

PHY 712 Electrodynamics

11-11:50 AM MWF in Olin 103

Class notes for Lecture 13:

Continue reading Chapter 5

A. Examples of magnetostatic fields

B. Magnetic dipoles

C. Hyperfine interaction

Course schedule for Spring 2022

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Mon: 01/10/2022	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/14/2022
2	Wed: 01/12/2022	Chap. 1	Electrostatic energy calculations	#2	01/19/2022
3	Fri: 01/14/2022	Chap. 1	Electrostatic energy calculations	#3	01/21/2022
	Mon: 01/17/2022		MLK Holiday -- no class		
4	Wed: 01/19/2022	Chap. 1 & 2	Electrostatic potentials and fields	#4	01/24/2022
5	Fri: 01/21/2022	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#5	01/26/2022
6	Mon: 01/24/2022	Chap. 1 - 3	Brief introduction to numerical methods	#6	01/28/2022
7	Wed: 01/26/2022	Chap. 2 & 3	Image charge constructions	#7	01/31/2022
8	Fri: 01/28/2022	Chap. 2 & 3	Cylindrical and spherical geometries	#8	02/02/2022
9	Mon: 01/31/2022	Chap. 3 & 4	Spherical geometry and multipole moments	#9	02/04/2022
	Wed: 02/02/2022	No class	Fire caution		
	Fri: 02/04/2022	No class	Fire caution		
10	Mon: 02/07/2022	Chap. 4	Dipoles and Dielectrics	#10	02/09/2022
11	Wed: 02/09/2022	Chap. 4	Dipoles and Dielectrics	#11	02/11/2022
12	Fri: 02/11/2022	Chap. 5	Magnetostatics	#12	02/14/2022
13	Mon: 02/14/2022	Chap. 5	Magnetic dipoles and hyperfine interaction	#13	02/16/2022
14	Wed: 02/16/2022	Chap. 5	Magnetic dipoles and dipolar fields		
16	Fri: 02/18/2022	Chap. 6	Maxwell's Equations		

PHY 712 -- Assignment #13

February 14, 2022

Continue reading Chapter 5 in **Jackson** .

1. Work problem 5.13 in **Jackson**.

Comment – in class, we will discuss a similar problem with a model current density is due to a rotating uniformly charged sphere

$$\mathbf{J}(\mathbf{r}) = \begin{cases} \rho_0 \boldsymbol{\omega} \times \mathbf{r} & \text{for } r \leq a \\ 0 & \text{otherwise} \end{cases}$$

For HW 13 the model current density is due to a rotating charged spherical shell:

$$\mathbf{J}(\mathbf{r}) = \sigma \boldsymbol{\omega} \times \mathbf{r} \delta(r - a)$$

For the model current density --

$$\mathbf{J}(\mathbf{r}) = \begin{cases} \rho_0 \boldsymbol{\omega} \times \mathbf{r} & \text{for } r \leq a \\ 0 & \text{otherwise} \end{cases}$$

