PHY 712 Electrodynamics
11-11:50 AM MWF Olin 103

Plan for Lecture 11:

Complete reading of Chapter 4

A. **Microscopic ←→ macroscopic polarizability and dielectric function**
B. **Clausius-Mossotti equation**
C. **Electrostatic energy in dielectric media**

PHYSICS COLLOQUIUM

4 PM in Olin 101

THURSDAY

•

FEBRUARY 10, 2022

"What we can Learn from Light: Observational Signatures of Rotating Black Holes"

As we enter the era of precision black hole imaging, identifying universal observational signatures which of the near horizon strong field region will become increasingly useful in constraining the parameters--mass, spin, spin axis orientations--of the black holes. In this talk we describe progress in analytic characterization of such signatures. These arise from two critical phenomena related to the Kerr geometry: the presence of the photon shell (a region of critical null geodesics), and the emergence of a throat of divergent proper length in the near horizon of near maximally rotating black holes.

We present two such observational signatures for electromagnetic emission:

First, we analytically predict the polarized near-horizon

Delilah Gates, Ph.D.

Associate Research Scholar
Princeton Gravity Initiative
Princeton University
Princeton, NJ

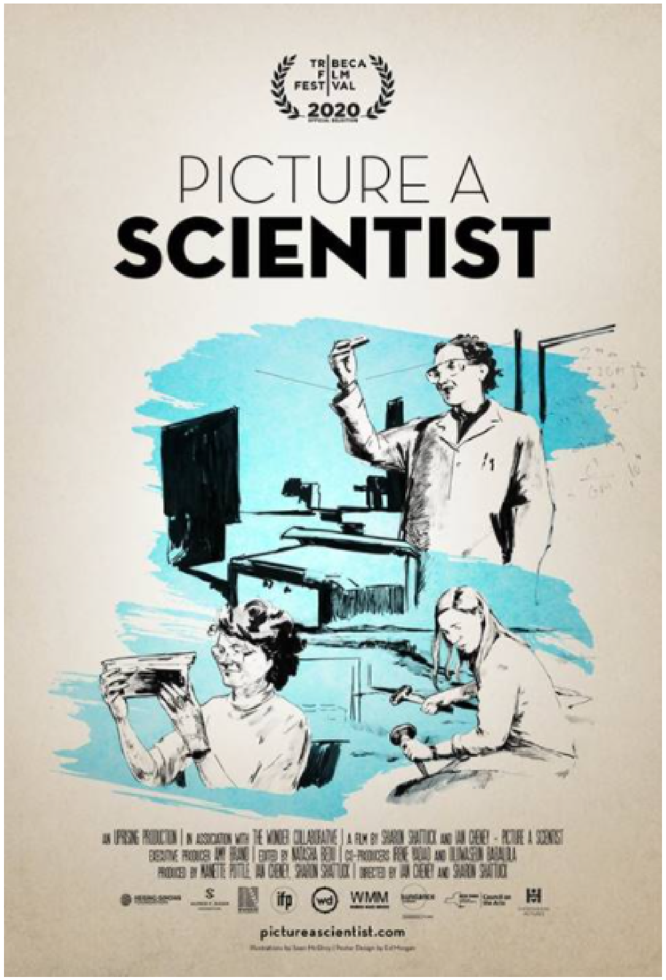
4:00 pm - Olin 101*

*Link provided for those unable to attend in person.

Note: For additional information on the seminar

Friday   Feb 11, 2022 at 4 PM
http://users.wfu.edu/shapiro/WIS.html

Celebration of the International Day of Women and Girls in Science



In celebration of the International Day of Women and Girls in Science Feb 11th, women-and-girls-in-science-day , we will be screening a film Picture a Scientist https://www.pictureascientist.com/ with a discussion and reception (with food) afterwards.

The film will be screened at 4 PM in Olin Physical Lab on Friday February 11, 2022. Small group discussions and a reception on the rooftop (penthouse) of the Olin Physical Laboratory will follow (around 5:30 to 6 PM)

Students from all academic disciplines are welcome. All gender identities and expressions are welcome and encouraged to attend.

PICTURE A SCIENTIST is a feature-length documentary film chronicling the groundswell of researchers who are writing a new chapter for women scientists. A biologist, a chemist and a geologist lead viewer on a journey deep into their own experiences in the sciences, overcoming brutal harassment, institutional discrimination, and years of subtle slights to revolutionize the culture of science. From cramped laboratories to spectacular field stations, we also encounter scientific luminaries who provide new perspectives on how to make science itself more diverse, equitable, and open to all.

The event is organized and sponsored by the Departments of Biology, Chemistry, Computer Science, Engineering, Mathematics, Statistics, Physics and Wake Forest's Center for Functiona Materials as well as the Wake Forest undergraduate Women in STEM group.

Register Here

Course schedule for Spring 2022

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Mon: 01/10/2022	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/14/2022
2	Wed: 01/12/2022	Chap. 1	Electrostatic energy calculations	#2	01/19/2022
3	Fri: 01/14/2022	Chap. 1	Electrostatic energy calculations	#3	01/21/2022
	Mon: 01/17/2022		MLK Holiday -- no class		
4	Wed: 01/19/2022	Chap. 1 & 2	Electrostatic potentials and fields	#4	01/24/2022
5	Fri: 01/21/2022	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#5	01/26/2022
6	Mon: 01/24/2022	Chap. 1 - 3	Brief introduction to numerical methods	#6	01/28/2022
7	Wed: 01/26/2022	Chap. 2 & 3	Image charge constructions	#7	01/31/2022
8	Fri: 01/28/2022	Chap. 2 & 3	Cylindrical and spherical geometries	#8	02/02/2022
9	Mon: 01/31/2022	Chap. 3 & 4	Spherical geometry and multipole moments	#9	02/04/2022
	Wed: 02/02/2022	No class	Fire caution		
	Fri: 02/04/2022	No class	Fire caution		
10	Mon: 02/07/2022	Chap. 4	Dipoles and Dielectrics	#10	02/09/2022
11	Wed: 02/09/2022	Chap. 4	Dipoles and Dielectrics	#11	02/11/2022
12	Fri: 02/11/2022	Chap. 5	Magnetostatics		

PHY 712 -- Assignment #11

February 9, 2022

Finish reading Chapter 4 in **Jackson** .

1. Work problem 4.9(a) in **Jackson**. It is probably most convenient to use a coordinate system with the origin at the center of the dielectric sphere.

