

## **Electrodynamics – PHY712**

### **Lecture 13 – magnetostatic examples**

**Reference: Chap. 5 in J. D. Jackson's textbook.**

#### **Calculation of the vector potential for a confined current density**

If the current density  $\mathbf{J}(\mathbf{r})$  is confined in space, the vector potential in the Coulomb gauge can be calculated from

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \quad (1)$$

## Simple example of current density from a rotating charged sphere

Consider the following example corresponding to a rotating charged sphere of radius  $a$ , with  $\rho_0$  denoting the uniform charge density within the sphere and  $\omega$  denoting the angular rotation of the sphere:

$$\mathbf{J}(\mathbf{r}') = \begin{cases} \rho_0 \boldsymbol{\omega} \times \mathbf{r}' & \text{for } r' \leq a \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

In order to evaluate the vector potential (1) for this problem, we can make use of the expansion:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^*(\hat{\mathbf{r}}'). \quad (3)$$

Noting that

$$\mathbf{r}' = r' \sqrt{\frac{4\pi}{3}} \left( Y_{1-1}(\hat{\mathbf{r}}') \frac{\hat{\mathbf{x}} + \mathbf{i}\hat{\mathbf{y}}}{\sqrt{2}} + Y_{11}(\hat{\mathbf{r}}') \frac{-\hat{\mathbf{x}} + \mathbf{i}\hat{\mathbf{y}}}{\sqrt{2}} + Y_{10}(\hat{\mathbf{r}}') \hat{\mathbf{z}} \right), \quad (4)$$

we see that the angular integral in Eq. (1) can be simplified with the use of the identity:

$$\int d\Omega' \sum_m Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^*(\hat{\mathbf{r}}') \mathbf{r}' = \frac{r'}{r} \mathbf{r} \delta_{l1}. \quad (5)$$

## Simple example of current density from a rotating charged sphere – continued

Therefore the vector potential for this system is:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 \rho_0 \boldsymbol{\omega} \times \mathbf{r}}{3r} \int_0^a dr' r'^3 \frac{r_{<}}{r_{>}^2}, \quad (6)$$

which can be evaluated as:

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \frac{\mu_0 \rho_0}{3} \boldsymbol{\omega} \times \mathbf{r} \left( \frac{a^2}{2} - \frac{3r^2}{10} \right) & \text{for } r \leq a \\ \frac{\mu_0 \rho_0}{3} \boldsymbol{\omega} \times \mathbf{r} \frac{a^5}{5r^3} & \text{for } r \geq a \end{cases}. \quad (7)$$

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) = \begin{cases} \frac{\mu_0 \rho_0}{3} \left[ \boldsymbol{\omega} \left( a^2 - \frac{6}{5} r^2 \right) + \frac{3}{5} \mathbf{r} (\boldsymbol{\omega} \cdot \mathbf{r}) \right] & \text{for } r \leq a \\ \frac{\mu_0 \rho_0}{3} \left[ -\boldsymbol{\omega} \frac{a^5}{5r^3} + \frac{3a^5}{5r^5} \mathbf{r} (\boldsymbol{\omega} \cdot \mathbf{r}) \right] & \text{for } r \geq a \end{cases}. \quad (8)$$

## Another example – current associated with an electron in a spherical atom

In this case, we assume that the current density is due to an electron in a bound atomic state with quantum numbers  $|nlm_l\rangle$ , as described by a wavefunction  $\psi_{nlm_l}(\mathbf{r})$ , where the azimuthal quantum number  $m_l$  is associated with a factor of the form  $e^{im_l\phi}$ . For such a wavefunction the quantum mechanical current density operator can be evaluated:

$$\mathbf{J}(\mathbf{r}) = \frac{-e\hbar}{2m_e i} (\psi_{nlm_l}^* \nabla \psi_{nlm_l} - \psi_{nlm_l} \nabla \psi_{nlm_l}^*) . \quad (9)$$

Since the only complex part of this wavefunction is associated with the azimuthal quantum number, this can be written:

$$\mathbf{J}(\mathbf{r}) = \frac{-e\hbar}{2m_e i r \sin \theta} \left( \psi_{nlm_l}^* \frac{\partial}{\partial \phi} \psi_{nlm_l} - \psi_{nlm_l} \frac{\partial}{\partial \phi} \psi_{nlm_l}^* \right) \hat{\phi} = \frac{-e\hbar m_l \hat{\phi}}{m_e r \sin \theta} |\psi_{nlm_l}|^2 . \quad (10)$$

where  $m_e$  denotes the electron mass and  $e$  denotes the magnitude of the electron charge.

