PHY 712 Electrodynamics
11-11:50 AM MWF in Olin 103

Plan for Lecture 14:

Finish reading Chapter 5

1. Recap of hyperfine interaction
2. Macroscopic magnetization density M
3. H field and its relation to B
4. Magnetic boundary values

PHYSICS COLLOQUIUM

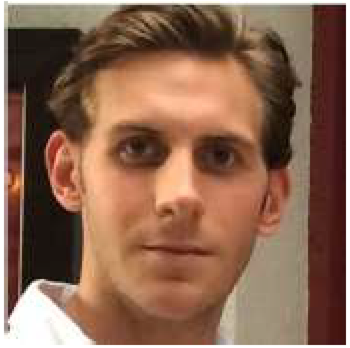
4 PM Olin 101

THURSDAY

•

FEBRUARY 17, 2022

"The Physicist's Secret Sauce: Why a Degree in the Sciences is Valuable on Wall Street"



Mr. Andrew Barelli

Senior Enterprise Risk Manager
Revolut
New York, NY

The finance career of a recent (2016) WFU physics graduate will be described, with particular attention to the skill set that is learned as a physics undergraduate and that provides an edge on Wall Street. Also discussed will be the current state of financial technology ("Fintech") and the ultimate goal of remaking the banking industry. Mr. Barelli will share specific projects from his tenure at Bloomberg and at Revolut. An analogy will be drawn between the thresholding of key risk indicators (KRIs) and the typical experimental design of an experimental or theoretical physics research project. Thus, it is unsurprising that students trained to design and troubleshoot research projects are uniquely well-equipped for success in the financial industry.

4:00 pm - Olin 101*

*Link provided for those unable to attend in person.
Note: For additional information on the seminar or to obtain the video conference link, contact wfuphys@wfu.edu

Reception at 3:30pm - Olin Lounge*

Course schedule for Spring 2022

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Mon: 01/10/2022	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/14/2022
2	Wed: 01/12/2022	Chap. 1	Electrostatic energy calculations	#2	01/19/2022
3	Fri: 01/14/2022	Chap. 1	Electrostatic energy calculations	#3	01/21/2022
	Mon: 01/17/2022		MLK Holiday -- no class		
4	Wed: 01/19/2022	Chap. 1 & 2	Electrostatic potentials and fields	#4	01/24/2022
5	Fri: 01/21/2022	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#5	01/26/2022
6	Mon: 01/24/2022	Chap. 1 - 3	Brief introduction to numerical methods	#6	01/28/2022
7	Wed: 01/26/2022	Chap. 2 & 3	Image charge constructions	#7	01/31/2022
8	Fri: 01/28/2022	Chap. 2 & 3	Cylindrical and spherical geometries	#8	02/02/2022
9	Mon: 01/31/2022	Chap. 3 & 4	Spherical geometry and multipole moments	#9	02/04/2022
	Wed: 02/02/2022	No class	Fire caution		
	Fri: 02/04/2022	No class	Fire caution		
10	Mon: 02/07/2022	Chap. 4	Dipoles and Dielectrics	#10	02/09/2022
11	Wed: 02/09/2022	Chap. 4	Dipoles and Dielectrics	#11	02/11/2022
12	Fri: 02/11/2022	Chap. 5	Magnetostatics	#12	02/14/2022
13	Mon: 02/14/2022	Chap. 5	Magnetic dipoles and hyperfine interaction	#13	02/16/2022
14	Wed: 02/16/2022	Chap. 5	Magnetic dipoles and dipolar fields	#14	02/18/2022
16	Fri: 02/18/2022	Chap. 6	Maxwell's Equations		
17	Mon: 02/21/2022	Chap. 1-6	Homework review & and presentations		

Proposed review session for Monday 2/21/2022

Rational:

1. Possible method of increasing engagement?
2. Facilitate understanding of outstanding homework assignments???

Plan:

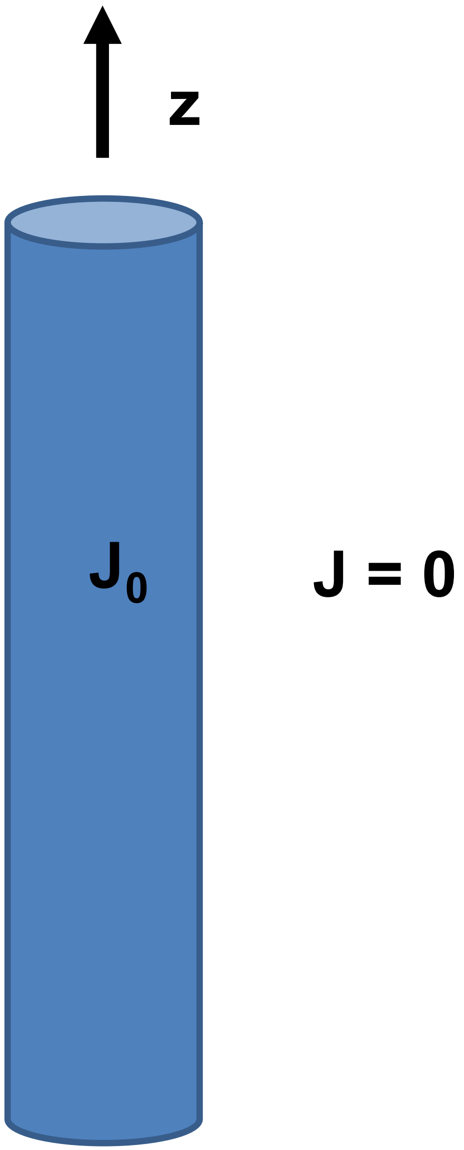
Use two class periods with student presentations of chosen homework solutions – one from PHY 712 and one from PHY 742, allotting ~20 minutes per student.