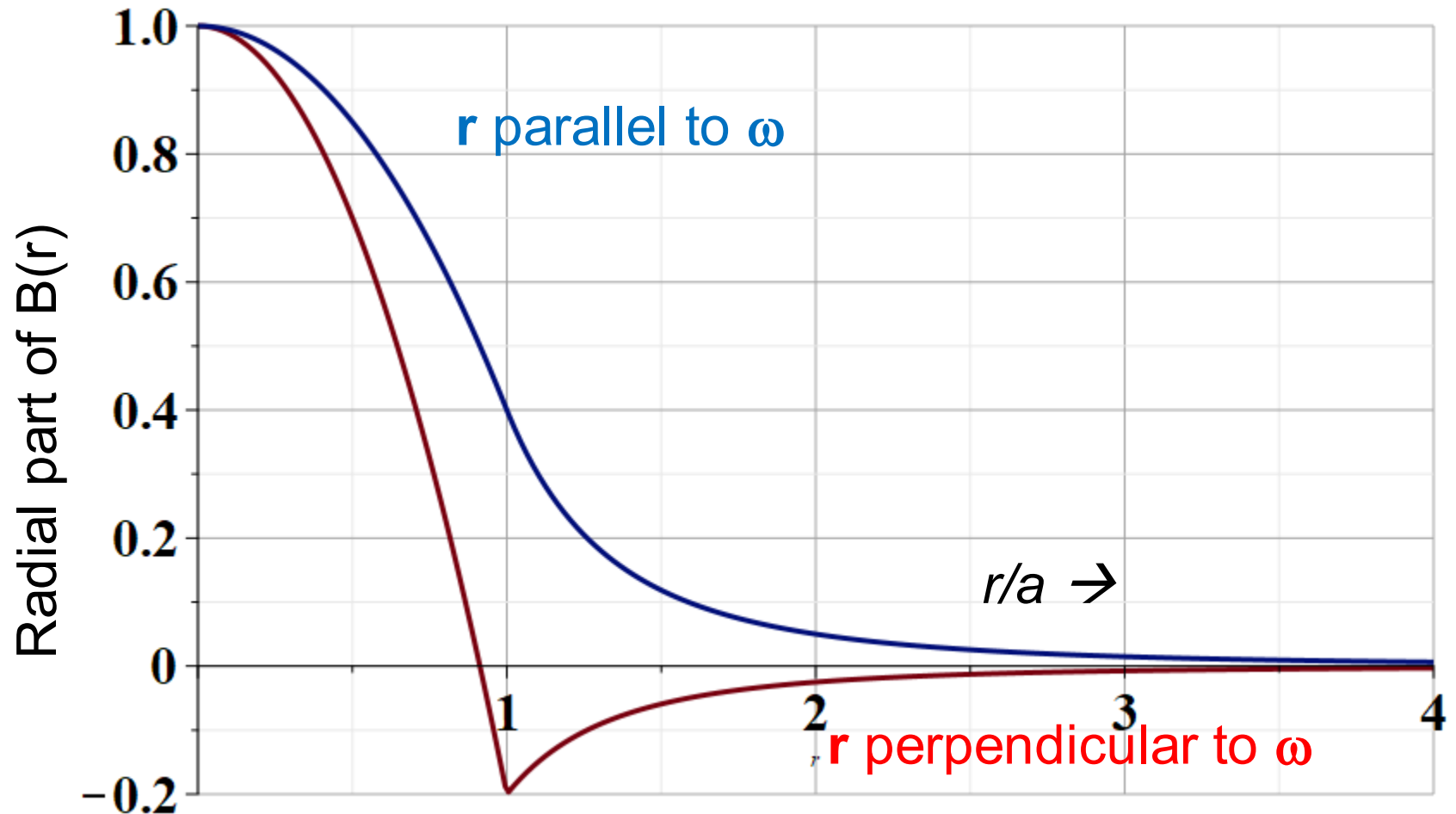
$$\mathbf{A}(\mathbf{r}) = \begin{cases} \frac{\mu_0 \rho_0}{3} \boldsymbol{\omega} \times \mathbf{r} \left(\frac{a^2}{2} - \frac{3r^2}{10} \right) & \text{for } r \leq a \\ \frac{\mu_0 \rho_0}{3} \boldsymbol{\omega} \times \mathbf{r} \frac{a^5}{5r^3} & \text{for } r \geq a \end{cases}.$$

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) = \begin{cases} \frac{\mu_0 \rho_0}{3} \left[\boldsymbol{\omega} \left(a^2 - \frac{6}{5} r^2 \right) + \frac{3}{5} \mathbf{r} (\boldsymbol{\omega} \cdot \mathbf{r}) \right] & \text{for } r \leq a \\ \frac{\mu_0 \rho_0}{3} \left[-\boldsymbol{\omega} \frac{a^5}{5r^3} + \frac{3a^5}{5r^5} \mathbf{r} (\boldsymbol{\omega} \cdot \mathbf{r}) \right] & \text{for } r \geq a \end{cases}$$

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \frac{\mu_0 \rho_0}{3} \boldsymbol{\omega} \times \mathbf{r} \left(\frac{a^2}{2} - \frac{3r^2}{10} \right) & \text{for } r \leq a \\ \frac{\mu_0 \rho_0}{3} \boldsymbol{\omega} \times \mathbf{r} \frac{a^5}{5r^3} & \text{for } r \geq a \end{cases} .$$



$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) = \begin{cases} \frac{\mu_0 \rho_0}{3} \left[\boldsymbol{\omega} \left(a^2 - \frac{6}{5} r^2 \right) + \frac{3}{5} \mathbf{r} (\boldsymbol{\omega} \cdot \mathbf{r}) \right] & \text{for } r \leq a \\ \frac{\mu_0 \rho_0}{3} \left[-\boldsymbol{\omega} \frac{a^5}{5r^3} + \frac{3a^5}{5r^5} \mathbf{r} (\boldsymbol{\omega} \cdot \mathbf{r}) \right] & \text{for } r \geq a \end{cases}$$