Review -- Focus on dipolar fields:

Dipole moment **p**:

$$\mathbf{p} \equiv \int d^3r' \, \mathbf{r}' \rho(\mathbf{r}')$$

For r outside the extent of $\rho(\mathbf{r})$:

Electrostatic potential from single dipole:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right)$$

Electrostatic field from single dipole:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{3\mathbf{r}\ (\mathbf{p}\cdot\mathbf{r}) - r^2\mathbf{p}}{r^5} - \frac{4\pi}{3}\mathbf{p}\delta^3(\mathbf{r})\right)$$

Now consider a distribution of dipoles and monopoles --

Electric polarization $\mathbf{P}(\mathbf{r})$ due to collection of dipoles :

$$\mathbf{P}(\mathbf{r}) \equiv \sum_i \mathbf{p}_i \delta^3(\mathbf{r}-\mathbf{r}_i)$$

Monopole electric charge density $\rho_{\text{mono}}(\mathbf{r})$:

$$\rho_{\text{mono}}(\mathbf{r}) \equiv \sum_i q_i \delta^3(\mathbf{r}-\mathbf{r}_i)$$

Define Displacement field: $\mathbf{D}(\mathbf{r}) \equiv \varepsilon_0 \mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r})$

Macroscopic form of Gauss's law: $\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho_{mono}(\mathbf{r})$

Review continued

Many materials are polarizable and produce a polarization field in the presence of an electric field with a proportionality constant χ_e :

$$\mathbf{P}(\mathbf{r}) = \varepsilon_0 \chi_e \mathbf{E}(\mathbf{r})$$

$$\mathbf{D}(\mathbf{r}) \equiv \varepsilon_0 \mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r}) = \varepsilon_0 (1 + \chi_e) \mathbf{E}(\mathbf{r}) \equiv \varepsilon \mathbf{E}(\mathbf{r})$$

ε represents the dielectric function of the material

Boundary value problems in dielectric materials

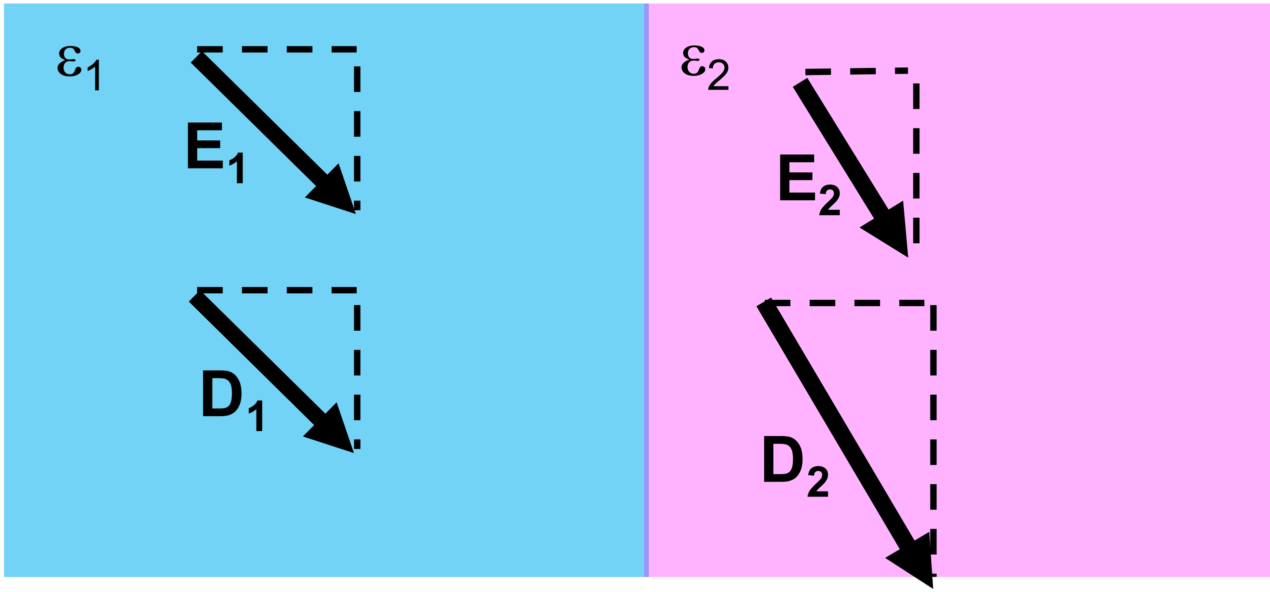
For $\rho_{\text{mono}}(\mathbf{r}) = 0$

$\nabla \cdot \mathbf{D}(\mathbf{r}) = 0$ and $\nabla \times \mathbf{E}(\mathbf{r}) = 0$

$\Rightarrow$ At a surface between two dielectrics, in terms of surface normal $\hat{\mathbf{r}}$:

$$\hat{\mathbf{r}} \cdot \mathbf{D}(\mathbf{r}) = \text{continuous} = \hat{\mathbf{r}} \times \mathbf{E}(\mathbf{r})$$

Boundary value problems in the presence of dielectrics – example:



For $\frac{\varepsilon_2}{\varepsilon_1} = 2$

For $\rho_{\text{mono}}(\mathbf{r}) = 0$: $\quad \nabla \cdot \mathbf{D}(\mathbf{r}) = 0$ and $\nabla \times \mathbf{E}(\mathbf{r}) = 0$

$\Rightarrow$ At a surface between two dielectrics, in terms of surface normal $\hat{\mathbf{r}}$:

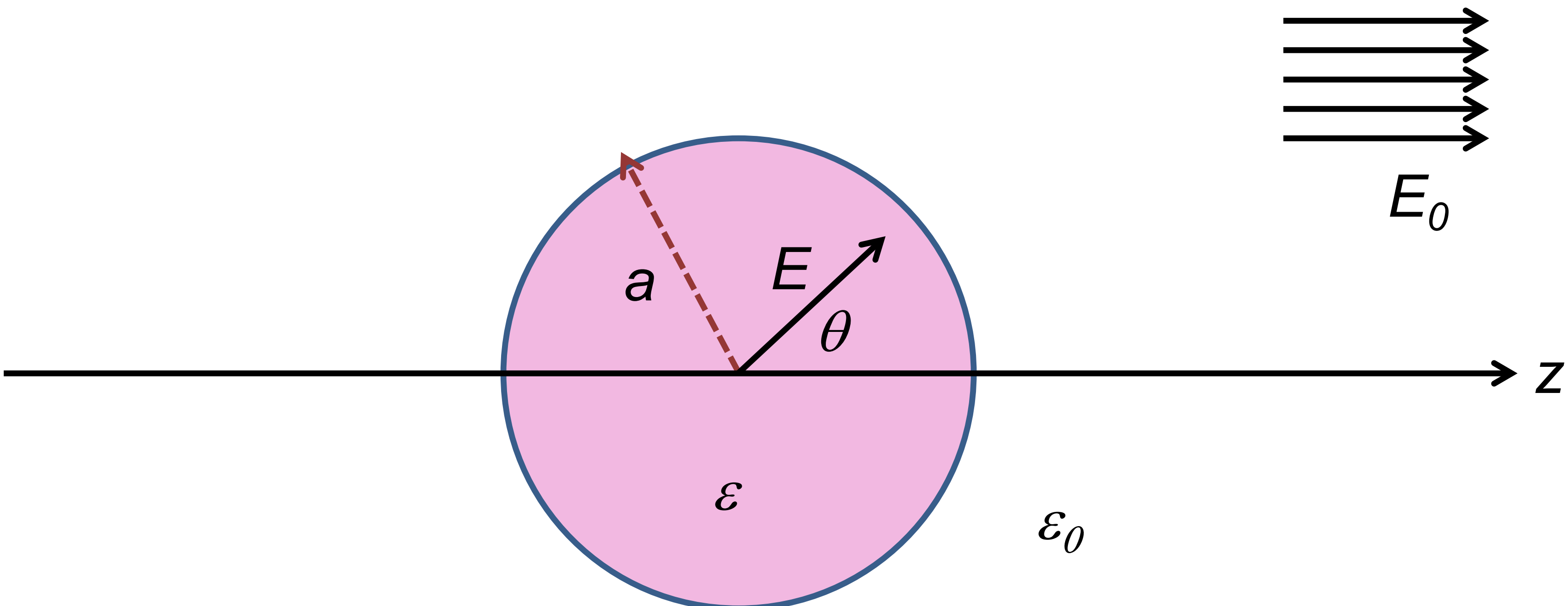
$\hat{\mathbf{r}} \cdot \mathbf{D}(\mathbf{r}) = \text{continuous} = \hat{\mathbf{r}} \times \mathbf{E}(\mathbf{r})$