## Current associated with an electron in a spherical atom – continued

$$\mathbf{J}(\mathbf{r}) = \frac{-e\hbar m_l \hat{\phi}}{m_e r \sin \theta} |\psi_{nlm_l}(\mathbf{r})|^2 = \frac{-e\hbar m_l \hat{\mathbf{z}} \times \mathbf{r}}{m_e r^2 \sin^2 \theta} |\psi_{nlm_l}(\mathbf{r})|^2. \quad (11)$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \left( \frac{-e\hbar m_l}{m_e} \right) \int d^3r' \frac{\hat{\mathbf{z}} \times \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2 \sin^2 \theta'}. \quad (12)$$

Note that for some atomic wavefunctions,  $\psi_{nlm_l}(\mathbf{r}')$ , the evaluation of the vector potential  $\mathbf{A}(\mathbf{r})$  simplifies.

## Current associated with an electron in a spherical atom – continued

For example, consider the  $|nlm = 211\rangle$  state of a H atom:

$$\psi_{211}(\mathbf{r}) = -\sqrt{\frac{1}{64\pi a^3}} \frac{r}{a} e^{-r/(2a)} \sin \theta e^{i\phi}, \quad (13)$$

and

$$\mathbf{J}(\mathbf{r}') = \frac{-e\hbar}{64m_e\pi a^5} e^{-r'/a} \hat{\mathbf{z}} \times \mathbf{r}', \quad (14)$$

where  $a$  here denotes the Bohr radius. Using arguments similar to those above, we find that

$$\mathbf{A}(\mathbf{r}) = \frac{-e\hbar\mu_0\hat{\mathbf{z}} \times \mathbf{r}}{192m_e\pi a^5 r} \int_0^\infty dr' r'^3 e^{-r'/a} \frac{r_{<}}{r_{>}^2}. \quad (15)$$

This expression can be integrated to give:

$$\mathbf{A}(\mathbf{r}) = \frac{-e\hbar\mu_0\hat{\mathbf{z}} \times \mathbf{r}}{8m_e\pi r^3} \left[ 1 - e^{-r/a} \left( 1 + \frac{r}{a} + \frac{r^2}{2a^2} + \frac{r^3}{8a^3} \right) \right]. \quad (16)$$

## Current associated with an electron in a spherical atom – continued

Previous result:

$$\mathbf{A}(\mathbf{r}) = \frac{-e\hbar\mu_0\hat{\mathbf{z}} \times \mathbf{r}}{8m_e\pi r^3} \left[ 1 - e^{-r/a} \left( 1 + \frac{r}{a} + \frac{r^2}{2a^2} + \frac{r^3}{8a^3} \right) \right]. \quad (17)$$

Note that for  $r \rightarrow \infty$  :

$$\mathbf{A}(\mathbf{r}) = \frac{-e\hbar\mu_0\hat{\mathbf{z}} \times \mathbf{r}}{8m_e\pi r^3} = \frac{\mu_0}{4\pi} \left( -\frac{e\hbar}{2m_e} \right) \frac{\hat{\mathbf{z}} \times \mathbf{r}}{r^3} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3}, \quad (18)$$

where

$$\mathbf{m} = \left( -\frac{e\hbar}{2m_e} \right) \hat{\mathbf{z}}. \quad (19)$$

More generally, the magnetic dipole moment is given by:

$$\mathbf{m} = \frac{1}{2} \int d^3r' \quad \mathbf{r}' \times \mathbf{J}(\mathbf{r}'). \quad (20)$$

## Current associated with an electron in a spherical atom – continued

Note that the general form of the current density for a spherical atom is given by:

$$\mathbf{J}(\mathbf{r}) = \frac{-e\hbar m_l \hat{\phi}}{m_e r \sin \theta} |\psi_{nlm_l}(\mathbf{r})|^2 = \frac{-e\hbar m_l}{m_e} \frac{\hat{\mathbf{z}} \times \hat{\mathbf{r}}}{r \sin^2 \theta} |\psi_{nlm_l}(\mathbf{r})|^2. \quad (21)$$