Signup schedule to follow --

Proposed schedule for Monday's HW presentations

Time	Name	PHY 712	PHY 742
11:05-11:25			
11:25-11:45			
11:45-12:05			
12:10-12:30			
12:30-12:50			

Comment about HW 12



Summary of hyperfine interaction form:

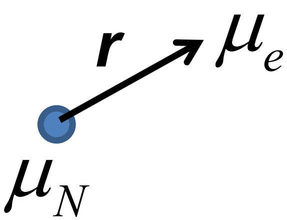
Interactions between magnetic dipoles

Sources of magnetic dipoles and other sources of magnetism in an atom:

- Intrinsic magnetic moment of a nucleus μ_N
- Intrinsic magnetic moment of an electron μ_e
- Magnetic field due to electron orbital current $\mathbf{J}_e(\mathbf{r})$

Interaction energy between a magnetic dipole ***m*** and a magnetic field **B:**

$$E_{int} = -\mathbf{m} \cdot \mathbf{B}$$



In this case: $E_{int} = -\mu_N \cdot \mathbf{B}_{\mu_e} - \mu_N \cdot \mathbf{B}_{\mathbf{J_e}}(0)$

$$\mathbf{B}_{\boldsymbol{\mu}_e}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left\{\frac{3\hat{\mathbf{r}}(\boldsymbol{\mu}_e \cdot \hat{\mathbf{r}}) - \boldsymbol{\mu}_e}{r^3} + \frac{8\pi}{3}\boldsymbol{\mu}_e \delta^3(\mathbf{r})\right\}$$

Hyperfine interaction energy: -- continued

$$E_{int} = -\mu_N \cdot \mathbf{B}_{\mu_e} - \mu_N \cdot \mathbf{B}_{\mathbf{J}_e}(0)$$

Here we assume that nuclear position is **r**=*0.*

Evaluation of the magnetic field at the nucleus due to the electron current density:

The vector potential associated with an electron in a bound state of an atom as described by a quantum mechanical wavefunction $\psi_{nlm_l}(\mathbf{r})$ can be written:

$$\mathbf{A}_{\mathbf{J}_e}(\mathbf{r}) = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \int d^3r' \, \frac{\hat{\mathbf{z}} \times \mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \frac{\left|\psi_{nlm_l}(\mathbf{r}')\right|^2}{r'^2 \sin^2\theta'}$$

We want to evaluate the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ in the vicinity of the nucleus $(\mathbf{r} \rightarrow 0)$.

Hyperfine interaction energy: -- continued

$$\mathbf{B}_{\mathbf{J}_e}(0) = \nabla \times \mathbf{A}_{\mathbf{J}_e}\Big|_{\mathbf{r}\to 0} = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \int d^3r' \nabla \times \frac{\hat{\mathbf{z}}\times\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\frac{\left|\psi_{nlm_l}(\mathbf{r}')\right|^2}{r'^2\sin^2\theta'}\Bigg|_{\mathbf{r}\to 0}$$

$$\mathbf{B}_{\mathbf{o}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \int d^3r' \frac{(\mathbf{r}-\mathbf{r}')\times(\hat{\mathbf{z}}\times\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\frac{\left|\psi_{nlm_l}(\mathbf{r}')\right|^2}{r'^2\sin^2\theta'}\Bigg|_{\mathbf{r}\to 0}$$

$$\mathbf{B}_{\mathbf{o}}(\mathbf{0}) = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \int d^3r' \frac{\mathbf{r}'\times(\hat{\mathbf{z}}\times\mathbf{r}')}{r'^3}\frac{\left|\psi_{nlm_l}(\mathbf{r}')\right|^2}{r'^2\sin^2\theta'}$$

$$\hat{\mathbf{r}}'\times(\hat{\mathbf{z}}\times\hat{\mathbf{r}}') = \hat{\mathbf{z}}(1-\cos^2\theta') - \hat{\mathbf{x}}\cos\theta'\sin\theta'\cos\phi' - \hat{\mathbf{y}}\cos\theta'\sin\theta'\sin\phi').$$

$$\mathbf{B}_{\mathbf{o}}(\mathbf{0}) = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \int d^3r' \frac{\hat{\mathbf{z}}r'^2\sin^2\theta'}{r'^3}\frac{\left|\psi_{nlm_l}(\mathbf{r}')\right|^2}{r'^2\sin^2\theta'} = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l\, \hat{\mathbf{z}}\int d^3r' \frac{\left|\psi_{nlm_l}(\mathbf{r}')\right|^2}{r'^3}$$

$$= -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l\, \hat{\mathbf{z}}\left\langle \frac{1}{r'^3}\right\rangle$$

Note that this field at the nucleus site is due to the electronic orbital angular momentum.