Comment about spherical polar coordinates

Ref: <https://www.cpp.edu/~ajm/materials/delsph.pdf>

Spherical Coordinates

Transforms

The forward and reverse coordinate transformations are

$$r = \sqrt{x^2 + y^2 + z^2}$$

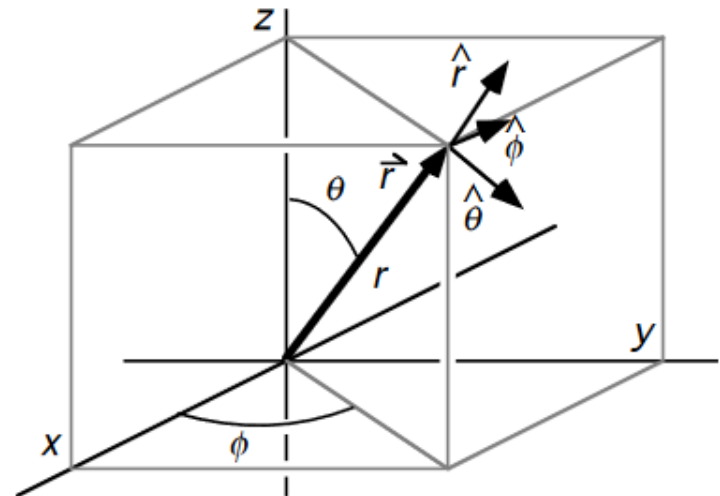
$$\theta = \arctan\left(\sqrt{x^2 + y^2}, z\right)$$

$$\phi = \arctan(y, x)$$

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$



where we *formally* take advantage of the *two argument* arctan function to eliminate quadrant confusion.

Unit Vectors

The unit vectors in the spherical coordinate system are functions of position. It is convenient to express them in terms of the *spherical* coordinates and the unit vectors of the *rectangular* coordinate system which are *not* themselves functions of position.

$$\hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{r} = \hat{x} \sin \theta \cos \phi + \hat{y} \sin \theta \sin \phi + \hat{z} \cos \theta$$

$$\hat{\phi} = \frac{\hat{z} \times \hat{r}}{\sin \theta} = -\hat{x} \sin \phi + \hat{y} \cos \phi$$

$$\hat{\theta} = \hat{\phi} \times \hat{r} = \hat{x} \cos \theta \cos \phi + \hat{y} \cos \theta \sin \phi - \hat{z} \sin \theta$$

Various forms of Ampere's law :

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r})$$

Vector potential: $\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$

For Coulomb gauge: $\nabla \cdot \mathbf{A}(\mathbf{r}) = 0$

$$\Rightarrow \nabla^2 \mathbf{A}(\mathbf{r}) = -\mu_0 \mathbf{J}(\mathbf{r})$$

For confined current density :

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3 r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

Other examples of current density sources:

Quantum mechanical expression for current density

for a particle of mass M and charge e and of probability amplitude $\Psi(\mathbf{r})$:

$$\mathbf{J}(\mathbf{r}) = -\frac{e\hbar}{2Mi} (\Psi^*(\mathbf{r})\nabla\Psi(\mathbf{r}) - \Psi(\mathbf{r})\nabla\Psi^*(\mathbf{r}))$$

For an electron in a spherical potential (such as in an atom):

$$\Psi(\mathbf{r}) \equiv \Psi_{nlm_l}(\mathbf{r}) = R_{nl}(r)Y_{lm_l}(\hat{\mathbf{r}})$$

$$\begin{aligned}\mathbf{J}(\mathbf{r}) &= \frac{e\hbar}{2Mi} |R_{nl}(r)|^2 \frac{1}{r \sin \theta} \left(Y_{lm_l}^*(\hat{\mathbf{r}}) \frac{\partial Y_{lm_l}(\hat{\mathbf{r}})}{\partial \varphi} - Y_{lm_l}(\hat{\mathbf{r}}) \frac{\partial Y_{lm_l}^*(\hat{\mathbf{r}})}{\partial \varphi} \right) \hat{\boldsymbol{\phi}} \\ &= \frac{e\hbar}{M} \frac{m_l}{r \sin \theta} |\Psi_{nlm_l}(\mathbf{r})|^2 \hat{\boldsymbol{\phi}}\end{aligned}$$

