}\,dr'\rho_{lm}(r')\right)$$

Why? -- Recall that

$$\int d\Omega' Y_{lm}{}^*(\theta',\varphi')Y_{\lambda\mu}(\theta',\varphi') = \delta_{l\lambda,m\mu}$$

The the multipole analysis has the following general behavior for r→infinity:

For r outside the extent of $\rho(\mathbf{r})$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{lm}\frac{4\pi}{2l+1}\frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}\left(\int d^3r' r'^l\, Y_{lm}^*(\theta',\varphi')\rho(\mathbf{r}')\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\sum_{lm}\frac{4\pi\, q_{lm}}{2l+1}\frac{Y_{lm}(\theta,\varphi)}{r^{l+1}} \qquad q_{lm} \equiv \int_0^\infty r'^{2+l}\, dr'\, \rho_{lm}(r')$$

In terms of Cartesian expansion :

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{r^3} + \frac{1}{2}\sum_{i,j} Q_{ij}\frac{r_i r_j}{r^5}\cdots\right)$$

Here q, p_i, and Q_{ij} are linearly proportional to the q_{lm} multipole values.

The multipole analysis also can be used to analyze the the electrostatic fields for r→0 as needed in the following example involving a very localized charge density $\rho(\mathbf{r})$ in a electrostatic field $\Phi(\mathbf{r})$ (such as a nucleus in the field produced by electrons in an atom).

charge density within nucleus

electrostatic potential due to electrons near the nucleus.

$$W = \int d^3r\, \rho(\mathbf{r})\Phi(\mathbf{r})$$

$$\approx \int d^3r\, \rho(\mathbf{r})\left(\Phi(0) + \mathbf{r}\cdot\nabla\Phi(\mathbf{r})\big|_{r=0} + \frac{1}{2}(\mathbf{r}\cdot\nabla)^2\Phi(\mathbf{r})\big|_{r=0} + \cdots \right)$$

$$= q\Phi(0) - \mathbf{p}\cdot\mathbf{E}(0) + \frac{1}{6}\sum_{i,j} Q_{ij}\,\frac{\partial^2\Phi(0)}{\partial r_i \partial r_j} + \cdots$$

Consider the following example of a charge distribution

$$\rho(\mathbf{r}) = \frac{q}{64\pi a^3}\left(\frac{r}{a}\right)^2 e^{-r/a}\sin^2\theta$$

Note that : $\sqrt{\frac{4\pi}{5}}Y_{20}(\theta,\phi) = \frac{3}{2}\cos^2\theta - \frac{1}{2} = 1 - \frac{3}{2}\sin^2\theta$

$$\sin^2\theta = \frac{2}{3} - \frac{2}{3}\sqrt{\frac{4\pi}{5}}Y_{20}(\theta,\phi) = \frac{2}{3}\sqrt{\frac{4\pi}{1}}Y_{00}(\theta,\phi) - \frac{2}{3}\sqrt{\frac{4\pi}{5}}Y_{20}(\theta,\phi)$$

$$\Rightarrow \rho(\mathbf{r}) = \rho_{00}(r)Y_{00}(\theta,\phi) + \rho_{20}(r)Y_{20}(\theta,\phi)$$

$$\Phi(\mathbf{r}) = \Phi_{00}(r)Y_{00}(\theta,\phi) + \Phi_{20}(r)Y_{20}(\theta,\phi)$$

$$\Phi_{lm} = \frac{1}{4\pi\varepsilon_0}\frac{4\pi}{2l+1}\left(\frac{1}{r^{l+1}}\int_0^r r'^{2+l}\,dr'\,\rho_{lm}(r') + r^l\int_r^\infty r'^{1-l}\,dr'\rho_{lm}(r')\right)$$

$$\rho_{00}(r) = \frac{2}{3}\sqrt{4\pi}\,\frac{q}{64\pi a^3}\left(\frac{r}{a}\right)^2 e^{-r/a} \qquad \rho_{20}(r) = -\frac{2}{3}\sqrt{\frac{4\pi}{5}}\,\frac{q}{64\pi a^3}\left(\frac{r}{a}\right)^2 e^{-r/a}$$