For isotropic dielectrics:

$$D_{1n} = D_{2n} \qquad \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$$

$$E_{1t} = E_{2t} \qquad \frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$

Boundary value problems in the presence of dielectrics – example:



$$\nabla \cdot \mathbf{D}(\mathbf{r}) = 0 \quad \text{and} \quad \nabla \times \mathbf{E}(\mathbf{r}) = 0$$

$$\text{For } r \leq a \quad \mathbf{D}(\mathbf{r}) = -\varepsilon \nabla \Phi(\mathbf{r})$$

$$\text{For } r > a \quad \mathbf{D}(\mathbf{r}) = -\varepsilon_0 \nabla \Phi(\mathbf{r})$$

$$\text{At } r = a: \quad \varepsilon \frac{\partial \Phi_<(\mathbf{r})}{\partial r} = \varepsilon_0 \frac{\partial \Phi_>(\mathbf{r})}{\partial r}$$

$$\frac{\partial \Phi_<(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_>(\mathbf{r})}{\partial \theta}$$

Boundary value problems in the presence of dielectrics – example -- continued:

$$\Phi_<(\mathbf{r}) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$\Phi_>(\mathbf{r}) = \sum_{l=0}^{\infty}\left(B_l r^l + \frac{C_l}{r^{l+1}}\right)P_l(\cos\theta)$$

At $r = a$: $\quad \varepsilon \dfrac{\partial \Phi_<(\mathbf{r})}{\partial r} = \varepsilon_0 \dfrac{\partial \Phi_>(\mathbf{r})}{\partial r}$

$$\frac{\partial \Phi_<(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_>(\mathbf{r})}{\partial \theta}$$

For $r \to \infty \quad \Phi_>(\mathbf{r}) = -E_0 r\cos\theta$

Solution -- only $l = 1$ contributes

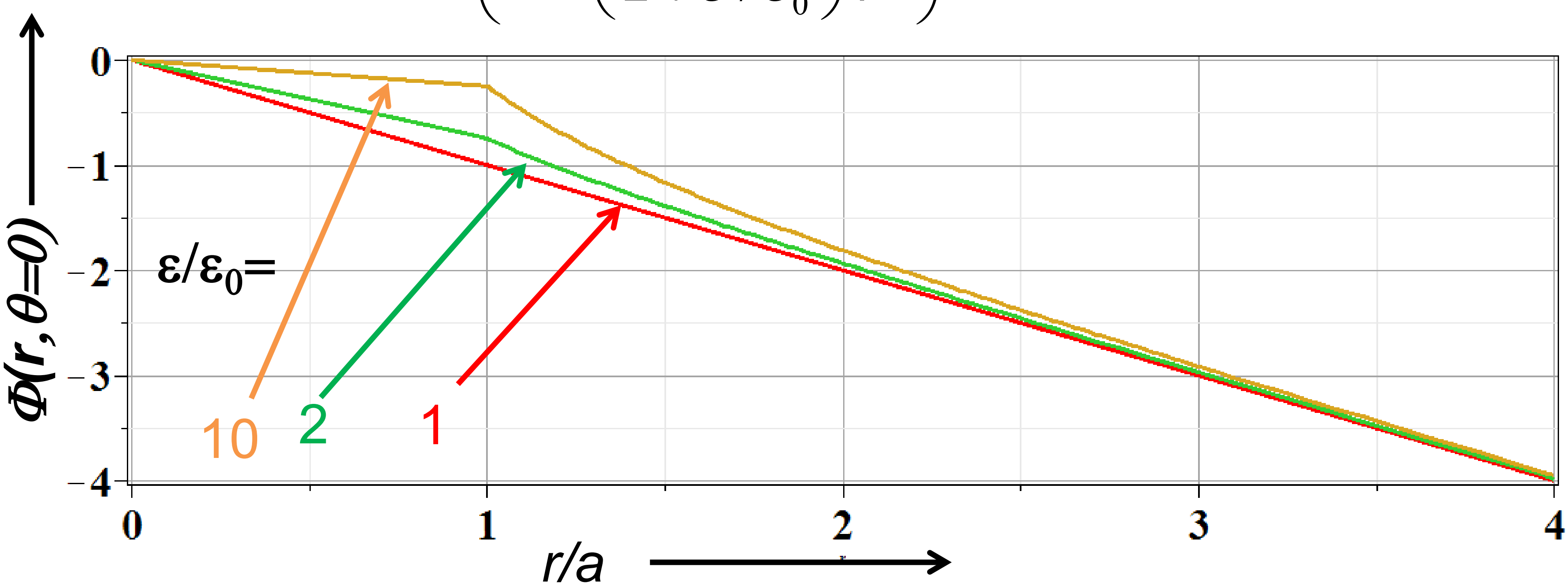
$$B_1 = -E_0$$

$$A_1 = -\left(\frac{3}{2 + \varepsilon/\varepsilon_0}\right)E_0 \qquad C_1 = \left(\frac{\varepsilon/\varepsilon_0 - 1}{2 + \varepsilon/\varepsilon_0}\right)a^3 E_0$$

Boundary value problems in the presence of dielectrics – example -- continued:

$$\Phi_<(\mathbf{r}) = -\left(\frac{3}{2+\varepsilon/\varepsilon_0}\right)E_0\, r\cos\theta$$