Using the general form of the magnetic dipole moment, for an electronic wavefunction of a spherical atom,

$$\mathbf{m} = \frac{1}{2} \int d^3 r' \quad \mathbf{r}' \times \mathbf{J}(\mathbf{r}') = -\frac{e\hbar m_l}{2m_e} \hat{\mathbf{z}} \int d^3 r' |\psi_{nlm_l}(\mathbf{r}')|^2 = -\frac{e\hbar}{2m_e} m_l \hat{\mathbf{z}}. \quad (22)$$



## Systematic multipole analysis of vector potential for a general confined current density $\mathbf{J}(\mathbf{r})$ (assuming $\nabla \cdot \mathbf{J}(\mathbf{r}) = 0$ ).

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \quad (23)$$

For field point  $\mathbf{r}$  outside of extent of current density:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^3} \dots \quad (24)$$

$$\mathbf{A}(\mathbf{r}) \approx \frac{\mu_0}{4\pi} \left( \frac{1}{r} \int d^3r' \mathbf{J}(\mathbf{r}') + \frac{\mathbf{r}}{r^3} \cdot \int d^3r' \mathbf{r}' \mathbf{J}(\mathbf{r}') \dots \right) \quad (25)$$

Note that

$$\int d^3r' \mathbf{J}(\mathbf{r}') = 0 \quad (26)$$

$$\mathbf{r} \cdot \int d^3r' \mathbf{r}' \mathbf{J}(\mathbf{r}') = -\frac{1}{2} \mathbf{r} \times \int d^3r' \mathbf{r}' \times \mathbf{J}(\mathbf{r}') \equiv \mathbf{m} \times \mathbf{r}. \quad (27)$$

## Magnetic dipolar field

The magnetic dipole moment is defined by

$$\mathbf{m} = \frac{1}{2} \int d^3r' \mathbf{r}' \times \mathbf{J}(\mathbf{r}'), \quad (28)$$

with the corresponding potential

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}, \quad (29)$$

and magnetostatic field

$$\mathbf{B}_{\mathbf{m}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \left\{ \frac{3\hat{\mathbf{r}}(\mathbf{m} \cdot \hat{\mathbf{r}}) - \mathbf{m}}{r^3} + \frac{8\pi}{3} \mathbf{m} \delta^3(\mathbf{r}) \right\}. \quad (30)$$

## Magnetic dipolar field – continued

Some details:

$$\nabla \times (s\mathbf{V}) = \nabla s \times \mathbf{V} + s\nabla \times \mathbf{V}. \quad (31)$$

$$\nabla \times (\mathbf{V}_1 \times \mathbf{V}_2) = \mathbf{V}_1(\nabla \cdot \mathbf{V}_2) - \mathbf{V}_2(\nabla \cdot \mathbf{V}_1) + (\mathbf{V}_2 \cdot \nabla)\mathbf{V}_1 - (\mathbf{V}_1 \cdot \nabla)\mathbf{V}_2. \quad (32)$$

For  $r > 0$ :

$$\nabla \times \left( \frac{\mathbf{m} \times \mathbf{r}}{r^3} \right) = \frac{3\mathbf{r}(\mathbf{m} \cdot \mathbf{r}) - r^2\mathbf{m}}{r^5}. \quad (33)$$

## Justification for the $\delta$ function contribution at the origin of the magnetic dipole

Note: This derivation is very similar to the analogous electrostatic case.

The evaluation of the field at the origin of the dipole is poorly defined, but we make the following approximation.

$$\mathbf{B}(\mathbf{r} \approx \mathbf{0}) \approx \left( \int_{\text{sphere}} \mathbf{B}(\mathbf{r}) d^3\mathbf{r} \right) \delta^3(\mathbf{r}). \quad (34)$$

First we note that

$$\int_{r \leq R} \mathbf{B}(\mathbf{r}) d^3r = R^2 \int_{r=R} \hat{\mathbf{r}} \times \mathbf{A}(\mathbf{r}) d\Omega. \quad (35)$$

This result follows from the divergence theorem:

$$\int_{\text{vol}} \nabla \cdot \mathcal{V} d^3\mathbf{r} = \int_{\text{surface}} \mathcal{V} \cdot d\mathbf{A}. \quad (36)$$