Writing out the details of the potential from evaluating integrals

$$\Phi_{00}(r) = \frac{1}{4\pi\varepsilon_0}\sqrt{4\pi}\,\frac{q}{r}\left(1 - e^{-r/a}\left(1 + \frac{3r}{4a} + \frac{r^2}{4a^2} + \frac{r^3}{24a^3}\right)\right)$$

$$\Phi_{20}(r) = -\frac{6}{4\pi\varepsilon_0}\sqrt{\frac{4\pi}{5}}\,\frac{qa^2}{r^3}\left(1 - e^{-r/a}\left(1 + \frac{r}{a} + \frac{r^2}{2a^2} + \frac{r^3}{6a^3} + \frac{r^4}{24a^3} + \frac{r^5}{144a^5}\right)\right)$$

For $r \to \infty$; in terms for Legendre polynomials:

$$\Phi(\mathbf{r}) \to \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{6a^2}{r^3}P_2(\cos\theta)\right)$$

$$Y_{l0}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)$$

For $r \to 0$; in terms for Legendre polynomials:

$$\Phi(\mathbf{r}) \to \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{4a} - \frac{r^2}{120a^3}P_2(\cos\theta)\right)$$

More details continued --

For $\quad r \to 0$; in terms for Legendre polynomials :

$$\Phi(\mathbf{r}) \to \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{4a} - \frac{r^2}{120a^3} P_2(\cos\theta)\right)$$

Implications for electric quadrupole interaction :

$$W = \frac{1}{6}\sum_{i,j} Q_{ij} \frac{\partial^2 \Phi(0)}{\partial r_i \partial r_j} + \cdots$$

$$P_2(\cos\theta) = \tfrac{3}{2}\cos^2\theta - \tfrac{1}{2} = \tfrac{1}{2r^2}\left(3z^2 - r^2\right)$$
$$= \tfrac{1}{2r^2}\left(2z^2 - x^2 - y^2\right)$$

For $\quad r \to 0$; in terms of Cartesian coordinates

$$\Phi(\mathbf{r}) \to \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{4a} - \frac{2z^2 - x^2 - y^2}{240a^3}\right)$$

$$\frac{\partial^2\Phi(0)}{\partial x^2} = \frac{\partial^2\Phi(0)}{\partial y^2} = -\frac{1}{2}\frac{\partial^2\Phi(0)}{\partial z^2} = \frac{q}{4\pi\varepsilon_0}\frac{1}{120a^3}$$

Another example of multipole distribution -- continued

Electric quadrupole interaction:

$$W=\frac{1}{6}\sum_{i,j}Q_{ij}\frac{\partial^2\Phi(0)}{\partial r_i\partial r_j}=\frac{1}{6}\left(Q_{xx}\frac{\partial^2\Phi(0)}{\partial x^2}+Q_{yy}\frac{\partial^2\Phi(0)}{\partial y^2}+Q_{zz}\frac{\partial^2\Phi(0)}{\partial z^2}\right)$$

For symmetric nuclei, $Q_{zz}\equiv Qq=-\frac{1}{2}Q_{xx}=-\frac{1}{2}Q_{yy}$

$$W\approx-\frac{q^2}{4\pi\varepsilon_0}\frac{Q}{240a^3}$$

Here q stands for the elementary charge
$q=e=1.602176634\text{x}10^{-19}\text{C}$

Notion of multipole moment:

In the spherical harmonic representation --

define the moment q_{lm} of the (confined) charge distribution $\rho(\mathbf{r})$:

$$q_{lm} \equiv \int d^3r'\, r'^l\, Y^*_{lm}(\theta',\phi')\rho(\mathbf{r}')$$

In the Cartesian representation --

define the monopole moment q:

$$q \equiv \int d^3r'\, \rho(\mathbf{r}')$$

define the dipole moment $\mathbf{p}$:

$$\mathbf{p} \equiv \int d^3r'\, \mathbf{r}'\, \rho(\mathbf{r}')$$

define the quadrupole moment components Q_{ij} $(i,j \rightarrow x,y,z)$:

$$Q_{\mathrm{ij}} \equiv \int d^3r' \left(3r'_i\, r'_j - r'^2\, \delta_{ij}\right)\rho(\mathbf{r}')$$

General form of electrostatic potential in terms of multipole moments:

For r outside the extent of $\rho(\mathbf{r})$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{lm}\frac{4\pi}{2l+1}\frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}\left(\int d^3r'\, r'^l\, Y_{lm}^*(\theta',\varphi')\rho(\mathbf{r}')\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\sum_{lm}\frac{4\pi\, q_{lm}}{2l+1}\frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}$$