Hyperfine interaction energy: -- continued

$$E_{int} \equiv H_{HF} = -\mu_N \cdot \mathbf{B}_{\mu_e} - \mu_N \cdot \mathbf{B}_{\mathbf{J_e}}(0)$$

Putting all of the terms together:

$$H_{\mathrm{HF}} = -\frac{\mu_0}{4\pi}\left(\left\langle \frac{3(\boldsymbol{\mu}_\mathbf{N}\cdot\hat{\mathbf{r}})(\boldsymbol{\mu}_\mathbf{e}\cdot\hat{\mathbf{r}}) - \boldsymbol{\mu}_\mathbf{N}\cdot\boldsymbol{\mu}_\mathbf{e}}{r^3} + \frac{8\pi}{3}\boldsymbol{\mu}_\mathbf{N}\cdot\boldsymbol{\mu}_\mathbf{e}\delta^3(\mathbf{r})\right\rangle + \frac{e}{m_e}\left\langle\frac{\mathbf{L}\cdot\boldsymbol{\mu}_\mathbf{N}}{r^3}\right\rangle\right).$$

In this expression the brackets $\langle\ \rangle$ indicate evaluating the expectation value relative to the electronic state.

Macroscopic dipolar effects --
Magnetic dipole moment

$$\mathbf{m} = \frac{1}{2}\int d^3r\, \mathbf{r}\times\mathbf{J}(\mathbf{r})$$

Note that the intrinsic spin of elementary particles is associated with a magnetic dipole moment, but we often do not have a detailed knowledge of its **J**(**r**).

Vector potential for magnetic dipole moment

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{|\mathbf{r}|^3}$$

Valid outside the extent of **J**(**r**)

Macroscopic magnetization

$$\mathbf{M}(\mathbf{r}) = \sum_i \mathbf{m}_i \delta^3(\mathbf{r} - \mathbf{r}_i)$$

Vector potential due to "free" current $\mathbf{J}_{\mathbf{free}}(\mathbf{r})$ and macroscopic magnetization $\mathbf{M}(\mathbf{r})$. Note: the designation $\mathbf{J}_{\mathbf{free}}(\mathbf{r})$ implies that this current does not also contribute to the magnetization density.

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \left(\frac{\mathbf{J}_{free}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \frac{\mathbf{M}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \right)$$

Vector potential contributions from macroscopic magnetization -- continued

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \left(\frac{\mathbf{J}_{free}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \frac{\mathbf{M}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \right)$$

Note that :

$$\frac{\mathbf{M}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = \mathbf{M}(\mathbf{r}')\times\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}$$

$$= -\nabla'\times\left(\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right) + \frac{\nabla'\times\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$\Rightarrow \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\mathbf{J}_{free}(\mathbf{r}') + \nabla'\times\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Vector potential contributions from macroscopic magnetization -- continued

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\mathbf{J}_{free}(\mathbf{r}') + \nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Note that for the case that $\nabla \cdot \mathbf{A} = 0$:

$$\nabla \times \mathbf{B}(\mathbf{r}) = \nabla \times (\nabla \times \mathbf{A}(\mathbf{r})) = -\nabla^2 \mathbf{A}(\mathbf{r})$$

$$= \frac{\mu_0}{4\pi}\int d^3r' \left(4\pi\delta^3(\mathbf{r}-\mathbf{r}')\right)\left(\mathbf{J}_{free}(\mathbf{r}') + \nabla' \times \mathbf{M}(\mathbf{r}')\right)$$

$$= \mu_0\left(\mathbf{J}_{free}(\mathbf{r}) + \nabla \times \mathbf{M}(\mathbf{r})\right)$$

$$\Rightarrow \nabla \times \left(\mathbf{B}(\mathbf{r}) - \mu_0\mathbf{M}(\mathbf{r})\right) = \mu_0 \mathbf{J}_{free}(\mathbf{r})$$

Magnetic field contributions

$$\nabla \times \left(\mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{M}(\mathbf{r})\right) = \mu_0 \mathbf{J}_{free}(\mathbf{r})$$

Define "new" magnetic field vector:

$$\mu_0 \mathbf{H}(\mathbf{r}) \equiv \mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{M}(\mathbf{r})$$

$$\Rightarrow \nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}_{free}(\mathbf{r})$$

$$\nabla \times \left(\mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{M}(\mathbf{r})\right) = \mu_0 \mathbf{J}_{free}(\mathbf{r})$$

Note that $\mathbf{B}(\mathbf{r}) \equiv$ the magnetic flux density

Define $\mathbf{H}(\mathbf{r}) \equiv$ the magnetic field

$$\mu_0 \mathbf{H}(\mathbf{r}) \equiv \mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{M}(\mathbf{r})$$

$$\Rightarrow \nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}_{free}(\mathbf{r})$$