## Singular contribution to dipolar field – continued

The divergence theorem can be used to prove Eq. (35) for each cartesian coordinate of  $\nabla \times \mathbf{A}$  since  $\nabla \times \mathbf{A} = \hat{\mathbf{x}} (\hat{\mathbf{x}} \cdot (\nabla \times \mathbf{A})) + \hat{\mathbf{y}} (\hat{\mathbf{y}} \cdot (\nabla \times \mathbf{A})) + \hat{\mathbf{z}} (\hat{\mathbf{z}} \cdot (\nabla \times \mathbf{A}))$ . Note that  $\hat{\mathbf{x}} \cdot (\nabla \times \mathbf{A}) = -\nabla \cdot (\hat{\mathbf{x}} \times \mathbf{A})$  and that we can use the Divergence theorem with  $\mathcal{V} \equiv \hat{\mathbf{x}} \times \mathbf{A}(\mathbf{r})$  for the  $x$ - component for example:

$$\int_{\text{vol}} \nabla \cdot (\hat{\mathbf{x}} \times \mathbf{A}) d^3r = \int_{\text{surface}} (\hat{\mathbf{x}} \times \mathbf{A}) \cdot \hat{\mathbf{r}} dA = \int_{\text{surface}} (\mathbf{A} \times \hat{\mathbf{r}}) \cdot \hat{\mathbf{x}} dA. \quad (37)$$

Therefore,

$$\int_{r \leq R} (\nabla \times \mathbf{A}) d^3r = - \int_{r=R} (\mathbf{A} \times \hat{\mathbf{r}}) \cdot (\hat{\mathbf{x}}\hat{\mathbf{x}} + \hat{\mathbf{y}}\hat{\mathbf{y}} + \hat{\mathbf{z}}\hat{\mathbf{z}}) dA = R^2 \int_{r=R} (\hat{\mathbf{r}} \times \mathbf{A}) d\Omega \quad (38)$$

which is identical to Eq. (35). We can use the identity (as in electrostatic case),

$$\int d\Omega \frac{\hat{\mathbf{r}}}{|\mathbf{r} - \mathbf{r}'|} = \frac{4\pi}{3} \frac{r_{<}}{r_{>}^2} \hat{\mathbf{r}}'. \quad (39)$$

## Singular contribution to dipolar field – continued

Now, expressing the vector potential in terms of the current density:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}, \quad (40)$$

the integral over  $\Omega$  in Eq. 35 becomes

$$R^2 \int_{r=R} (\hat{\mathbf{r}} \times \mathbf{A}) d\Omega = \frac{4\pi R^2}{3} \frac{\mu_0}{4\pi} \int d^3r' \frac{r_{<}}{r_{>}^2} \hat{\mathbf{r}}' \times \mathbf{J}(\mathbf{r}'). \quad (41)$$

If the sphere  $R$  contains the entire current distribution, then  $r_{>} = R$  and  $r_{<} = r'$  so that (41) becomes

$$R^2 \int_{r=R} (\hat{\mathbf{r}} \times \mathbf{A}) d\Omega = \frac{4\pi}{3} \frac{\mu_0}{4\pi} \int d^3r' \mathbf{r}' \times \mathbf{J}(\mathbf{r}') \equiv \frac{8\pi}{3} \frac{\mu_0}{4\pi} \mathbf{m}, \quad (42)$$

which thus justifies the delta-function contribution in Eq. 30 and results so-called “Fermi contact” contribution in the “hyperfine” interaction.

## Magnetic field due to electrons in the vicinity of a nucleus

### Contribution due to “orbital” magnetism in a spherical atom

The current density associated with an electron in a bound state of an atom as described by a quantum mechanical wavefunction  $\psi_{nlm_l}(\mathbf{r})$  can be written:

$$\mathbf{J}(\mathbf{r}) = \frac{-e\hbar m_l \hat{\phi}}{m_e r \sin \theta} |\psi_{nlm_l}(\mathbf{r})|^2. \quad (43)$$

In the following, it will be convenient to represent the azimuthal unit vector  $\hat{\phi}$  in terms of cartesian coordinates:

$$\hat{\phi} = -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}} = \frac{\hat{\mathbf{z}} \times \mathbf{r}}{r \sin \theta}. \quad (44)$$

