PHY 712 Electrodynamics
11-11:50 AM MWF in Olin 103

Notes for Lecture 10:

Reading Chapter 4 in JDJ --
Dipolar fields and dielectrics

A. Electric field due to a dipole

B. Electric polarization **P**

C. Electric displacement **D** and dielectric functions

Course schedule for Spring 2022

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Mon: 01/10/2022	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/14/2022
2	Wed: 01/12/2022	Chap. 1	Electrostatic energy calculations	#2	01/19/2022
3	Fri: 01/14/2022	Chap. 1	Electrostatic energy calculations	#3	01/21/2022
	Mon: 01/17/2022		MLK Holiday -- no class		
4	Wed: 01/19/2022	Chap. 1 & 2	Electrostatic potentials and fields	#4	01/24/2022
5	Fri: 01/21/2022	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#5	01/26/2022
6	Mon: 01/24/2022	Chap. 1 - 3	Brief introduction to numerical methods	#6	01/28/2022
7	Wed: 01/26/2022	Chap. 2 & 3	Image charge constructions	#7	01/31/2022
8	Fri: 01/28/2022	Chap. 2 & 3	Cylindrical and spherical geometries	#8	02/02/2022
9	Mon: 01/31/2022	Chap. 3 & 4	Spherical geometry and multipole moments	#9	02/04/2022
	Wed: 02/02/2022	No class	Fire caution		
	Fri: 02/04/2022	No class	Fire caution		
10	Mon: 02/07/2022	Chap. 4	Dipoles and Dielectrics	#10	02/09/2022
11	Wed: 02/09/2022	Chap. 4	Dipoles and Dielectrics		

PHY 712 -- Assignment #10

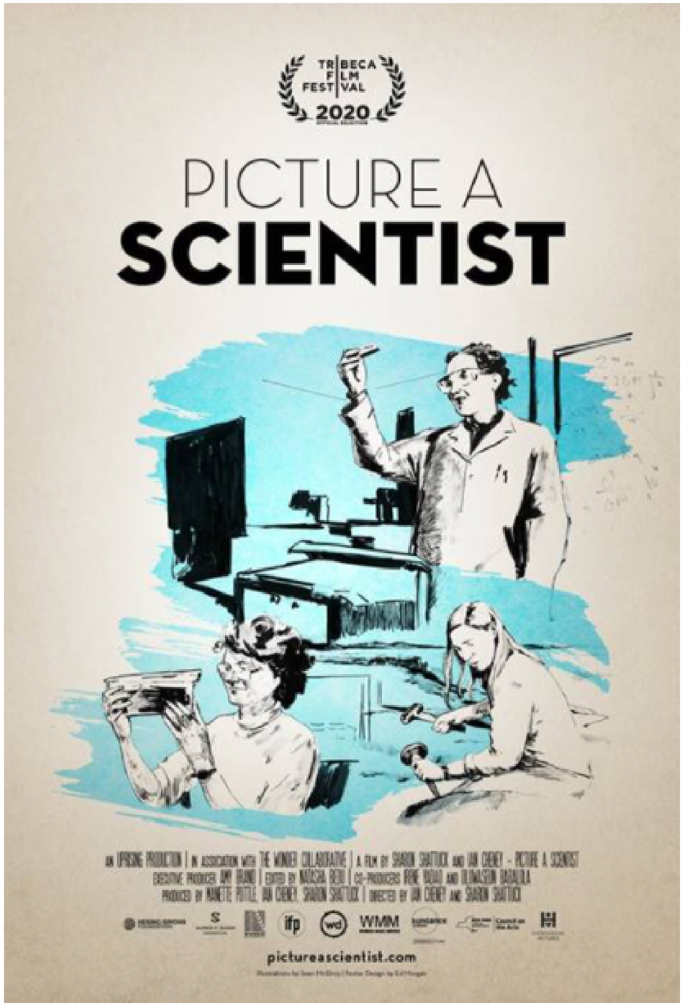
February 2, 2022

Continue reading Chapter 4 in **Jackson** .

1. Find the monopole, dipole, and quadrupole moments of the charge distribution shown in the figure in problem 4.1(b) of **Jackson**. You can use either the Cartesian or spherical polar forms for the moments as you prefer.

Colloquium schedule and other important events – On Friday 2/11/2022 -- http://users.wfu.edu/shapiro/WIS.html

Celebration of the International Day of Women and Girls in Science



In celebration of the International Day of Women and Girls in Science Feb 11th, women-and-girls-in-science-day , we will be screening a film Picture a Scientist https://www.pictureascientist.com/ with a discussion and reception (with food) afterwards.

The film will be screened at 4 PM in Olin Physical Lab on Friday February 11, 2022. Small group discussions and a reception on the rooftop (penthouse) of the Olin Physical Laboratory will follow (around 5:30 to 6 PM)

Students from all academic disciplines are welcome. All gender identities and expressions are welcome and encouraged to attend.