Energy associated with magnetic fields

Note: We previously used without proof --

the force on a magnetic dipole $\mathbf{m}$ in an external $\mathbf{B}$ field is:

$$\mathbf{F} = \nabla\left(\mathbf{m} \cdot \mathbf{B}\right)$$

This implies that energy associated with aligning a magnetic dipole $\mathbf{m}$ in an external $\mathbf{B}$ field is given by:

$$E_{\text{int}} = -\mathbf{m} \cdot \mathbf{B}$$

Macroscopic energies --

It can be shown that: $$W_B = \frac{1}{2}\int d^3r \; \mathbf{B}(\mathbf{r}) \cdot \mathbf{H}(\mathbf{r})$$

In analogy to: $$W_E = \frac{1}{2}\int d^3r \; \mathbf{E}(\mathbf{r}) \cdot \mathbf{D}(\mathbf{r})$$

Summary of equations of magnetostatics :

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}_{total}(\mathbf{r})$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}_{free}(\mathbf{r})$$

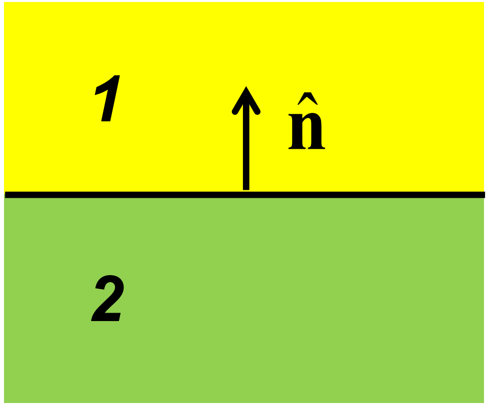
$$\mathbf{B}(\mathbf{r}) = \mu_0 (\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

For the case that $\mathbf{J}_{free}(\mathbf{r}) = 0$:

$$\nabla \times \mathbf{H}(\mathbf{r}) = 0$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$



For the case that $\mathbf{J}_{free}(\mathbf{r}) = 0:$

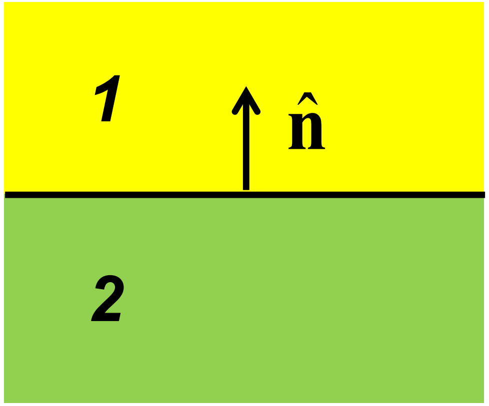
$$\nabla \times \mathbf{H}(\mathbf{r}) = 0$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

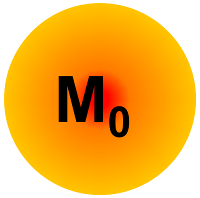
At boundary:

$$\mathbf{H}_1 \times \hat{\mathbf{n}} = \mathbf{H}_2 \times \hat{\mathbf{n}}$$

$$\mathbf{B}_1 \cdot \hat{\mathbf{n}} = \mathbf{B}_2 \cdot \hat{\mathbf{n}}$$



Example magnetostatic boundary value problem



$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0 \hat{\mathbf{z}} & r \leq a \\ 0 & r > a \end{cases}$$

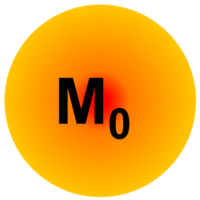
$$\nabla \times \mathbf{H}(\mathbf{r}) = 0 \qquad \Rightarrow \mathbf{H}(\mathbf{r}) = -\nabla \Phi_H(\mathbf{r})$$

$$\mathbf{B}(\mathbf{r}) = \mu_0 \left(\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}) \right)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0 = \mu_0 \nabla \cdot \left(\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}) \right)$$

$$\Rightarrow \nabla^2 \Phi_H(\mathbf{r}) = \nabla \cdot \mathbf{M}(\mathbf{r})$$

Example magnetostatic boundary value problem -- continued



$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0 \hat{\mathbf{z}} & r \le a \\ 0 & r > a \end{cases}$$

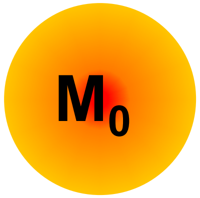
$$\nabla^2 \Phi_H(\mathbf{r}) = \nabla \cdot \mathbf{M}(\mathbf{r})$$

$$\Rightarrow \Phi_H(\mathbf{r}) = -\frac{1}{4\pi} \int d^3r' \frac{\nabla' \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

$$= -\frac{1}{4\pi} \int d^3r' \left[\nabla' \cdot \left(\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) - \mathbf{M}(\mathbf{r}') \cdot \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right]$$

$$= -\frac{1}{4\pi} \nabla \cdot \int d^3r' \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

Example magnetostatic boundary value problem -- continued



$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0 \hat{\mathbf{z}} & r \leq a \\ 0 & r > a \end{cases} \qquad \Phi_H(\mathbf{r}) = -\frac{1}{4\pi} \nabla \cdot \int d^3r' \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