The vector potential for this current density can be written

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{4\pi} \frac{e\hbar}{m_e} m_l \int d^3 r' \frac{\hat{\mathbf{z}} \times \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2 \sin^2 \theta'} \quad (45)$$

## Contribution due to “orbital” magnetism in a spherical atom – continued

We want to evaluate the magnetic field  $B = \nabla \times A$  in the vicinity of the nucleus ( $\mathbf{r} \rightarrow 0$ ). Taking the curl of the Eq. 45, we obtain

$$\mathbf{B}_o(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{e\hbar}{m_e} m_l \int d^3r' \frac{(\mathbf{r} - \mathbf{r}') \times (\hat{\mathbf{z}} \times \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2 \sin^2 \theta'} \quad (46)$$

Evaluating this expression with ( $\mathbf{r} \rightarrow 0$ ), we obtain

$$\mathbf{B}_o(\mathbf{0}) = -\frac{\mu_0}{4\pi} \frac{e\hbar}{m_e} m_l \int d^3r' \frac{\mathbf{r}' \times (\hat{\mathbf{z}} \times \mathbf{r}')}{r'^3} \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2 \sin^2 \theta'} \quad (47)$$



## Contribution due to “orbital” magnetism in a spherical atom – continued

$$\mathbf{B}_o(\mathbf{0}) = -\frac{\mu_0}{4\pi} \frac{e\hbar}{m_e} m_l \int d^3r' \frac{\mathbf{r}' \times (\hat{\mathbf{z}} \times \mathbf{r}')}{r'^3} \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2 \sin^2 \theta'} \quad (48)$$

Expanding the cross product and expressing the result in spherical polar coordinates, we obtain in the numerator

$$\hat{\mathbf{r}}' \times (\hat{\mathbf{z}} \times \hat{\mathbf{r}}') = \hat{\mathbf{z}}(1 - \cos^2 \theta') - \hat{\mathbf{x}} \cos \theta' \sin \theta' \cos \phi' - \hat{\mathbf{y}} \cos \theta' \sin \theta' \sin \phi'.$$

In evaluating the integration over the azimuthal variable  $\phi'$ , the  $\hat{\mathbf{x}}$  and  $\hat{\mathbf{y}}$  components vanish which reduces to

$$\mathbf{B}_o(\mathbf{0}) = -\frac{\mu_0}{4\pi} \frac{e\hbar}{m_e} m_l \int d^3r' \frac{\hat{\mathbf{z}} r'^2 \sin^2 \theta'}{r'^3} \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2 \sin^2 \theta'} \quad (49)$$

and

$$\mathbf{B}_o(\mathbf{0}) = -\frac{\mu_0 e \hbar m_l \hat{\mathbf{z}}}{4\pi m_e} \int d^3r' |\psi_{nlm_l}|^2 \frac{1}{r'^3} \equiv -\frac{\mu_0 e}{4\pi m_e} L_z \hat{\mathbf{z}} \left\langle \frac{1}{r'^3} \right\rangle. \quad (50)$$

## “Hyperfine” interaction

The so-called “hyperfine” interaction results from the magnetic dipole moment of a nucleus  $\mu_{\text{N}}$  responding to the magnetic field formed by the magnetic dipole of the electron spin ( $\mu_{\text{e}}$ ) as well as the electron orbital current contribution.

$$\mathcal{H}_{\text{HF}} = -\mu_{\text{N}} \cdot (\mathbf{B}_{\mu_{\text{e}}} + \mathbf{B}_o(0)). \quad (51)$$

$$\mathcal{H}_{\text{HF}} = -\frac{\mu_0}{4\pi} \left( \frac{3(\mu_{\text{N}} \cdot \hat{\mathbf{r}})(\mu_{\text{e}} \cdot \hat{\mathbf{r}}) - \mu_{\text{N}} \cdot \mu_{\text{e}}}{r^3} + \frac{8\pi}{3} \mu_{\text{N}} \cdot \mu_{\text{e}} \delta^3(\mathbf{r}) + \frac{e}{m_e} \left\langle \frac{\mathbf{L} \cdot \mu_{\text{N}}}{r^3} \right\rangle \right). \quad (52)$$