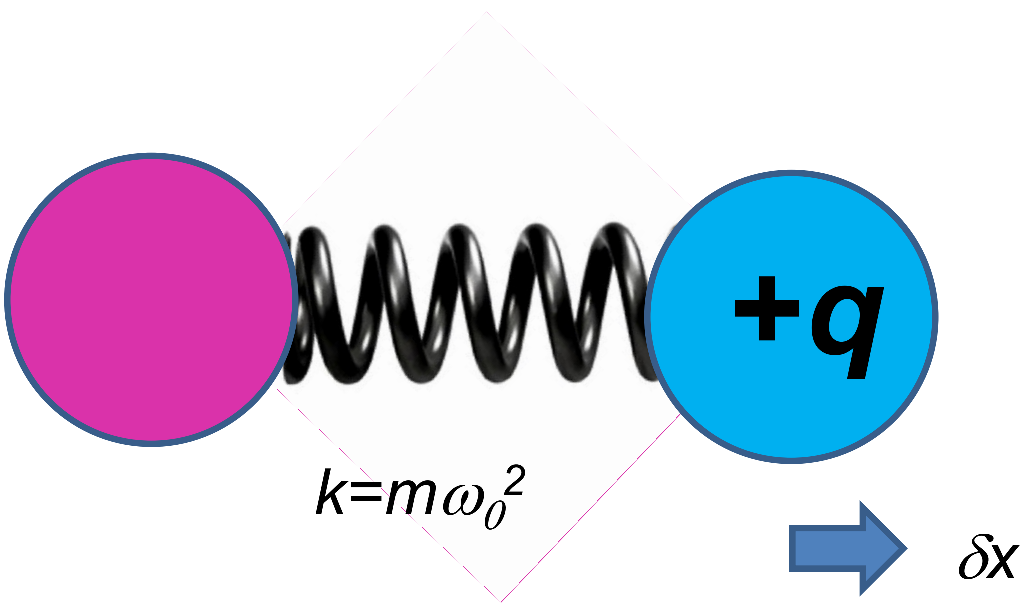
$$\Phi_>(\mathbf{r}) = -\left(r - \left(\frac{\varepsilon/\varepsilon_0 - 1}{2+\varepsilon/\varepsilon_0}\right)\frac{a^3}{r^2}\right)E_0 \cos\theta$$

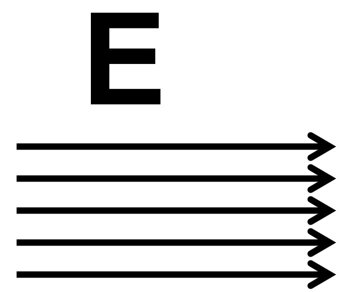


Microscopic origin of dipole moments

- Polarizable atoms/molecules
- Anisotropic charged molecules aligned in random directions

Polarizable isotropic atoms/molecules



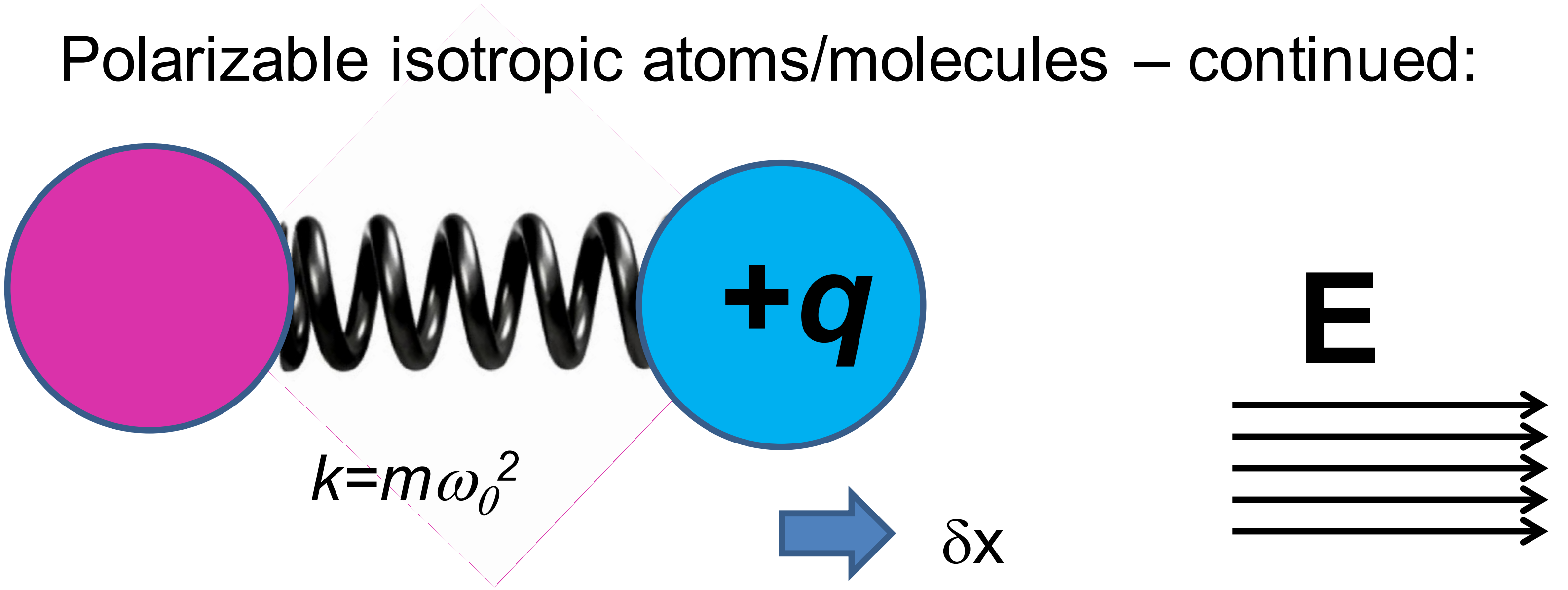


At equilibrium :

$$q\mathbf{E} - m\omega_0^2 \delta\mathbf{x} = 0$$

$$\delta\mathbf{x} = \frac{q\mathbf{E}}{m\omega_0^2}$$

Polarizable isotropic atoms/molecules – continued:



At equilibrium:

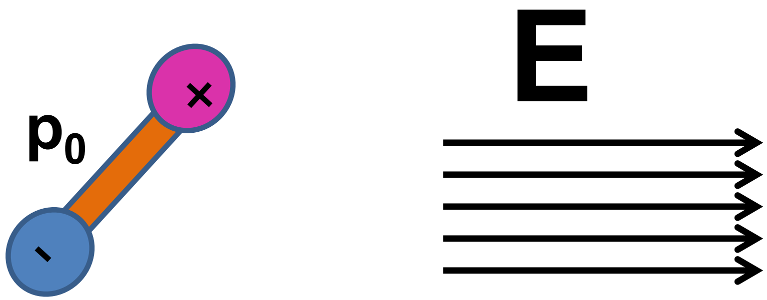
$$q\mathbf{E} - m\omega_0^2 \delta\mathbf{x} = 0$$

$$\delta\mathbf{x} = \frac{q\mathbf{E}}{m\omega_0^2}$$

Induced dipole moment:

$$p = q\delta\mathbf{x} = \frac{q^2}{m\omega_0^2}\mathbf{E} \equiv \varepsilon_0 \gamma_{mol} \mathbf{E} \quad \Rightarrow \gamma_{mol} = \frac{q^2}{m\omega_0^2 \varepsilon_0}$$

Alignment of molecules with permanent dipoles $\mathbf{p}_0$:

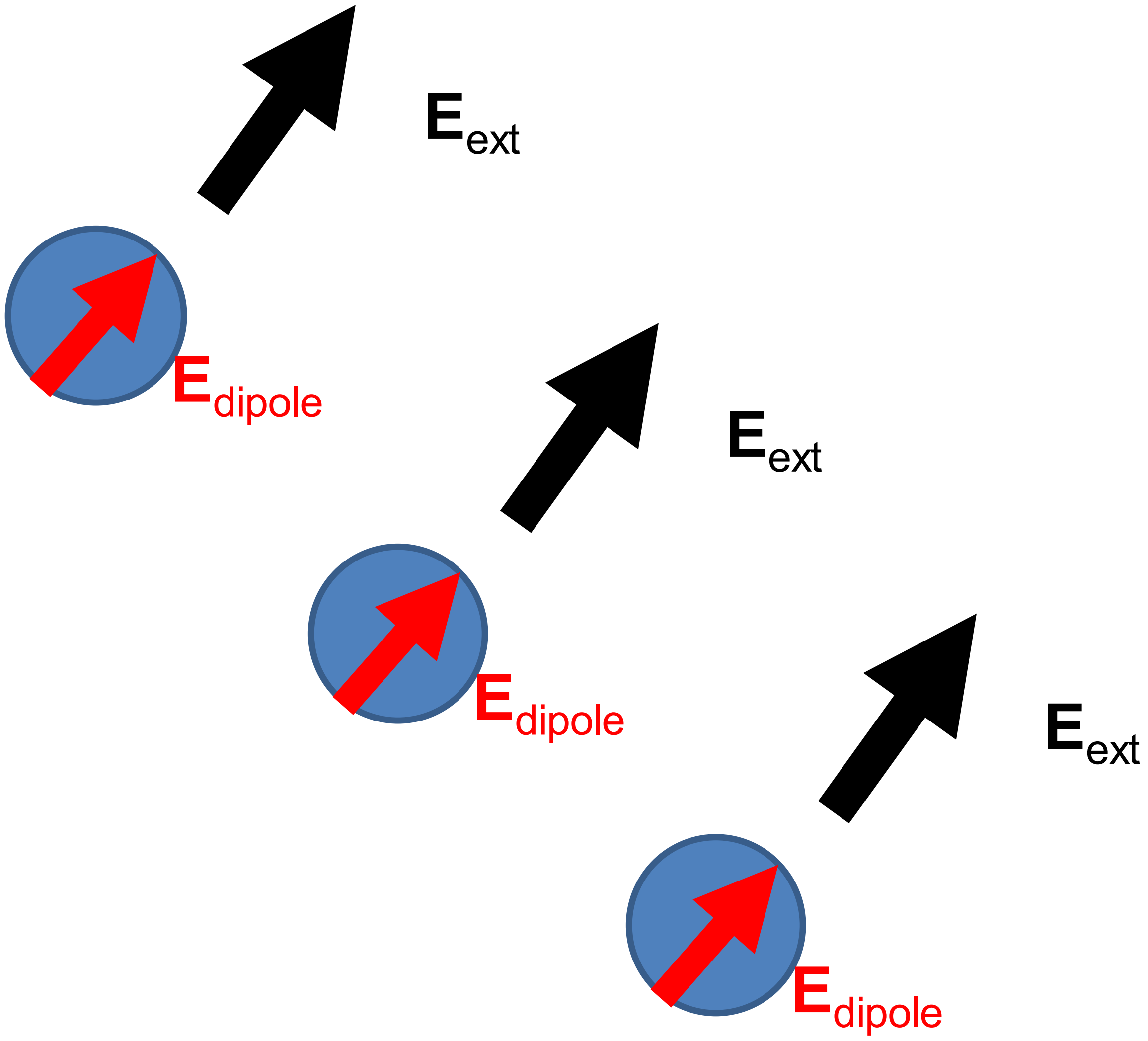


For a freely rotating dipole its average moment in an electric field, estimated assuming a Boltzmann distribution:

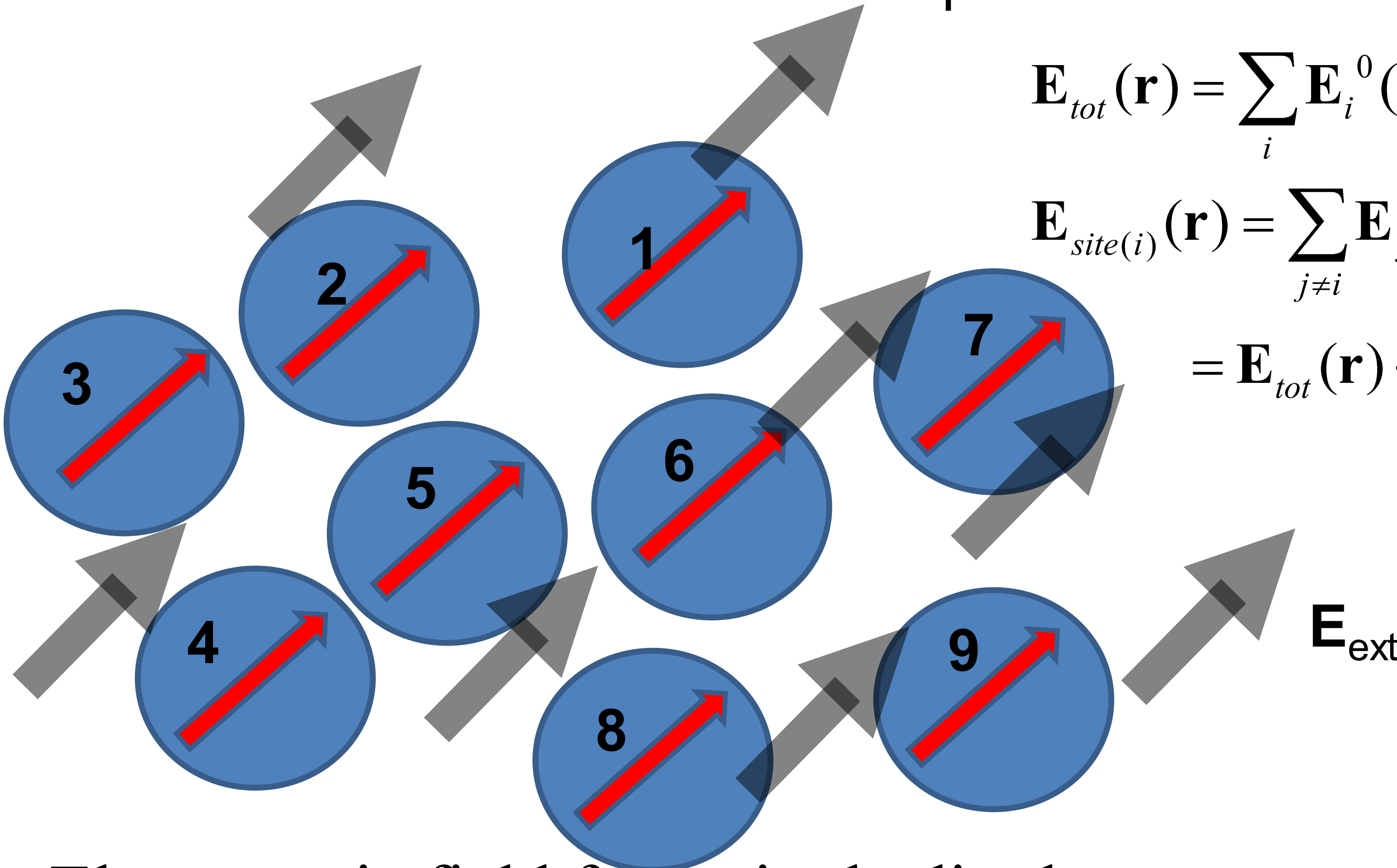
$$\langle \mathbf{p}_{mol} \rangle = \frac{\int d\Omega \;\; p_0 \cos\theta e^{p_0 E \cos\theta / kT}}{\int d\Omega e^{p_0 E \cos\theta / kT}} \approx \frac{1}{3} \frac{p_0^2}{kT} \mathbf{E} \text{ for } \frac{p_0 E}{kT} \ll 1$$