Note that: $\hat{\boldsymbol{\phi}} = -\sin \varphi \hat{\mathbf{x}} + \cos \varphi \hat{\mathbf{y}} = \frac{\hat{\mathbf{z}} \times \mathbf{r}}{r \sin \theta}$

$$\mathbf{J}(\mathbf{r}) = \frac{e\hbar}{M} \frac{m_l}{r^2 \sin^2 \theta} |\Psi_{nlm_l}(\mathbf{r})|^2 (\hat{\mathbf{z}} \times \mathbf{r})$$

Details of the electron orbital magnetic dipole moment

$$\mathbf{J}(\mathbf{r}) = \frac{e\hbar}{m_e} \frac{m_l}{r \sin \theta} \left| \Psi_{nlm_l}(\mathbf{r}) \right|^2 \hat{\boldsymbol{\phi}}$$

Note that: $\hat{\boldsymbol{\phi}} = -\sin \varphi \hat{\mathbf{x}} + \cos \varphi \hat{\mathbf{y}}$

Magnetic dipole moment:

$$\begin{aligned} \mathbf{m} &= \frac{1}{2} \int d^3 r' \mathbf{r}' \times \mathbf{J}(\mathbf{r}') = -\frac{e\hbar m_l}{2m_e} \int d^3 r' \frac{\mathbf{r}' \times \hat{\boldsymbol{\phi}}'}{r' \sin \theta'} \left| \Psi_{nlm_l}(\mathbf{r}') \right|^2 \\ &= -\frac{e\hbar m_l}{2m_e} \int d^3 r' \frac{-r' \hat{\boldsymbol{\theta}}'}{r' \sin \theta'} \left| \Psi_{nlm_l}(\mathbf{r}') \right|^2 \end{aligned}$$

Note that: $\hat{\boldsymbol{\theta}} = \cos \theta \cos \varphi \hat{\mathbf{x}} + \cos \theta \sin \varphi \hat{\mathbf{y}} - \sin \theta \hat{\mathbf{z}}$

$$\begin{aligned} \mathbf{m} &= -\frac{e\hbar m_l \hat{\mathbf{z}}}{2m_e} \int d^3 r' \left| \Psi_{nlm_l}(\mathbf{r}') \right|^2 \\ &= -\frac{e\hbar m_l}{2m_e} \hat{\mathbf{z}} \end{aligned}$$

Summary of magnetic field generated by point magnetic dipole moment discussed in the detailed notes:

$$\mathbf{B}_{\mu_e}(\mathbf{r}) = \frac{\mu_0}{4\pi} \left(\frac{3\hat{\mathbf{r}}(\boldsymbol{\mu}_e \cdot \hat{\mathbf{r}}) - \boldsymbol{\mu}_e}{r^3} + \frac{8\pi}{3} \boldsymbol{\mu}_e \delta(\mathbf{r}) \right)$$

Magnetic field near nucleus due to orbiting electron:

$$\mathbf{B}_O(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{e}{m_e} L_z \hat{\mathbf{z}} \left\langle \frac{1}{r^3} \right\rangle$$

"Hyperfine" interaction energy:

$$\begin{aligned} \mathcal{H}_{HF} &= -\boldsymbol{\mu}_N \cdot (\mathbf{B}_{\mu_e}(\mathbf{r}) + \mathbf{B}_O(\mathbf{r})) \\ &= \frac{\mu_0}{4\pi} \left(\frac{3(\boldsymbol{\mu}_N \cdot \hat{\mathbf{r}})(\boldsymbol{\mu}_e \cdot \hat{\mathbf{r}}) - \boldsymbol{\mu}_N \cdot \boldsymbol{\mu}_e}{r^3} + \frac{8\pi}{3} \boldsymbol{\mu}_N \cdot \boldsymbol{\mu}_e \delta(\mathbf{r}) + \frac{e}{m_e} \left\langle \frac{\mathbf{L} \cdot \boldsymbol{\mu}_N}{r^3} \right\rangle \right) \end{aligned}$$

$$\mathcal{H}_{HF} = \frac{\mu_0}{4\pi} \left(\frac{3(\boldsymbol{\mu}_N \cdot \hat{\mathbf{r}})(\boldsymbol{\mu}_e \cdot \hat{\mathbf{r}}) - \boldsymbol{\mu}_N \cdot \boldsymbol{\mu}_e}{r^3} + \frac{8\pi}{3} \boldsymbol{\mu}_N \cdot \boldsymbol{\mu}_e \delta(\mathbf{r}) + \frac{e}{m_e} \left\langle \frac{\mathbf{L} \cdot \boldsymbol{\mu}_N}{r^3} \right\rangle \right)$$