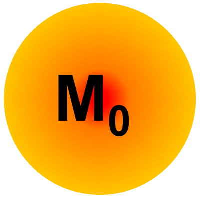
For this example:

$$\Phi_H(\mathbf{r}) = -\frac{M_0}{4\pi} \frac{\partial}{\partial z} \left(4\pi \int_0^a r'^2 \, dr' \frac{1}{r_>} \right)$$

For $r \leq a$: $$\Phi_H(\mathbf{r}) = -M_0 \frac{\partial}{\partial z} \left(\frac{a^2}{2} - \frac{r^2}{6} \right) = \frac{M_0 z}{3}$$

For $r > a$: $$\Phi_H(\mathbf{r}) = -M_0 \frac{\partial}{\partial z} \left(\frac{a^3}{3r} \right) = \frac{M_0 a^3 z}{3r^3}$$

Example magnetostatic boundary value problem -- continued



$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0\hat{\mathbf{z}} & r \le a \\ 0 & r > a \end{cases}$$

For $r \le a$: $\quad \Phi_H(\mathbf{r}) = \dfrac{M_0 z}{3} \qquad \mathbf{H}(\mathbf{r}) = -\nabla\Phi_H(\mathbf{r}) = -\dfrac{M_0}{3}\hat{\mathbf{z}}$

For $r > a$: $\quad \Phi_H(\mathbf{r}) = \dfrac{M_0 a^3 z}{3r^3} \qquad \mathbf{H}(\mathbf{r}) = -\nabla\Phi_H(\mathbf{r}) = -\dfrac{M_0 a^3}{3}\left(\dfrac{\hat{\mathbf{z}}}{r^3} - \dfrac{3z\mathbf{r}}{r^5}\right)$

$$\mathbf{B}(\mathbf{r}) = \mu_0\left(\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r})\right)$$

For $r \le a$: $\quad \mathbf{H}(\mathbf{r}) = -\dfrac{M_0\hat{\mathbf{z}}}{3} \qquad \mathbf{B}(\mathbf{r}) = \mu_0\dfrac{2M_0\hat{\mathbf{z}}}{3}$

For $r > a$: $\quad \mathbf{H}(\mathbf{r}) = -\dfrac{M_0 a^3}{3}\left(\dfrac{\hat{\mathbf{z}}}{r^3} - \dfrac{3z\mathbf{r}}{r^5}\right)$

$$\mathbf{B}(\mathbf{r}) = -\mu_0\frac{M_0 a^3}{3}\left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5}\right)$$

Check boundary values:

For $r \le a$: $$\mathbf{H}(\mathbf{r}) = -\frac{M_0 \hat{\mathbf{z}}}{3} \qquad \mathbf{H}(a\hat{\mathbf{r}}) \times \hat{\mathbf{r}} = -\frac{M_0}{3} \hat{\mathbf{z}} \times \hat{\mathbf{r}}$$

For $r > a$: $$\mathbf{H}(\mathbf{r}) = -\frac{M_0 a^3}{3} \left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5} \right)$$

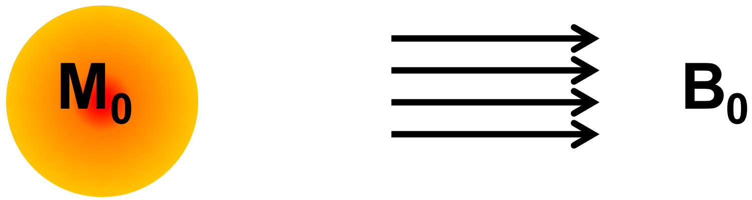
$$\mathbf{H}(a\hat{\mathbf{r}}) \times \hat{\mathbf{r}} = -\frac{M_0 a^3}{3} \frac{\hat{\mathbf{z}} \times \hat{\mathbf{r}}}{a^3}$$

For $r \le a$: $$\mathbf{B}(\mathbf{r}) = \mu_0 \frac{2M_0 \hat{\mathbf{z}}}{3} \qquad \mathbf{B}(a\hat{\mathbf{r}}) \cdot \hat{\mathbf{r}} = \mu_0 \frac{2M_0}{3} \hat{\mathbf{z}} \cdot \hat{\mathbf{r}}$$

For $r > a$: $$\mathbf{B}(\mathbf{r}) = -\mu_0 \frac{M_0 a^3}{3} \left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5} \right)$$

$$\mathbf{B}(a\hat{\mathbf{r}}) \cdot \hat{\mathbf{r}} = -\mu_0 \frac{M_0 a^3}{3} \hat{\mathbf{z}} \cdot \hat{\mathbf{r}} \left(\frac{1}{a^3} - \frac{3a^2}{a^5} \right)$$

Variation; magnetic sphere plus external field $\mathbf{B}_0$



$$\mathbf{M}(\mathbf{r}) = \begin{cases} \mathbf{M}_0 & r \le a \\ 0 & r > a \end{cases}$$

By superposition:

For $r \le a$:

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}_0 + \mu_0 \frac{2}{3}\mathbf{M}_0$$