In terms of Cartesian expansion:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r}+\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}+\frac{1}{2}\sum_{i,j}Q_{ij}\frac{r_i r_j}{r^5}\cdots\right)$$

Focus on dipolar contributions:

For r outside the extent of $\rho(\mathbf{r})$:

Electrostatic potential:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right)$$

Electrostatic field:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{3\mathbf{r}\ (\mathbf{p}\cdot\mathbf{r}) - r^2\mathbf{p}}{r^5} - \frac{4\pi}{3}\mathbf{p}\delta^3(\mathbf{r})\right)$$

Poorly defined for $r \to 0$ | Correct value for $r \to 0$

“Justification” of surprizing δ-function term in dipole electric field -- Assuming dipole is located at *r=0*, we need to need to evaluate the electrostatic field near *r=0*:

We will use the approximation:

$$\mathbf{E}(\mathbf{r} \approx \mathbf{0}) \approx \left(\int_{\text{sphere}} \mathbf{E}(\mathbf{r}) d^3 r \right) \delta^3(\mathbf{r}).$$

First we note that:

$$\int_{r \leq R} \mathbf{E}(\mathbf{r}) d^3 r = -R^2 \int_{r=R} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\mathbf{\Omega}.$$

Some details:

$$\int_{r\le R}\mathbf{E}(\mathbf{r})d^3r = -R^2\int_{r=R}\Phi(\mathbf{r})\hat{\mathbf{r}}d\mathbf{\Omega}.$$

This result follows from the divergence theorem:

$$\int_{\text{vol}}\nabla\cdot\mathcal{V}d^3r = \int_{\text{surface}}\mathcal{V}\cdot d\mathbf{A}.$$

In our case, this theorem can be used for each cartesian coordinate if we choose $\mathcal{V}\equiv\hat{\mathbf{x}}\Phi(\mathbf{r})$ for the x component, etc.

$$\int_{r\le R}\nabla\Phi(\mathbf{r})d^3r = \hat{\mathbf{x}}\int_{r\le R}\nabla\cdot(\hat{\mathbf{x}}\Phi)d^3r + \hat{\mathbf{y}}\int_{r\le R}\nabla\cdot(\hat{\mathbf{y}}\Phi)d^3r + \hat{\mathbf{z}}\int_{r\le R}\nabla\cdot(\hat{\mathbf{z}}\Phi)d^3r,$$

which is equal to:

$$\int_{r=R}\Phi(\mathbf{r})R^2d\Omega\left((\hat{\mathbf{x}}\cdot\hat{\mathbf{r}})\hat{\mathbf{x}} + (\hat{\mathbf{y}}\cdot\hat{\mathbf{r}})\hat{\mathbf{y}} + (\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})\hat{\mathbf{z}}\right) = \int_{r=R}\Phi(\mathbf{r})R^2d\Omega\hat{\mathbf{r}}.$$

Therefore --

$$\int_{r\le R}\mathbf{E}(\mathbf{r})d^3r = -\int_{r\le R}\nabla\Phi(\mathbf{r})d^3r = -R^2\int_{r=R}\Phi(\mathbf{r})\hat{\mathbf{r}}d\mathbf{\Omega}.$$

More details

$$\int_{r\leq R} \mathbf{E}(\mathbf{r}) d^3r = -R^2 \int_{r=R} \Phi(\mathbf{r})\hat{\mathbf{r}} d\mathbf{\Omega}.$$

Now, we notice that the electrostatic potential can be determined from the charge density $\rho(\mathbf{r})$ according to:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{4\pi\epsilon_0}\sum_{lm}\frac{4\pi}{2l+1}\int d^3r' \rho(\mathbf{r}')\frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\hat{\mathbf{r}})Y_{lm}(\hat{\mathbf{r}}').$$