PICTURE A SCIENTIST is a feature-length documentary film chronicling the groundswell of researchers who are writing a new chapter for women scientists. A biologist, a chemist and a geologist lead viewers on a journey deep into their own experiences in the sciences, overcoming brutal harassment, institutional discrimination, and years of subtle slights to revolutionize the culture of science. From cramped laboratories to spectacular field stations, we also encounter scientific luminaries who provide new perspectives on how to make science itself more diverse, equitable, and open to all.

The event is organized and sponsored by the Departments of Biology, Chemistry, Computer Science, Engineering, Mathematics, Statistics, Physics and Wake Forest's Center for Functional Materials as well as the Wake Forest undergraduate Women in STEM group.

Register Here

Changes to colloquium schedule --

Thurs. Feb. 3, 2022 (rescheduled for Feb. 23)

Thurs. Feb. 10, 2022 — Dr. Delilah Gates, Princeton University — "What We can Learn from Light: Observational Signatures of Rotating Black Holes" (host: D. Kim-Shapiro)

Thurs. Feb. 17, 2022 — Mr. Andrew Barelli, WFU physics alumn, — (host: J. Macosko)

Wed. Feb. 23, 2022 (Please note different day of the week) — Professor Wendu Ding, Chemistry Department WFU — "Plasmon-Coupled Resonance Energy Transfer" (host: S. Winter)

Thurs. Feb. 24, 2022 **(Please note this is just one day after the previous colloquium)** — Dr. Ken Cousins, Research Principal at Earth Economics — "An Ecological Economics: Concepts, Application, and Potential" — (hosts: J. Macosko, A. Cottrell, from Economics)

Thurs. Mar. 3, 2022 — Dr. Jess McIver, University of British Columbia — "What can we learn from gravitational waves?" —(host: G. Cook)

Review: General results for a multipole analysis of the electrostatic potential due to an isolated charge distribution:

General form of electrostatic potential with boundary value $\Phi(r \to \infty) = 0$

for confined charge density $\rho(\mathbf{r})$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

$$= \frac{1}{4\pi\varepsilon_0} \int d^3r' \rho(\mathbf{r}') \left(\sum_{lm} \frac{4\pi}{2l+1} \frac{r_<^{\,l}}{r_>^{\,l+1}} Y_{lm}(\theta,\varphi) Y_{lm}^*(\theta',\varphi') \right)$$

Suppose that $\rho(\mathbf{r}) = \sum_{lm} \rho_{lm}(r) Y_{lm}(\theta,\varphi)$

$$\Rightarrow \Phi(\mathbf{r}) = \frac{1}{\varepsilon_0} \sum_{lm} \frac{1}{2l+1} Y_{lm}(\theta,\varphi) \left(\frac{1}{r^{l+1}} \int_0^r r'^{2+l} dr' \rho_{lm}(r') + r^l \int_r^\infty r'^{1-l} dr' \rho_{lm}(r') \right)$$

For $r \to \infty$: $\Phi(\mathbf{r}) = \frac{1}{\varepsilon_0} \sum_{lm} \frac{1}{2l+1} Y_{lm}(\theta,\varphi) \frac{1}{r^{l+1}} \underbrace{\int_0^\infty r'^{2+l} dr' \rho_{lm}(r')}_{q_{lm}}$

Comment --

Acts like a projection operator

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$= \frac{1}{4\pi\varepsilon_0}\int d^3r' \rho(\mathbf{r}')\left(\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^{\,l}}{r_>^{\,l+1}} Y_{lm}(\theta,\varphi) Y_{lm}^{\;*}(\theta',\varphi')\right)$$

Suppose that $\rho(\mathbf{r}') = \sum_{lm} \rho_{lm}(r') Y_{lm}(\theta',\varphi')$

$$\Rightarrow \Phi(\mathbf{r}) = \frac{1}{\varepsilon_0}\sum_{lm}\frac{1}{2l+1} Y_{lm}(\theta,\varphi)\left(\frac{1}{r^{l+1}}\int_0^r r'^{2+l}\, dr' \rho_{lm}(r') + r^l \int_r^\infty r'^{1-l}\, dr' \rho_{lm}(r')\right)$$

Why? -- Recall that

$$\int d\Omega' Y_{lm}^{\;*}(\theta',\varphi') Y_{\lambda\mu}(\theta',\varphi') = \delta_{l\lambda,m\mu}$$

The the multipole analysis has the following general behavior for r→infinity:

For r outside the extent of $\rho(\mathbf{r})$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{lm}\frac{4\pi}{2l+1}\frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}\left(\int d^3r' r'^l\, Y_{lm}^*(\theta',\varphi')\rho(\mathbf{r}')\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\sum_{lm}\frac{4\pi\, q_{lm}}{2l+1}\frac{Y_{lm}(\theta,\varphi)}{r^{l+1}} \qquad q_{lm} \equiv \int_0^\infty r'^{2+l}\, dr'\, \rho_{lm}(r')$$

In terms of Cartesian expansion :

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{r^3} + \frac{1}{2}\sum_{i,j} Q_{ij}\frac{r_i r_j}{r^5}\cdots\right)$$

Here q, p_i, and Q_{ij} are linearly proportional to the q_{lm} multipole values.