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\mu_0}\mathbf{B}_0 - \frac{1}{3}\mathbf{M}_0$$

$$\mathbf{B}(\mathbf{r}) + 2\mu_0 \mathbf{H}(\mathbf{r}) = 3\mathbf{B}_0$$

For an isotropic "paramagnetic" material, $\mathbf{B}(\mathbf{r}) = \mu \mathbf{H}(\mathbf{r})$

$$\mathbf{M}_0 = \frac{3}{\mu_0}\left(\frac{\mu - \mu_0}{\mu + 2\mu_0}\right)\mathbf{B}_0$$

Summary of equations of magnetostatics :

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}_{total}(\mathbf{r})$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}_{free}(\mathbf{r})$$

$$\mathbf{B}(\mathbf{r}) = \mu_0 (\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

For the case that $\mathbf{J}_{free}(\mathbf{r})$:

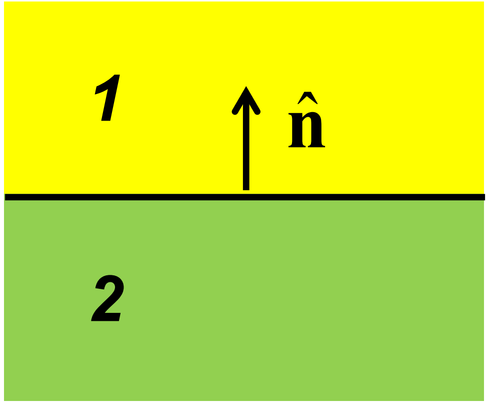
$$\nabla \times \mathbf{H}(\mathbf{r}) = 0$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

At boundary :

$$\mathbf{H}_1 \times \hat{\mathbf{n}} = \mathbf{H}_2 \times \hat{\mathbf{n}}$$

$$\mathbf{B}_1 \cdot \hat{\mathbf{n}} = \mathbf{B}_2 \cdot \hat{\mathbf{n}}$$



Magnetism in materials

$$\mathbf{B}(\mathbf{r}) = \mu_0\left(\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r})\right)$$

For materials with linear magnetism:

$$\mathbf{B} = \mu\mathbf{H}$$

$$\mu > \mu_0 \quad \Rightarrow \text{paramagnetic material}$$

$$\mu < \mu_0 \quad \Rightarrow \text{diamagnetic material}$$

For ferromagnetic, antiferromagnetic materials

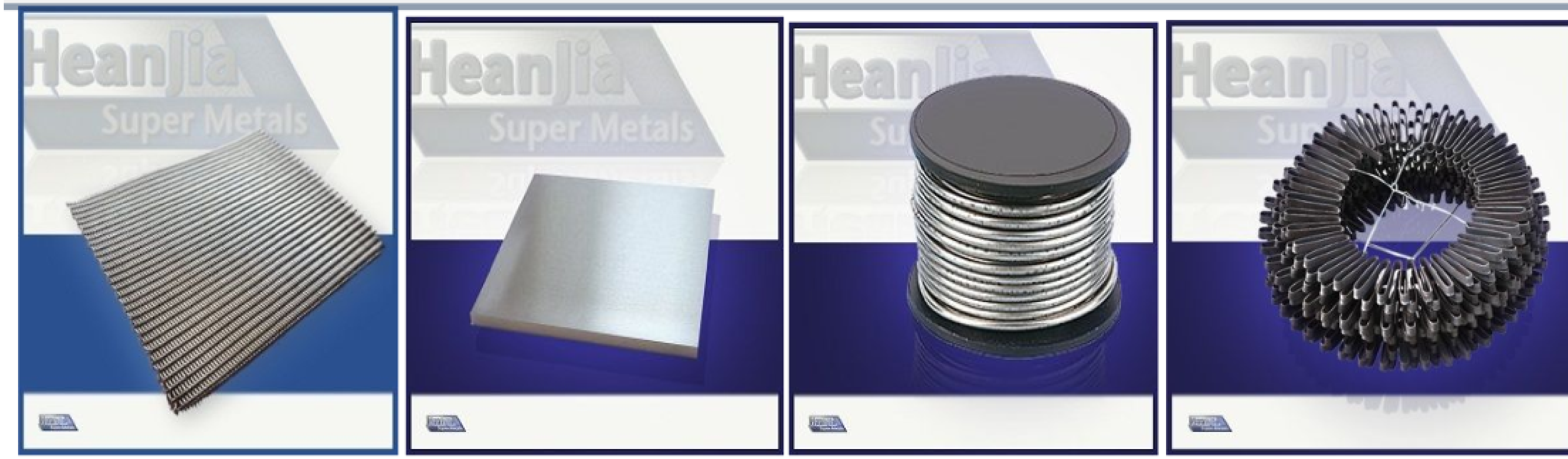
$$\mathbf{B} = f(\mathbf{H}) \quad \text{(with hysteresis)}$$

https://en.wikipedia.org/wiki/Permeability_(electromagnetism)