We also note that the unit vector can be written in terms of spherical harmonic functions:

$$\hat{\mathbf{r}} = \begin{cases} \sin(\theta)\cos(\phi)\hat{\mathbf{x}} + \sin(\theta)\sin(\phi)\hat{\mathbf{y}} + \cos(\theta)\hat{\mathbf{z}} \\ \sqrt{\frac{4\pi}{3}}\left(Y_{1-1}(\hat{\mathbf{r}})\frac{\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{\sqrt{2}} + Y_{11}(\hat{\mathbf{r}})\frac{-\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{\sqrt{2}} + Y_{10}(\hat{\mathbf{r}})\hat{\mathbf{z}} \right) \end{cases}$$

$$\int_{r=R} \Phi(\mathbf{r})\hat{\mathbf{r}} d\mathbf{\Omega} = \frac{1}{3\varepsilon_0}\int d^3r' \rho(\mathbf{r}')\frac{r_<}{r_>^2}\sqrt{\frac{4\pi}{3}}\left(Y_{1-1}(\hat{\mathbf{r}}')\frac{\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{\sqrt{2}} + Y_{11}(\hat{\mathbf{r}}')\frac{-\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{\sqrt{2}} + Y_{10}(\hat{\mathbf{r}}')\hat{\mathbf{z}} \right)$$

$$= \frac{1}{3\varepsilon_0}\int d^3r' \rho(\mathbf{r}')\frac{r_<}{r_>^2}\hat{\mathbf{r}}'$$

More details continued --

When we evaluate the integral over solid angle $d\Omega$, only the $l = 1$ terms contribute, and the result of the integration reduces to:

$$-R^2 \int_{r=R} \Phi(\mathbf{r})\hat{\mathbf{r}} d\boldsymbol{\Omega} = -\frac{1}{4\pi\epsilon_0}\frac{4\pi R^2}{3}\int d^3r' \rho(\mathbf{r}')\frac{r_<}{r_>^2}\hat{\mathbf{r}}'.$$

The choice of $r_<$ and $r_>$ is a choice between the integration variables r' and the sphere radius R. If the sphere encloses the charge distribution, $\rho(\mathbf{r}')$, then $r_< = r'$ and $r_> = R$ so that the result is:

$$-R^2 \int_{r=R} \Phi(\mathbf{r})\hat{\mathbf{r}} d\boldsymbol{\Omega} = -\frac{1}{4\pi\epsilon_0}\frac{4\pi R^2}{3}\frac{1}{R^2}\int d^3r' \rho(\mathbf{r}')r'\hat{\mathbf{r}}' \equiv -\frac{\mathbf{p}}{3\varepsilon_0}.$$

Otherwise, if the charge distribution $\rho(\mathbf{r}')$ lies outside of the sphere, then $r_< = R$ and $r_> = r'$ and the result is:

$$-R^2 \int_{r=R} \Phi(\mathbf{r})\hat{\mathbf{r}} d\boldsymbol{\Omega} = -\frac{1}{4\pi\epsilon_0}\frac{4\pi R^2}{3}R\int d^3r' \frac{\rho(\mathbf{r}')}{r'^2}\hat{\mathbf{r}}' \equiv \frac{4\pi R^3}{3}\mathbf{E}(0).$$

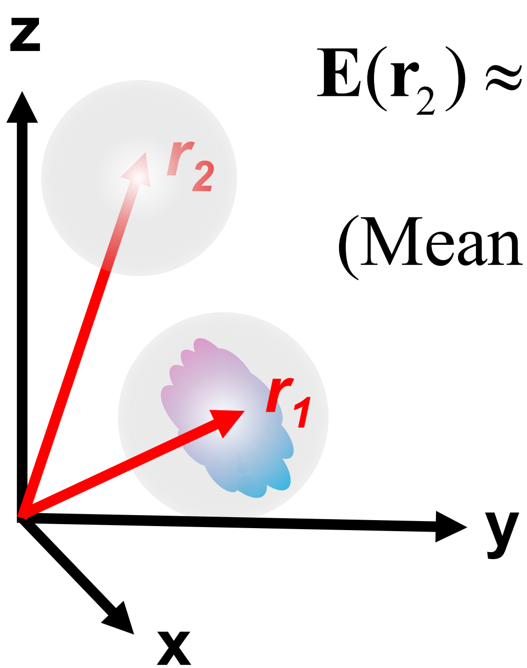
In summary --

Electrostatic dipolar field for dipole moment **p** at **r**=0:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{3\mathbf{r}\ (\mathbf{p}\cdot\mathbf{r}) - r^2\mathbf{p}}{r^5} - \frac{4\pi}{3}\mathbf{p}\delta^3(\mathbf{r})\right)$$

Note: The lecture ran out of time at this slide. The discussion will continue on Wednesday.