$$\langle \mathbf{p}_{mol} \rangle \approx \frac{1}{3} \frac{p_0^2}{kT} \mathbf{E} \equiv \varepsilon_0 \gamma_{mol} \mathbf{E} \qquad \Rightarrow \gamma_{mol} \approx \frac{1}{3} \frac{p_0^2}{kT \varepsilon_0}$$

Now consider a superposition of dipoles in an electric field



Field due to collection of induced dipoles



$$\mathbf{E}_{tot}(\mathbf{r}) = \sum_i \mathbf{E}_i^{\ 0}(\mathbf{r}) + \mathbf{E}_{ext}(\mathbf{r})$$

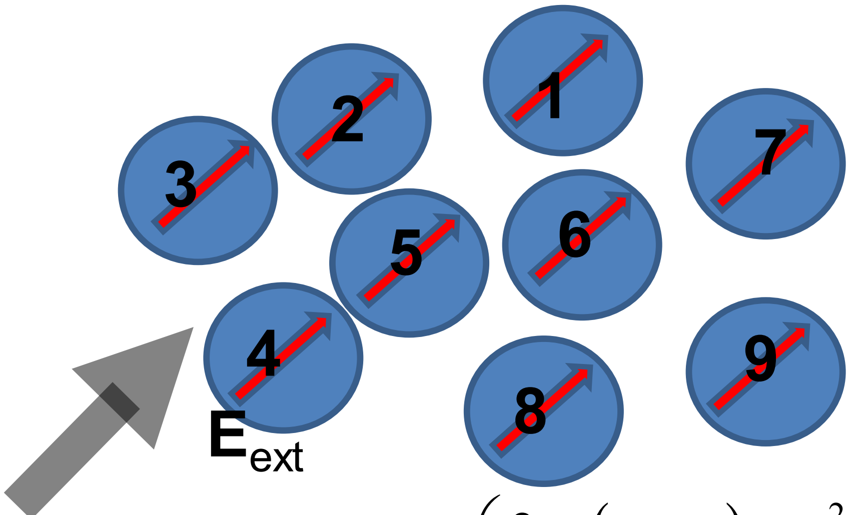
$$\mathbf{E}_{site(i)}(\mathbf{r}) = \sum_{j \neq i} \mathbf{E}_j^{\ 0}(\mathbf{r}) + \mathbf{E}_{ext}(\mathbf{r})$$

$$= \mathbf{E}_{tot}(\mathbf{r}) - \mathbf{E}_i^{\ 0}(\mathbf{r})$$

Electrostatic field from single dipole:

$$\mathbf{E}_i^0(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{3\mathbf{r}\ (\mathbf{p}_i \cdot \mathbf{r}) - r^2\mathbf{p}_i}{r^5} - \frac{4\pi}{3}\mathbf{p}_i\delta^3(\mathbf{r}-\mathbf{r}_i)\right)$$

Field due to collection of induced dipoles -- continued



$$\mathbf{E}_{tot}(\mathbf{r}) = \sum_i \mathbf{E}_i^{\,0}(\mathbf{r}) + \mathbf{E}_{ext}(\mathbf{r})$$

$$\mathbf{E}_{site(i)}(\mathbf{r}) = \sum_{j\neq i} \mathbf{E}_j^{\,0}(\mathbf{r}) + \mathbf{E}_{ext}(\mathbf{r})$$

$$= \mathbf{E}_{tot}(\mathbf{r}) - \mathbf{E}_i^{\,0}(\mathbf{r})$$

$$\mathbf{E}(\mathbf{r})_{tot} = \frac{1}{4\pi\varepsilon_0}\sum_i \left(\frac{3\mathbf{r}\,\left(\mathbf{p}_i\cdot\mathbf{r}\right) - r^2\mathbf{p}_i}{r^5} - \frac{4\pi}{3}\mathbf{p}_i\delta^3\left(\mathbf{r}-\mathbf{r}_i\right) \right) + \mathbf{E}_{ext}(\mathbf{r})$$

$$\mathbf{E}(\mathbf{r})_{site(i)} = \frac{1}{4\pi\varepsilon_0}\left(\sum_{j\neq i} \frac{3\mathbf{r}\,\left(\mathbf{p}_j\cdot\mathbf{r}\right) - r^2\mathbf{p}_j}{r^5} \right) + \mathbf{E}_{ext}(\mathbf{r}) = \mathbf{E}(\mathbf{r})_{tot} - \left(\mathbf{E}_i^0(\mathbf{r})\right)_{site(i)}$$

$$\left\langle \mathbf{E}_{site(i)} \right\rangle = \left\langle \mathbf{E}_{tot} \right\rangle + \frac{1}{V}\frac{1}{3\varepsilon_0}\left\langle \mathbf{p} \right\rangle = \left\langle \mathbf{E}_{tot} \right\rangle + \frac{1}{3\varepsilon_0}\left\langle \mathbf{P} \right\rangle$$

Averaging:

$$\langle\ \rangle \rightarrow \frac{1}{V}\int_V d^3r$$

Field due to collection of induced dipoles -- continued

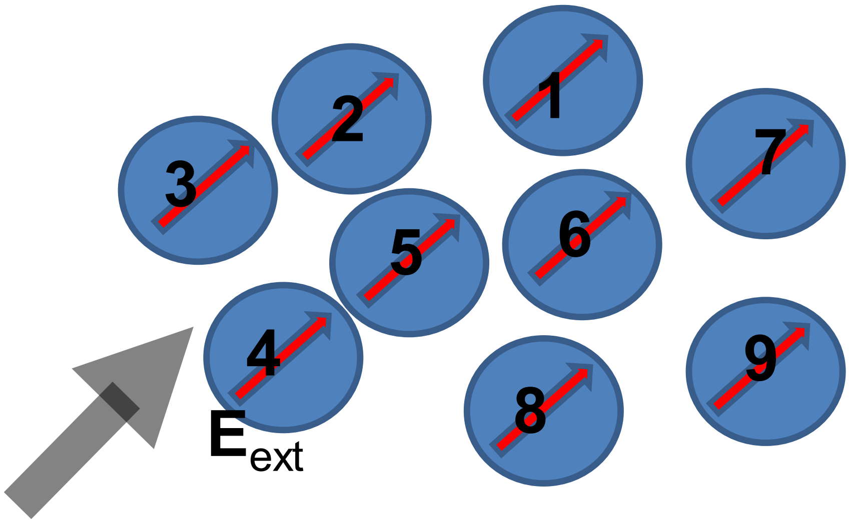
$$\mathbf{E}(\mathbf{r})_{site(i)} = \mathbf{E}(\mathbf{r})_{tot} - \left(\mathbf{E}_i^0(\mathbf{r})\right)_{site(i)}$$