The multipole analysis also can be used to analyze the the electrostatic fields for r→0 as needed in the following example involving a very localized charge density $\rho(\mathbf{r})$ in a electrostatic field $\Phi(\mathbf{r})$ (such as a nucleus in the field produced by electrons in an atom).

charge density within nucleus

electrostatic potential due to electrons near the nucleus.

$$W = \int d^3r\ \rho(\mathbf{r})\Phi(\mathbf{r})$$

$$\approx \int d^3r\ \rho(\mathbf{r})\left(\Phi(0) + \mathbf{r}\cdot\nabla\Phi(\mathbf{r})\big|_{r=0} + \frac{1}{2}(\mathbf{r}\cdot\nabla)^2\Phi(\mathbf{r})\big|_{r=0} + \cdots \right)$$

$$= q\Phi(0) - \mathbf{p}\cdot\mathbf{E}(0) + \frac{1}{6}\sum_{i,j} Q_{ij}\frac{\partial^2\Phi(0)}{\partial r_i \partial r_j} + \cdots$$

The following results were presented on Monday and are summarized here for a more complicated charge distribution:

$$\rho(\mathbf{r}) = \frac{q}{64\pi a^3}\left(\frac{r}{a}\right)^2 e^{-r/a}\sin^2\theta$$

$$\text{Note that :}\quad \sqrt{\frac{4\pi}{5}}Y_{20}(\theta,\phi) = \frac{3}{2}\cos^2\theta - \frac{1}{2} = 1 - \frac{3}{2}\sin^2\theta$$

$$\sin^2\theta = \frac{2}{3} - \frac{2}{3}\sqrt{\frac{4\pi}{5}}Y_{20}(\theta,\phi) = \frac{2}{3}\sqrt{\frac{4\pi}{1}}Y_{00}(\theta,\phi) - \frac{2}{3}\sqrt{\frac{4\pi}{5}}Y_{20}(\theta,\phi)$$

$$\Rightarrow \rho(\mathbf{r}) = \rho_{00}(r)Y_{00}(\theta,\phi) + \rho_{20}(r)Y_{20}(\theta,\phi)$$

$$\Phi(\mathbf{r}) = \Phi_{00}(r)Y_{00}(\theta,\phi) + \Phi_{20}(r)Y_{20}(\theta,\phi)$$

$$\Phi_{lm} = \frac{1}{4\pi\varepsilon_0}\frac{4\pi}{2l+1}\left(\frac{1}{r^{l+1}}\int_0^r r'^{2+l}\,dr'\,\rho_{lm}(r') + r^l\int_r^\infty r'^{1-l}\,dr'\rho_{lm}(r')\right)$$

$$\rho_{00}(r) = \frac{2}{3}\sqrt{4\pi}\,\frac{q}{64\pi a^3}\left(\frac{r}{a}\right)^2 e^{-r/a} \qquad \rho_{20}(r) = -\frac{2}{3}\sqrt{\frac{4\pi}{5}}\,\frac{q}{64\pi a^3}\left(\frac{r}{a}\right)^2 e^{-r/a}$$

Writing out the details of the potential from evaluating integrals

$$\Phi_{00}(r) = \frac{1}{4\pi\varepsilon_0}\sqrt{4\pi}\,\frac{q}{r}\left(1 - e^{-r/a}\left(1 + \frac{3r}{4a} + \frac{r^2}{4a^2} + \frac{r^3}{24a^3}\right)\right)$$

$$\Phi_{20}(r) = -\frac{6}{4\pi\varepsilon_0}\sqrt{\frac{4\pi}{5}}\,\frac{qa^2}{r^3}\left(1 - e^{-r/a}\left(1 + \frac{r}{a} + \frac{r^2}{2a^2} + \frac{r^3}{6a^3} + \frac{r^4}{24a^3} + \frac{r^5}{144a^5}\right)\right)$$

For $r \to \infty$; in terms for Legendre polynomials:

$$\Phi(\mathbf{r}) \to \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{6a^2}{r^3}P_2(\cos\theta)\right)$$

$$Y_{l0}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)$$

For $r \to 0$; in terms for Legendre polynomials:

$$\Phi(\mathbf{r}) \to \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{4a} - \frac{r^2}{120a^3}P_2(\cos\theta)\right)$$

More details continued --

For $r \to 0$; in terms for Legendre polynomials :

$$\Phi(\mathbf{r}) \to \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{4a} - \frac{r^2}{120a^3}P_2(\cos\theta)\right)$$