Magnetic susceptibility and permeability data for selected materials

Medium	Susceptibility, volumetric, SI, χ_m	Permeability, μ (H/m)	Relative permeability, max., μ/μ_0	Magnetic field
Metglas 2714A (annealed)		1.26×10^{0}	1 000 000[10]	At 0.5 T
Iron (99.95% pure Fe annealed in H)		2.5×10^{-1}	200 000[11]	
NANOPERM®		1.0×10^{-1}	80 000[12]	At 0.5 T
Mu-metal		2.5×10^{-2}	20 000[13]	At 0.002 T
Mu-metal		6.3×10^{-2}	50 000[14]	
Cobalt-iron (high permeability strip material)		2.3×10^{-2}	18 000[15]	
Permalloy	8000	1.0×10^{-2}	8000[13]	At 0.002 T

Mumetal Magnetic Shielding



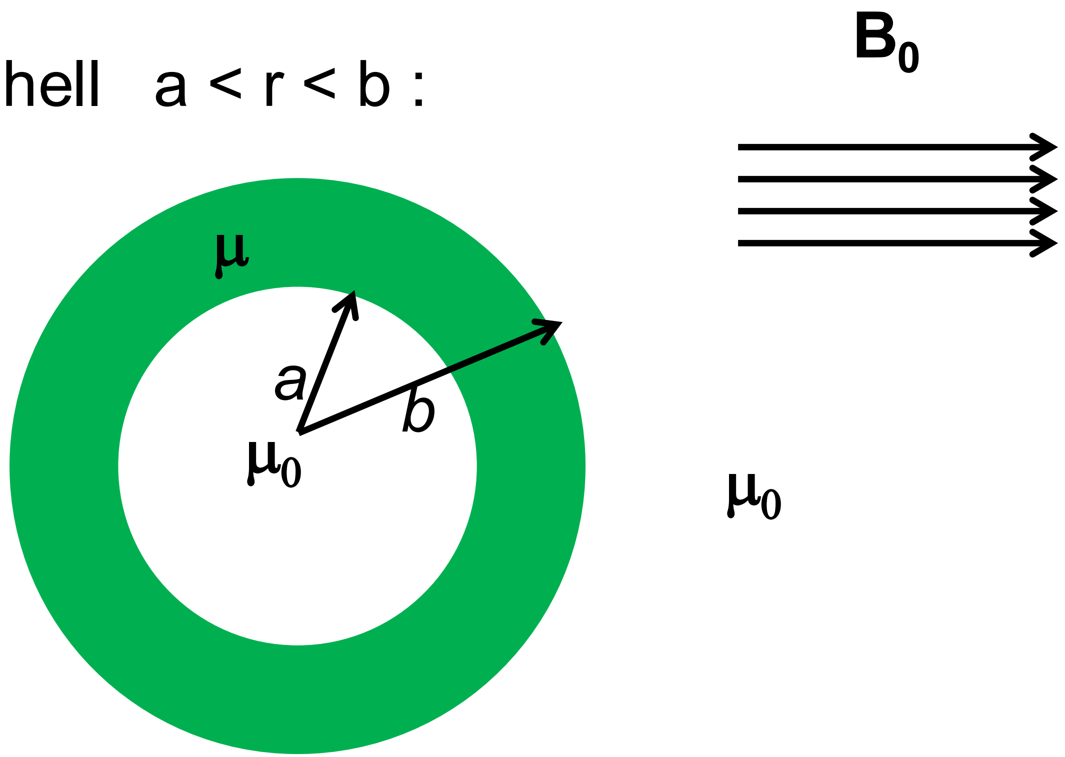
Mu metal is a soft ferromagnetic alloy that has extremely high initial and maximum magnetic permeability. It is used in electric transformer, storage disks, magnetic phonographs, resonance devices and superconducting circuits.

Mumetal alloy generally attributes relative permeability about 80,000 to 100,000 than the normal steel alloy. It is also called as soft magnetic alloy and offers low magnetic anisotropy and magnetostriction providing low corceivity to saturate the low magnetic fields. It provides nominal hysteresis loses when the alloy is employed in the AC magnetic circuits.

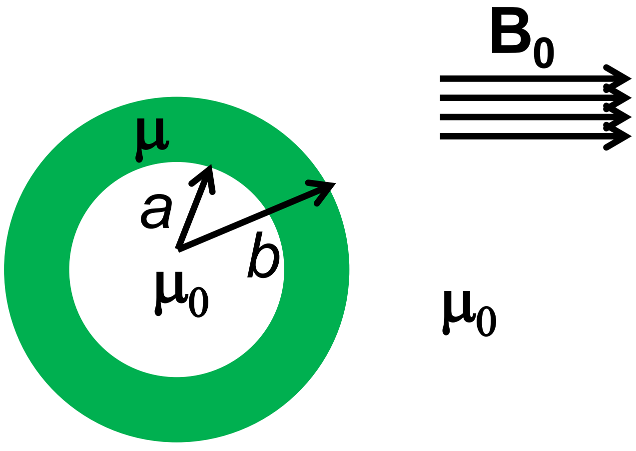
Composed of 80% Ni, 15% Fe, 5% Mo+other materials

Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$

Spherical shell $a < r < b$:



Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$ -- continued



For this case:

$$\nabla \times \mathbf{H}(\mathbf{r}) = 0$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