$$= \mathbf{E}(\mathbf{r})_{tot} - \frac{1}{4\pi\varepsilon_0}\left(\frac{3\mathbf{r}\ (\mathbf{p}_i \cdot \mathbf{r}) - r^2\mathbf{p}_i}{r^5} - \frac{4\pi}{3}\mathbf{p}_i\delta^3(\mathbf{r}-\mathbf{r}_i)\right)$$

$$\left\langle \mathbf{E}_{site(i)} \right\rangle = \left\langle \mathbf{E}_{tot} \right\rangle - \left\langle \frac{1}{4\pi\varepsilon_0}\left(\frac{3\mathbf{r}\ (\mathbf{p}_i \cdot \mathbf{r}) - r^2\mathbf{p}_i}{r^5} - \frac{4\pi}{3}\mathbf{p}_i\delta^3(\mathbf{r}-\mathbf{r}_i)\right)\right\rangle$$

$$\left\langle \mathbf{E}_{site(i)} \right\rangle = \left\langle \mathbf{E}_{tot} \right\rangle + \frac{1}{V}\frac{1}{3\varepsilon_0}\left\langle \mathbf{p} \right\rangle = \left\langle \mathbf{E}_{tot} \right\rangle + \frac{1}{3\varepsilon_0}\left\langle \mathbf{P} \right\rangle$$

Field due to collection of induced dipoles -- continued



$$\langle \mathbf{E}_{site(i)} \rangle = \langle \mathbf{E}_{tot} \rangle + \frac{1}{3\varepsilon_0} \langle \mathbf{P} \rangle$$

$$\langle \mathbf{p} \rangle = \varepsilon_0 \gamma_{mol} \langle \mathbf{E}_{site} \rangle$$

$$\langle \mathbf{P} \rangle = \frac{1}{V} \langle \mathbf{p} \rangle = \frac{\varepsilon_0 \gamma_{mol}}{V} \left(\langle \mathbf{E}_{tot} \rangle + \frac{1}{3\varepsilon_0} \langle \mathbf{P} \rangle \right)$$

$$\langle \mathbf{P} \rangle = \frac{\varepsilon_0 \gamma_{mol}}{V} \frac{\langle \mathbf{E}_{tot} \rangle}{1 - \frac{\gamma_{mol}}{3V}} = \varepsilon_0 \chi_e \langle \mathbf{E}_{tot} \rangle$$

Claussius-Mossotti equation

$$\chi_e = \frac{\frac{\gamma_{mol}}{V}}{1 - \frac{\gamma_{mol}}{3V}} = \frac{\varepsilon}{\varepsilon_0} - 1 \qquad \gamma_{mol} = 3V \left(\frac{\varepsilon / \varepsilon_0 - 1}{\varepsilon / \varepsilon_0 + 2} \right)$$

Example of the Clausius-Mossotti equation –

Pentane (C_5H_{12}) at various densities

Density (g/cm3)	Mol/m3	ε/ε0	3V*(ε/ε0-1)/(ε/ε0+2)
0.613	5.12536E+27	1.82	1.25646E-28
0.701	5.86114E+27	1.96	1.24084E-28
0.796	6.65544E+27	2.12	1.22536E-28
0.865	7.23236E+27	2.24	1.2131E-28
0.907	7.58353E+27	2.33	1.2151E-28

$$\gamma_{mol} = 1.2 \times 10^{-28}\, m^3 = 0.12\, nm^3$$

Re-examination of electrostatic energy in dielectric media

$$W = \frac{1}{2}\int d^3r \rho_{mono}(\mathbf{r})\Phi(\mathbf{r})$$

In terms of displacement field:

$$\nabla \cdot \mathbf{D} = \rho_{mono}(\mathbf{r})$$

$$W = \frac{1}{2}\int d^3r \ \nabla \cdot \mathbf{D} \ \Phi(\mathbf{r}) = \frac{1}{2}\int d^3r \ \nabla \cdot \left(\mathbf{D}(\mathbf{r}) \ \Phi(\mathbf{r})\right) - \frac{1}{2}\int d^3r \ \mathbf{D}(\mathbf{r}) \cdot \nabla\Phi(\mathbf{r})$$

$$= \qquad 0 \qquad + \ \frac{1}{2}\int d^3r \ \mathbf{D}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})$$

$$W = \frac{1}{2}\int d^3r \ \mathbf{D}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})$$

Comment on the "Modern Theory of Polarization"

Some references:

- R. D.King-Smith and D. Vanderbilt, Phys. Rev. B 47, 1651 (1993)
- R. Resta, Rev. Mod. Physics **66**, 699 (1994)
- R. Resta, J. Phys. Condens. Matter 23, 123201 (2010)
- N. A. Spaldin, J. Solid State Chem. **195**, 2 (2012)

Basic equations:

$$\varepsilon_0 \nabla \cdot \mathbf{E} = \rho_{tot} = \rho_{bound} + \rho_{mono}$$

$$\nabla \cdot \mathbf{P} = \rho_{bound}$$

$$\nabla \cdot \mathbf{D} = \rho_{mono}$$

$$\varepsilon_0 \mathbf{E} = \mathbf{D} + \mathbf{P}$$

Note: In general **P** is highly dependent on the boundary values; often it is more convenient/meaningful to calculate **ΔP**.

Comment on the "Modern Theory of Polarization" -- continued

$$\nabla \cdot \Delta \mathbf{P} = \Delta\rho_{bound} = \Delta\rho^{nuclear}_{bound} + \Delta\rho^{electronic}_{bound}$$

$$\Delta \mathbf{P}^{electronic} = -\frac{e}{V_{crystal}} \sum_n \langle w_{n0} | \mathbf{r} | w_{n0} \rangle$$

Note: The concept of the polarization of a periodic solid is not unique:

N.A. Spaldin / Journal of Solid State Chemistry 195 (2012) 2–10 3

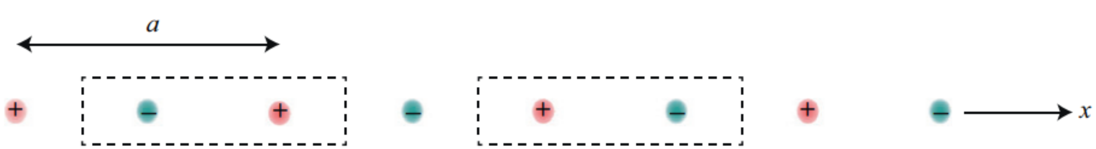
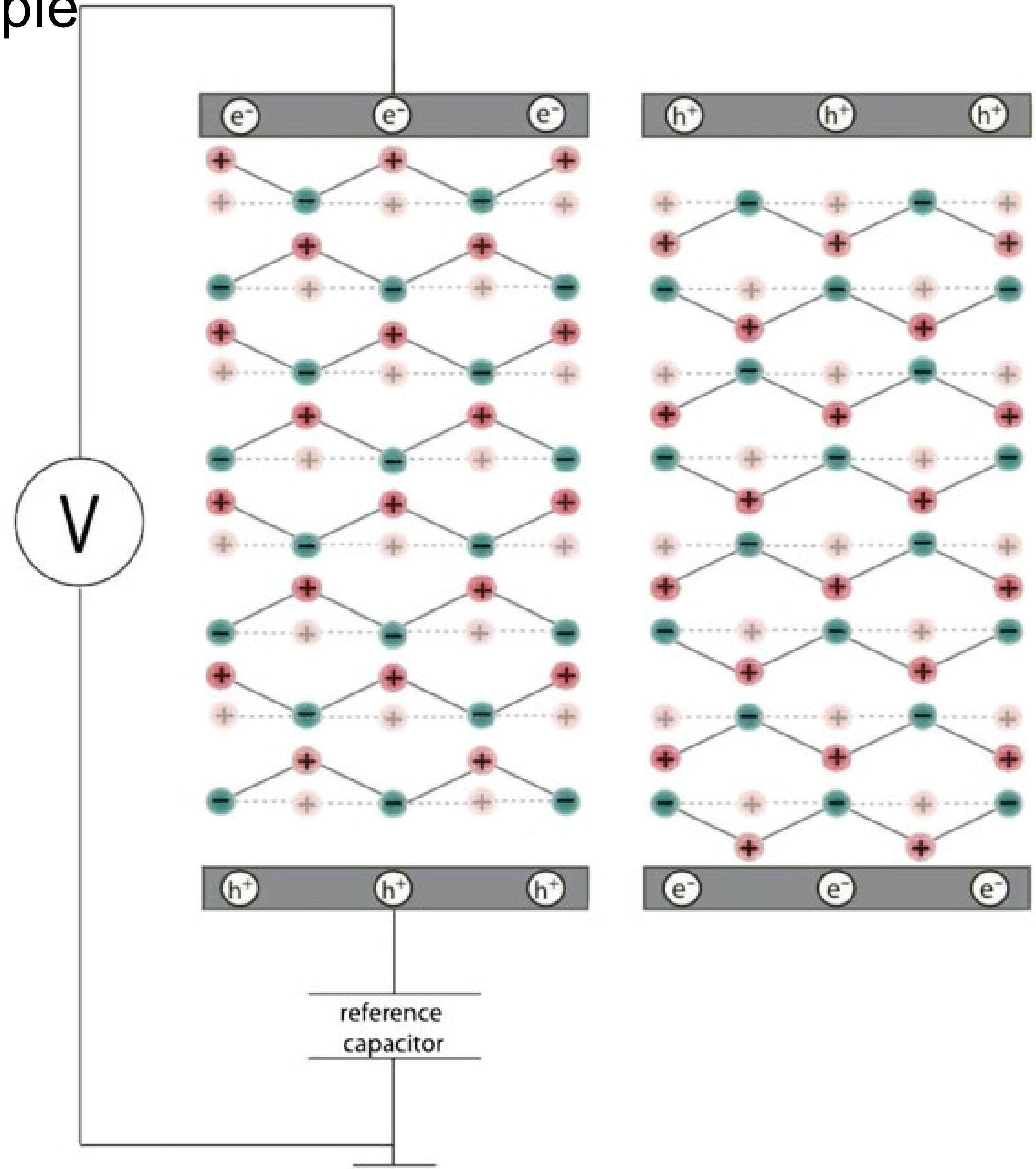


Fig. 1. One-dimensional chain of alternating anions and cations, spaced a distance $a/2$ apart, where a is the lattice constant. The dashed lines indicate two representative unit cells which are used in the text for calculation of the polarization.

ΔP example



ΔP example -- linear visualization

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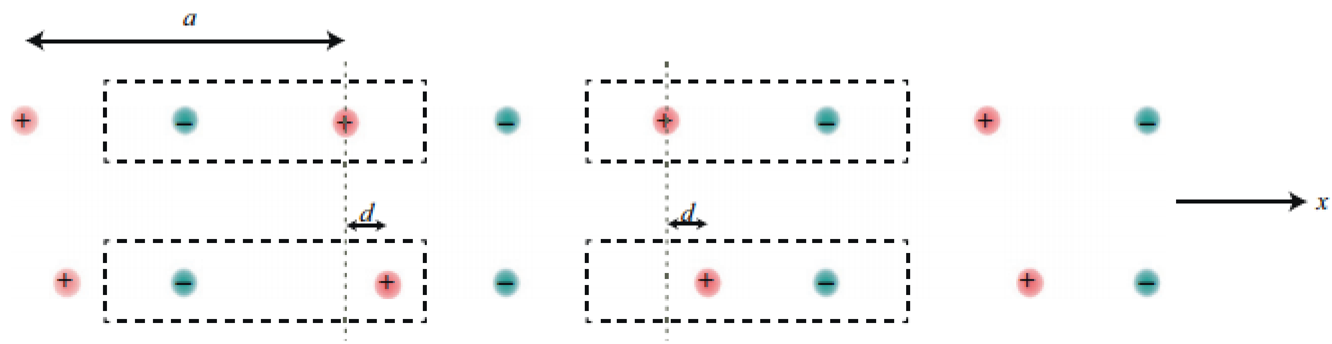
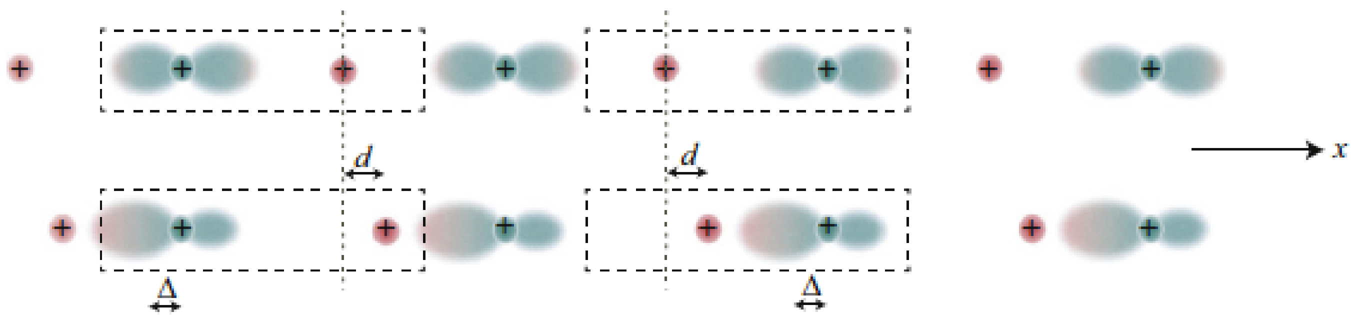


Effects on the electronic distribution



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