Implications for electric quadrupole interaction :

$$W = \frac{1}{6}\sum_{i,j} Q_{ij}\frac{\partial^2\Phi(0)}{\partial r_i \partial r_j} + \cdots$$

$$P_2(\cos\theta) = \tfrac{3}{2}\cos^2\theta - \tfrac{1}{2} = \tfrac{1}{2r^2}\left(3z^2 - r^2\right)$$
$$= \tfrac{1}{2r^2}\left(2z^2 - x^2 - y^2\right)$$

For $r \to 0$; in terms of Cartesian coordinates

$$\Phi(\mathbf{r}) \to \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{4a} - \frac{2z^2 - x^2 - y^2}{240a^3}\right)$$

$$\frac{\partial^2\Phi(0)}{\partial x^2} = \frac{\partial^2\Phi(0)}{\partial y^2} = -\frac{1}{2}\frac{\partial^2\Phi(0)}{\partial z^2} = \frac{q}{4\pi\varepsilon_0}\frac{1}{120a^3}$$

Example of multipole distribution continued --

Electric quadrupole interaction:

$$W = \frac{1}{6}\sum_{i,j} Q_{ij} \frac{\partial^2 \Phi(0)}{\partial r_i \partial r_j} = \frac{1}{6}\left(Q_{xx}\frac{\partial^2 \Phi(0)}{\partial x^2} + Q_{yy}\frac{\partial^2 \Phi(0)}{\partial y^2} + Q_{zz}\frac{\partial^2 \Phi(0)}{\partial z^2} \right)$$

For symmetric nuclei, $Q_{zz} \equiv Qq = -\frac{1}{2}Q_{xx} = -\frac{1}{2}Q_{yy}$

$$W \approx -\frac{q^2}{4\pi\varepsilon_0}\frac{Q}{240a^3}$$

Here q stands for the elementary charge
q=e=1.602176634x10^{-19}C

Summary -- Notion of multipole moment:

In the spherical harmonic representation --

define the moment q_{lm} of the (confined) charge distribution $\rho(\mathbf{r})$:

$$q_{lm} \equiv \int d^3r'\, r'^l\, Y^*_{lm}(\theta', \phi')\rho(\mathbf{r}')$$

In the Cartesian representation --

define the monopole moment q:

$$q \equiv \int d^3r'\, \rho(\mathbf{r}')$$

define the dipole moment $\mathbf{p}$:

$$\mathbf{p} \equiv \int d^3r'\, \mathbf{r}'\, \rho(\mathbf{r}')$$

define the quadrupole moment components Q_{ij} $(i,j \rightarrow x,y,z)$:

$$Q_{\mathrm{ij}} \equiv \int d^3r' \left(3r'_i\, r'_j - r'^2\, \delta_{ij}\right)\rho(\mathbf{r}')$$

General form of electrostatic potential in terms of multipole moments:

For r outside the extent of $\rho(\mathbf{r})$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{lm}\frac{4\pi}{2l+1}\frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}\left(\int d^3r'\, r'^l\, Y_{lm}^*(\theta',\varphi')\rho(\mathbf{r}')\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\sum_{lm}\frac{4\pi\, q_{lm}}{2l+1}\frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}$$

In terms of Cartesian expansion :

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r}+\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}+\frac{1}{2}\sum_{i,j}Q_{ij}\,\frac{r_i r_j}{r^5}\cdots\right)$$

Focus on dipolar contributions:

For r outside the extent of $\rho(\mathbf{r})$:

Electrostatic potential:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right)$$

Electrostatic field:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{3\mathbf{r}\;(\mathbf{p}\cdot\mathbf{r}) - r^2\mathbf{p}}{r^5} - \frac{4\pi}{3}\mathbf{p}\delta^3(\mathbf{r})\right)$$

Poorly defined for $r \to 0$ Correct value for $r \to 0$

“Justification” of surprising δ-function term in dipole electric field -- Assuming dipole is located at *r=0*, we need to need to evaluate the electrostatic field near *r=0*:

We will use the approximation:

$$\mathbf{E}(\mathbf{r} \approx \mathbf{0}) \approx \left(\int_{\text{sphere}} \mathbf{E}(\mathbf{r}) d^3r \right) \delta^3(\mathbf{r}).$$

First we note that:

$$\int_{r \leq R} \mathbf{E}(\mathbf{r}) d^3r = -R^2 \int_{r=R} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\boldsymbol{\Omega}.$$

Some details -- amplifying discussion in JDJ:

$$\int_{r\le R}\mathbf{E}(\mathbf{r})d^3r = -R^2\int_{r=R}\Phi(\mathbf{r})\hat{\mathbf{r}}d\boldsymbol{\Omega}.$$

This result follows from the divergence theorem:

$$\int_{\text{vol}}\nabla\cdot\mathcal{V}d^3r = \int_{\text{surface}}\mathcal{V}\cdot d\mathbf{A}.$$