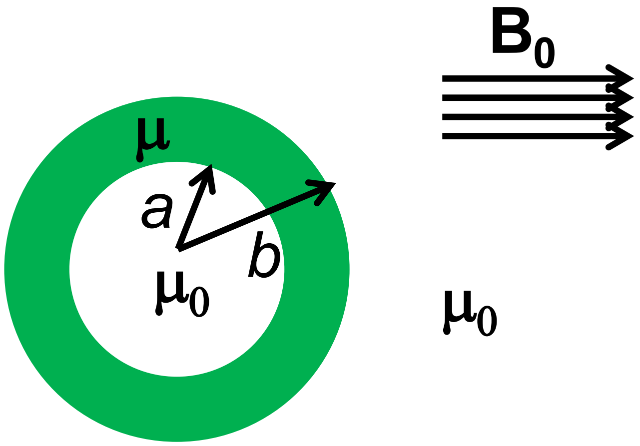
$$\mathbf{B}(\mathbf{r}) = \mu \mathbf{H}(\mathbf{r})$$

Continuity at boundaries:

$$\mathbf{H} \times \hat{\mathbf{n}} = \text{continuous}$$

$$\mathbf{B} \cdot \hat{\mathbf{n}} = \text{continuous}$$

Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$ -- continued



$$\text{Let:} \quad \mathbf{H}(\mathbf{r}) = -\nabla\Phi_H(\mathbf{r})$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0 \qquad \Rightarrow \nabla^2\Phi_H(\mathbf{r}) = 0$$

$$\text{For} \quad 0 \le r \le a \qquad \Phi_H(\mathbf{r}) = \sum_l \delta_l r^l P_l(\cos\theta)$$

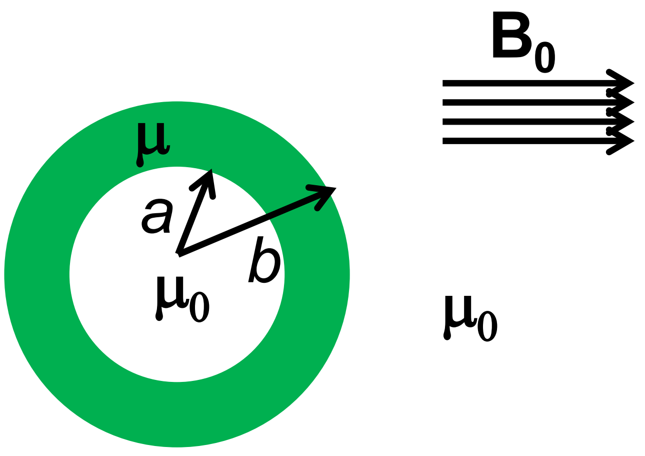
$$\text{For} \quad a \le r \le b \qquad \Phi_H(\mathbf{r}) = \sum_l \left(\beta_l r^l + \frac{\gamma_l}{r^{l+1}} \right) P_l(\cos\theta)$$

$$\text{For} \quad r \ge b \qquad \Phi_H(\mathbf{r}) = -\frac{B_0}{\mu_0} r\cos\theta + \sum_l \frac{\alpha_l}{r^{l+1}} P_l(\cos\theta)$$

Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$ -- continued

Applying boundary conditions

(only $l=1$ terms contribute):



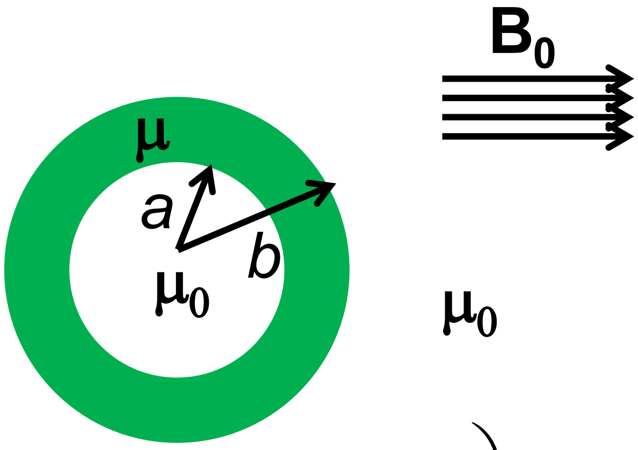
At $r=a$
$$\delta_1 = \frac{\mu}{\mu_0}\left(\beta_1 - 2\frac{\gamma_1}{a^3}\right)$$

$$a\delta_1 = a\beta_1 + \frac{\gamma_1}{a^2}$$

At $r=b$
$$\frac{\mu}{\mu_0}\left(\beta_1 - 2\frac{\gamma_1}{b^3}\right) = -\frac{B_0}{\mu_0} - 2\frac{\alpha_1}{b^3}$$

$$b\beta_1 + \frac{\gamma_1}{b^2} = -b\frac{B_0}{\mu_0} + \frac{\alpha_1}{b^2}$$

Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$ -- continued



When the dust clears:

$$\delta_1 = \left(\frac{-9\mu/\mu_0}{(2\mu/\mu_0+1)(\mu/\mu_0+2)-2(a/b)^3(\mu/\mu_0-1)^2} \right)\frac{B_0}{\mu_0}$$

$$\approx \frac{1}{\mu/\mu_0}\left(\frac{-9/2}{\left(1-(a/b)^3\right)}\frac{B_0}{\mu_0} \right)$$