Summary of key argument:



$$\mathbf{E}(\mathbf{r}_2) \approx \frac{3}{4\pi R^3}\int_{r\le R} d^3r\ \mathbf{E}(\mathbf{r}_2+\mathbf{r}) = \mathbf{E}(\mathbf{r}_2)$$

(Mean value theorem for Laplace equation)

$$\mathbf{E}(\mathbf{r}_1) \approx \frac{3}{4\pi R^3}\int_{r\le R} d^3r\ \mathbf{E}(\mathbf{r}_1+\mathbf{r})$$

$$\approx \frac{3}{4\pi R^3}\left(-\frac{\mathbf{p}}{3\epsilon_0}\right) \Rightarrow -\frac{\mathbf{p}}{3\epsilon_0}\delta^3(\mathbf{r}-\mathbf{r}_1)$$

Summary:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{3\mathbf{r}\ (\mathbf{p}\cdot\mathbf{r}) - r^2\mathbf{p}}{r^5} - \frac{4\pi}{3}\mathbf{p}\delta^3(\mathbf{r})\right)$$

Coarse grain representation of macroscopic distribution of dipoles:

Electric polarization $\mathbf{P}(\mathbf{r})$ due to collection of dipoles :

$$\mathbf{P}(\mathbf{r}) \equiv \sum_i \mathbf{p}_i \delta^3 (\mathbf{r} - \mathbf{r}_i)$$

Monopole electric charge density $\rho_{\text{mono}}(\mathbf{r})$:

$$\rho_{\text{mono}}(\mathbf{r}) \equiv \sum_i q_i \delta^3 (\mathbf{r} - \mathbf{r}_i)$$

Electrostatic potential for a single monopole charge q

and a single dipole $\mathbf{p}$:

$$\Phi(\mathbf{r}) \quad = \frac{1}{4\pi\varepsilon_0} \left(\frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{r^3} \right)$$

Coarse grain representation of macroscopic distribution of dipoles -- continued:

Electrostatic potential for a single monopole charge q

and a single dipole $\mathbf{p}$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right)$$

Electrostatic potential for collections of monopole charges q_i

and dipoles $\mathbf{p}_i$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\int d^3r' \frac{\rho_{mono}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \int d^3r' \frac{\mathbf{P}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\right)$$

Note: $$\int d^3r' \frac{\mathbf{P}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = \int d^3r' \; \mathbf{P}(\mathbf{r}')\cdot\nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\int d^3r' \frac{\nabla'\cdot\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Coarse grain representation of macroscopic distribution of dipoles -- continued:

Electrostatic potential for collections of monopole charges q_i

and dipoles $\mathbf{p}_i$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\int d^3r' \frac{\rho_{mono}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} - \int d^3r' \frac{\nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right)$$

$$-\nabla^2\Phi(\mathbf{r}) = \nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{1}{\varepsilon_0}\left(\rho_{mono}(\mathbf{r}) - \nabla \cdot \mathbf{P}(\mathbf{r})\right)$$

$$\Rightarrow \nabla \cdot \left(\varepsilon_0 \mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r})\right) = \rho_{mono}(\mathbf{r})$$

Define Displacement field: $\mathbf{D}(\mathbf{r}) \equiv \varepsilon_0 \mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r})$

Macroscopic form of Gauss's law: $\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho_{mono}(\mathbf{r})$

Coarse grain representation of macroscopic distribution of dipoles -- continued:

Many materials are polarizable and produce a polarization field in the presence of an electric field with a proportionality constant χ_e :

$$\mathbf{P}(\mathbf{r}) = \varepsilon_0 \chi_e \mathbf{E}(\mathbf{r})$$

$$\mathbf{D}(\mathbf{r}) \equiv \varepsilon_0 \mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r}) = \varepsilon_0 (1 + \chi_e) \mathbf{E}(\mathbf{r}) \equiv \varepsilon \mathbf{E}(\mathbf{r})$$