In our case, this theorem can be used for each cartesian coordinate if we choose $\mathcal{V}\equiv\hat{\mathbf{x}}\Phi(\mathbf{r})$ for the x component, etc.

$$\int_{r\le R}\nabla\Phi(\mathbf{r})d^3r = \hat{\mathbf{x}}\int_{r\le R}\nabla\cdot(\hat{\mathbf{x}}\Phi)d^3r + \hat{\mathbf{y}}\int_{r\le R}\nabla\cdot(\hat{\mathbf{y}}\Phi)d^3r + \hat{\mathbf{z}}\int_{r\le R}\nabla\cdot(\hat{\mathbf{z}}\Phi)d^3r,$$

which is equal to:

$$\int_{r=R}\Phi(\mathbf{r})R^2d\Omega\left((\hat{\mathbf{x}}\cdot\hat{\mathbf{r}})\hat{\mathbf{x}} + (\hat{\mathbf{y}}\cdot\hat{\mathbf{r}})\hat{\mathbf{y}} + (\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})\hat{\mathbf{z}}\right) = \int_{r=R}\Phi(\mathbf{r})R^2d\Omega\hat{\mathbf{r}}.$$

Therefore --

$$\int_{r\le R}\mathbf{E}(\mathbf{r})d^3r = -\int_{r\le R}\nabla\Phi(\mathbf{r})d^3r = -R^2\int_{r=R}\Phi(\mathbf{r})\hat{\mathbf{r}}d\boldsymbol{\Omega}.$$

More details

$$\int_{r\le R} \mathbf{E}(\mathbf{r}) d^3r = -R^2 \int_{r=R} \Phi(\mathbf{r})\hat{\mathbf{r}} d\mathbf{\Omega}.$$

Now, we notice that the electrostatic potential can be determined from the charge density $\rho(\mathbf{r})$ according to:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{4\pi\epsilon_0}\sum_{lm}\frac{4\pi}{2l+1}\int d^3r' \rho(\mathbf{r}')\frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\hat{\mathbf{r}}) Y_{lm}(\hat{\mathbf{r}}').$$

We also note that the unit vector can be written in terms of spherical harmonic functions:

$$\hat{\mathbf{r}} = \begin{cases} \sin(\theta)\cos(\phi)\hat{\mathbf{x}} + \sin(\theta)\sin(\phi)\hat{\mathbf{y}} + \cos(\theta)\hat{\mathbf{z}} \\ \sqrt{\frac{4\pi}{3}}\left(Y_{1-1}(\hat{\mathbf{r}})\frac{\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{\sqrt{2}} + Y_{11}(\hat{\mathbf{r}})\frac{-\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{\sqrt{2}} + Y_{10}(\hat{\mathbf{r}})\hat{\mathbf{z}} \right) \end{cases}$$

$$\int_{r=R} \Phi(\mathbf{r})\hat{\mathbf{r}} d\mathbf{\Omega} = \frac{1}{3\varepsilon_0}\int d^3r' \rho(\mathbf{r}')\frac{r_<}{r_>^2}\sqrt{\frac{4\pi}{3}}\left(Y_{1-1}(\hat{\mathbf{r}}')\frac{\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{\sqrt{2}} + Y_{11}(\hat{\mathbf{r}}')\frac{-\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{\sqrt{2}} + Y_{10}(\hat{\mathbf{r}}')\hat{\mathbf{z}} \right)$$

$$= \frac{1}{3\varepsilon_0}\int d^3r' \rho(\mathbf{r}')\frac{r_<}{r_>^2}\hat{\mathbf{r}}'$$

More details continued --

When we evaluate the integral over solid angle $d\Omega$, only the $l = 1$ terms contribute, and the result of the integration reduces to:

$$-R^2 \int_{r=R} \Phi(\mathbf{r})\hat{\mathbf{r}} d\mathbf{\Omega} = -\frac{1}{4\pi\epsilon_0}\frac{4\pi R^2}{3}\int d^3r' \rho(\mathbf{r}')\frac{r_<}{r_>^2}\hat{\mathbf{r}}'.$$

The choice of $r_<$ and $r_>$ is a choice between the integration variables r' and the sphere radius R. If the sphere encloses the charge distribution, $\rho(\mathbf{r}')$, then $r_< = r'$ and $r_> = R$ so that the result is:

$$-R^2 \int_{r=R} \Phi(\mathbf{r})\hat{\mathbf{r}} d\mathbf{\Omega} = -\frac{1}{4\pi\epsilon_0}\frac{4\pi R^2}{3}\frac{1}{R^2}\int d^3r' \rho(\mathbf{r}')r'\hat{\mathbf{r}}' \equiv -\frac{\mathbf{p}}{3\varepsilon_0}.$$

Otherwise, if the charge distribution $\rho(\mathbf{r}')$ lies outside of the sphere, then $r_< = R$ and $r_> = r'$ and the result is:

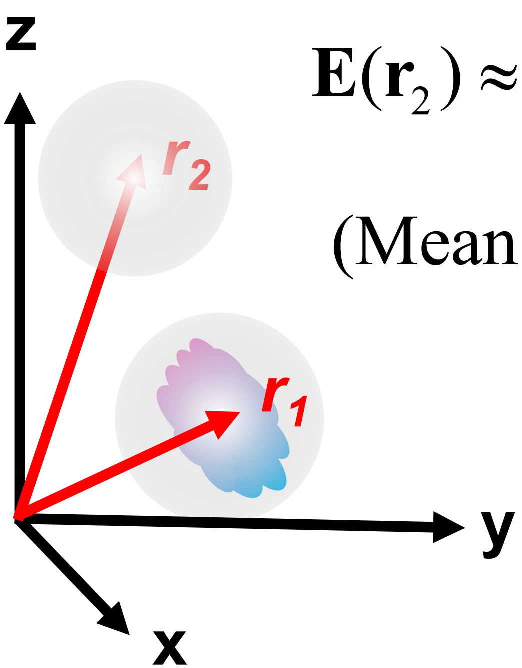
$$-R^2 \int_{r=R} \Phi(\mathbf{r})\hat{\mathbf{r}} d\mathbf{\Omega} = -\frac{1}{4\pi\epsilon_0}\frac{4\pi R^2}{3}R\int d^3r' \frac{\rho(\mathbf{r}')}{r'^2}\hat{\mathbf{r}}' \equiv \frac{4\pi R^3}{3}\mathbf{E}(0).$$

In summary --

Electrostatic dipolar field for dipole moment **p** at **r**=0:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{3\mathbf{r}\ (\mathbf{p}\cdot\mathbf{r}) - r^2\mathbf{p}}{r^5} - \frac{4\pi}{3}\mathbf{p}\delta^3(\mathbf{r}) \right)$$

Summary of key argument:



$$\mathbf{E}(\mathbf{r}_2) \approx \frac{3}{4\pi R^3} \int_{r \le R} d^3r \; \mathbf{E}(\mathbf{r}_2 + \mathbf{r}) = \mathbf{E}(\mathbf{r}_2)$$

(Mean value theorem for Laplace equation)

$$\mathbf{E}(\mathbf{r}_1) \approx \frac{3}{4\pi R^3} \int_{r \le R} d^3r \; \mathbf{E}(\mathbf{r}_1 + \mathbf{r})$$

$$\approx \frac{3}{4\pi R^3}\left(-\frac{\mathbf{p}}{3\epsilon_0}\right) \Rightarrow -\frac{\mathbf{p}}{3\epsilon_0}\delta^3(\mathbf{r} - \mathbf{r}_1)$$

Summary:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{3\mathbf{r}\;(\mathbf{p}\cdot\mathbf{r}) - r^2\mathbf{p}}{r^5} - \frac{4\pi}{3}\mathbf{p}\delta^3(\mathbf{r})\right)$$

Coarse grain representation of macroscopic distribution of dipoles:

Electric polarization $\mathbf{P}(\mathbf{r})$ due to collection of dipoles :

$$\mathbf{P}(\mathbf{r}) \equiv \sum_i \mathbf{p}_i \delta^3(\mathbf{r} - \mathbf{r}_i)$$

Monopole electric charge density $\rho_{\text{mono}}(\mathbf{r})$:

$$\rho_{\text{mono}}(\mathbf{r}) \equiv \sum_i q_i \delta^3(\mathbf{r} - \mathbf{r}_i)$$

Electrostatic potential for a single monopole charge q

and a single dipole $\mathbf{p}$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right)$$

Coarse grain representation of macroscopic distribution of dipoles -- continued:

Electrostatic potential for a single monopole charge q

and a single dipole $\mathbf{p}$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right)$$

Electrostatic potential for collections of monopole charges q_i

and dipoles $\mathbf{p}_i$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\int d^3r' \frac{\rho_{mono}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \int d^3r' \frac{\mathbf{P}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\right)$$

Note: $$\int d^3r' \frac{\mathbf{P}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = \int d^3r' \, \mathbf{P}(\mathbf{r}')\cdot\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\int d^3r' \frac{\nabla'\cdot\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Coarse grain representation of macroscopic distribution of dipoles -- continued:

Electrostatic potential for collections of monopole charges q_i

and dipoles $\mathbf{p}_i$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\int d^3r' \frac{\rho_{mono}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} - \int d^3r' \frac{\nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \right)$$

$$-\nabla^2\Phi(\mathbf{r}) = \nabla\cdot\mathbf{E}(\mathbf{r}) = \frac{1}{\varepsilon_0}\left(\rho_{mono}(\mathbf{r}) - \nabla\cdot\mathbf{P}(\mathbf{r})\right)$$

$$\Rightarrow \nabla\cdot\left(\varepsilon_0\mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r})\right) = \rho_{mono}(\mathbf{r})$$

Define Displacement field: $\mathbf{D}(\mathbf{r}) \equiv \varepsilon_0\mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r})$

Macroscopic form of Gauss's law: $\nabla\cdot\mathbf{D}(\mathbf{r}) = \rho_{mono}(\mathbf{r})$

Coarse grain representation of macroscopic distribution of dipoles -- continued:

Many materials are polarizable and produce a polarization field in the presence of an electric field with a proportionality constant χ_e :

$$\mathbf{P}(\mathbf{r}) = \varepsilon_0 \chi_e \mathbf{E}(\mathbf{r})$$

$$\mathbf{D}(\mathbf{r}) \equiv \varepsilon_0 \mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r}) = \varepsilon_0 (1 + \chi_e) \mathbf{E}(\mathbf{r}) \equiv \varepsilon \mathbf{E}(\mathbf{r})$$

ε represents the dielectric function of the material

Boundary value problems in dielectric materials

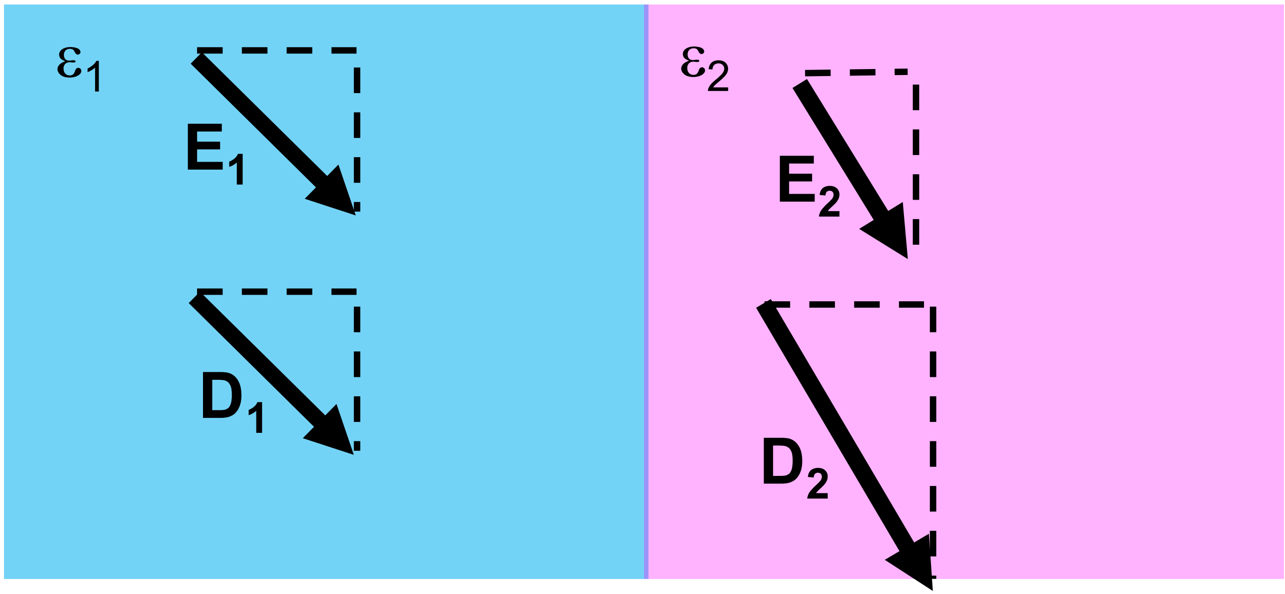
For $\rho_{mono}(\mathbf{r}) = 0$

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = 0 \quad \text{and} \quad \nabla \times \mathbf{E}(\mathbf{r}) = 0$$

$\Rightarrow$ At a surface between two dielectrics, in terms of surface normal $\hat{\mathbf{r}}$:

$$\hat{\mathbf{r}} \cdot \mathbf{D}(\mathbf{r}) = \text{continuous} \;=\; \hat{\mathbf{r}} \times \mathbf{E}(\mathbf{r})$$

Boundary value problems in the presence of dielectrics – example:



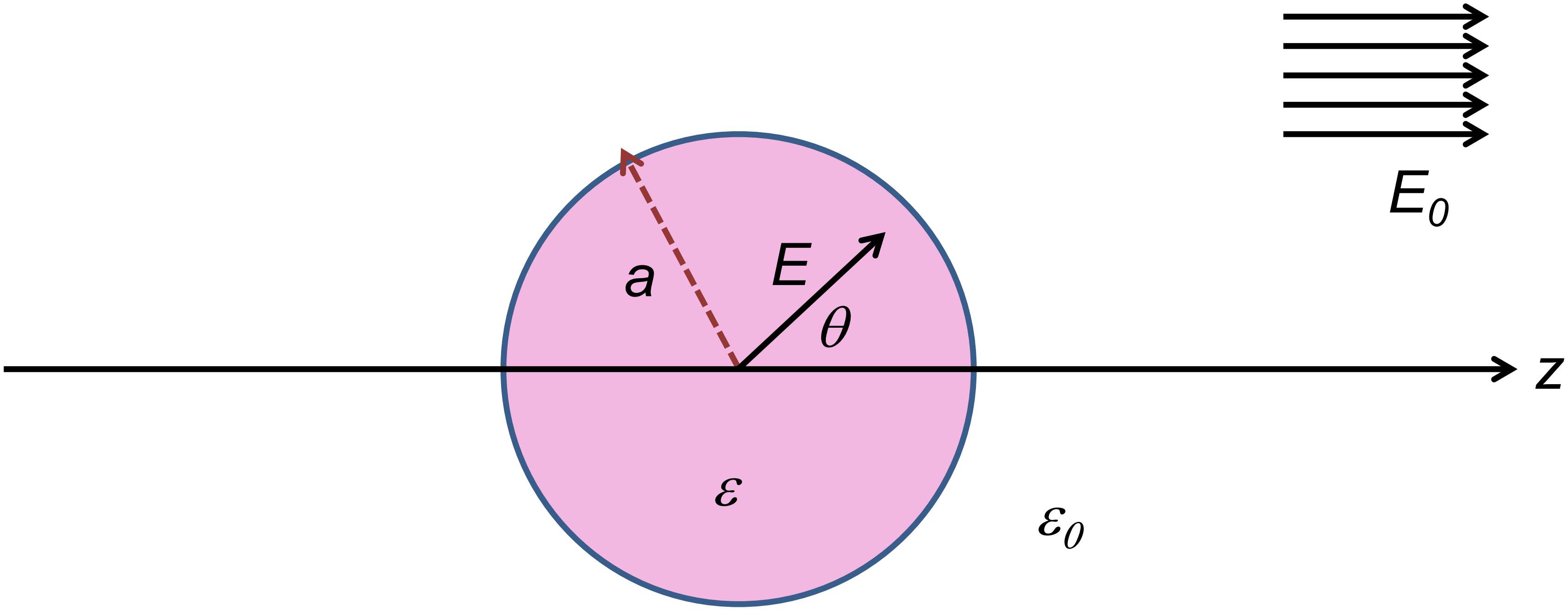
For $\frac{\varepsilon_2}{\varepsilon_1} = 2$

For isotropic dielectrics:

$$D_{1n} = D_{2n} \qquad \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$$

$$E_{1t} = E_{2t} \qquad \frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$

Boundary value problems in the presence of dielectrics – example:



$$\nabla \cdot \mathbf{D}(\mathbf{r}) = 0 \quad \text{and} \quad \nabla \times \mathbf{E}(\mathbf{r}) = 0$$

For $r \leq a \quad \mathbf{D}(\mathbf{r}) = -\varepsilon \nabla \Phi(\mathbf{r})$

For $r > a \quad \mathbf{D}(\mathbf{r}) = -\varepsilon_0 \nabla \Phi(\mathbf{r})$

At $r = a$:

$$\varepsilon \frac{\partial \Phi_<(\mathbf{r})}{\partial r} = \varepsilon_0 \frac{\partial \Phi_>(\mathbf{r})}{\partial r}$$

$$\frac{\partial \Phi_<(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_>(\mathbf{r})}{\partial \theta}$$

Boundary value problems in the presence of dielectrics – example -- continued:

$$\Phi_<(\mathbf{r}) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$\Phi_>(\mathbf{r}) = \sum_{l=0}^{\infty} \left(B_l r^l + \frac{C_l}{r^{l+1}} \right) P_l(\cos\theta)$$

At $r = a$: $$\varepsilon \frac{\partial \Phi_<(\mathbf{r})}{\partial r} = \varepsilon_0 \frac{\partial \Phi_>(\mathbf{r})}{\partial r}$$

$$\frac{\partial \Phi_<(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_>(\mathbf{r})}{\partial \theta}$$

For $r \to \infty$ $\quad \Phi_>(\mathbf{r}) = -E_0 r \cos\theta$

Solution -- only $l = 1$ contributes

$$B_1 = -E_0$$

$$A_1 = -\left(\frac{3}{2 + \varepsilon/\varepsilon_0} \right) E_0 \qquad C_1 = \left(\frac{\varepsilon/\varepsilon_0 - 1}{2 + \varepsilon/\varepsilon_0} \right) a^3 E_0$$

Boundary value problems in the presence of dielectrics – example -- continued:

$$\Phi_<(\mathbf{r}) = -\left(\frac{3}{2+\varepsilon/\varepsilon_0}\right)E_0\, r\cos\theta$$

$$\Phi_>(\mathbf{r}) = -\left(r - \left(\frac{\varepsilon/\varepsilon_0 - 1}{2+\varepsilon/\varepsilon_0}\right)\frac{a^3}{r^2}\right)E_0 \cos\theta$$