ε represents the dielectric function of the material

Boundary value problems in dielectric materials

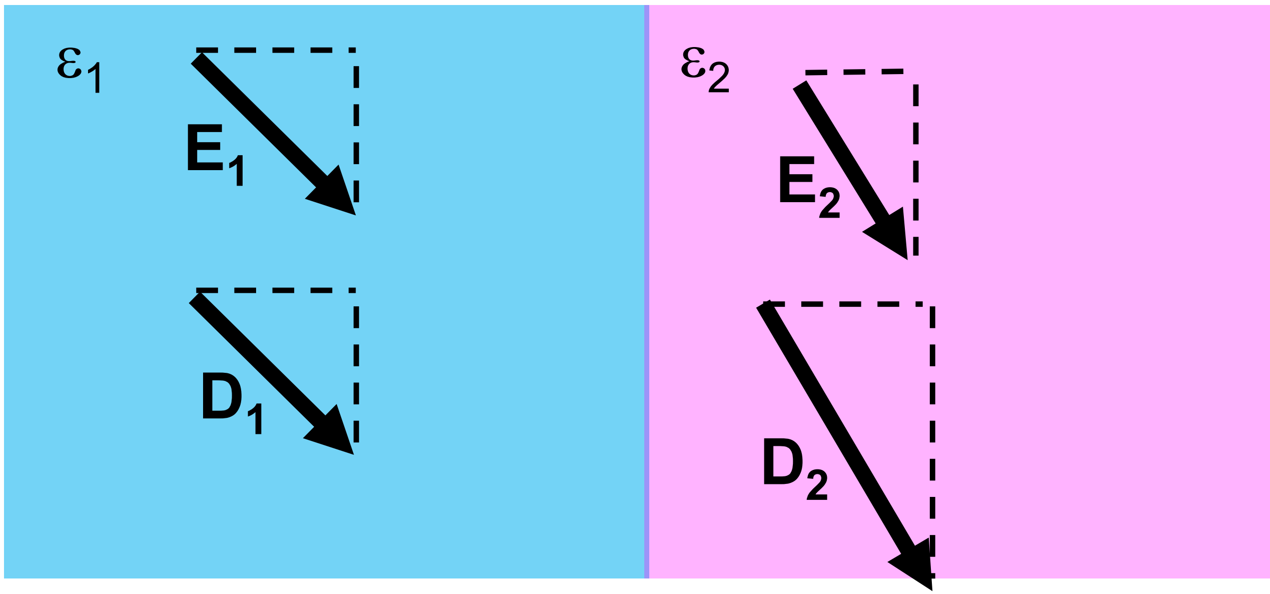
For $\rho_{\text{mono}}(\mathbf{r}) = 0$

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = 0 \quad \text{and} \quad \nabla \times \mathbf{E}(\mathbf{r}) = 0$$

$\Rightarrow$ At a surface between two dielectrics, in terms of surface normal $\hat{\mathbf{r}}$:

$$\hat{\mathbf{r}} \cdot \mathbf{D}(\mathbf{r}) = \text{continuous} = \hat{\mathbf{r}} \times \mathbf{E}(\mathbf{r})$$

Boundary value problems in the presence of dielectrics – example:



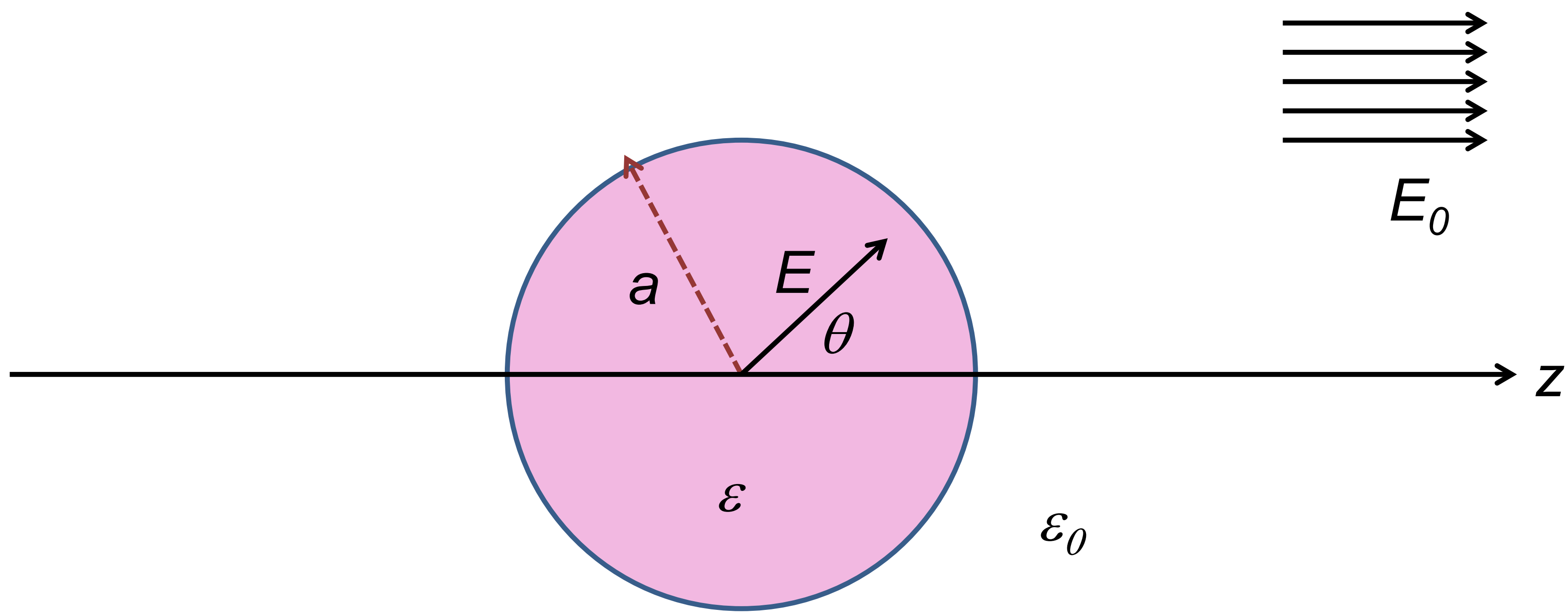
For $\frac{\varepsilon_2}{\varepsilon_1} = 2$

For isotropic dielectrics:

$$D_{1n} = D_{2n} \qquad \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$$

$$E_{1t} = E_{2t} \qquad \frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$

Boundary value problems in the presence of dielectrics – example:



$$\nabla \cdot \mathbf{D}(\mathbf{r}) = 0 \quad \text{and} \quad \nabla \times \mathbf{E}(\mathbf{r}) = 0$$

For $r \le a \quad \mathbf{D}(\mathbf{r}) = -\varepsilon \nabla \Phi(\mathbf{r})$

For $r > a \quad \mathbf{D}(\mathbf{r}) = -\varepsilon_0 \nabla \Phi(\mathbf{r})$

At $r = a$:

$$\varepsilon \frac{\partial \Phi_<(\mathbf{r})}{\partial r} = \varepsilon_0 \frac{\partial \Phi_>(\mathbf{r})}{\partial r}$$

$$\frac{\partial \Phi_<(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_>(\mathbf{r})}{\partial \theta}$$

Boundary value problems in the presence of dielectrics – example -- continued:

$$\Phi_<(\mathbf{r}) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$\Phi_>(\mathbf{r}) = \sum_{l=0}^{\infty} \left(B_l r^l + \frac{C_l}{r^{l+1}} \right) P_l(\cos\theta)$$

At $r = a$: $\quad \varepsilon \dfrac{\partial \Phi_<(\mathbf{r})}{\partial r} = \varepsilon_0 \dfrac{\partial \Phi_>(\mathbf{r})}{\partial r}$

$$\frac{\partial \Phi_<(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_>(\mathbf{r})}{\partial \theta}$$

For $r \to \infty \quad \Phi_>(\mathbf{r}) = -E_0 r \cos\theta$

Solution -- only $l = 1$ contributes

$$B_1 = -E_0$$

$$A_1 = -\left(\frac{3}{2 + \varepsilon / \varepsilon_0} \right) E_0 \qquad C_1 = \left(\frac{\varepsilon / \varepsilon_0 - 1}{2 + \varepsilon / \varepsilon_0} \right) a^3 E_0$$

Boundary value problems in the presence of dielectrics – example -- continued:

$$\Phi_<(\mathbf{r}) = -\left(\frac{3}{2+\varepsilon/\varepsilon_0}\right)E_0\, r\cos\theta$$

$$\Phi_>(\mathbf{r}) = -\left(r - \left(\frac{\varepsilon/\varepsilon_0 - 1}{2+\varepsilon/\varepsilon_0}\right)\frac{a^3}{r^2}\right)E_0 \cos\theta$$