PHY 712 Electrodynamics
11-11:50 AM MWF in Olin 103

Lecture Notes for Lecture 15:

Start reading Chapter 6

1. **Maxwell's full equations; effects of time varying fields and sources**
2. **Gauge choices and transformations**
3. **Green's function for vector and scalar potentials**

Course schedule for Spring 2022

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Mon: 01/10/2022	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/14/2022
2	Wed: 01/12/2022	Chap. 1	Electrostatic energy calculations	#2	01/19/2022
3	Fri: 01/14/2022	Chap. 1	Electrostatic energy calculations	#3	01/21/2022
	Mon: 01/17/2022		MLK Holiday -- no class		
4	Wed: 01/19/2022	Chap. 1 & 2	Electrostatic potentials and fields	#4	01/24/2022
5	Fri: 01/21/2022	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#5	01/26/2022
6	Mon: 01/24/2022	Chap. 1 - 3	Brief introduction to numerical methods	#6	01/28/2022
7	Wed: 01/26/2022	Chap. 2 & 3	Image charge constructions	#7	01/31/2022
8	Fri: 01/28/2022	Chap. 2 & 3	Cylindrical and spherical geometries	#8	02/02/2022
9	Mon: 01/31/2022	Chap. 3 & 4	Spherical geometry and multipole moments	#9	02/04/2022
	Wed: 02/02/2022	No class	Fire caution		
	Fri: 02/04/2022	No class	Fire caution		
10	Mon: 02/07/2022	Chap. 4	Dipoles and Dielectrics	#10	02/09/2022
11	Wed: 02/09/2022	Chap. 4	Dipoles and Dielectrics	#11	02/11/2022
12	Fri: 02/11/2022	Chap. 5	Magnetostatics	#12	02/14/2022
13	Mon: 02/14/2022	Chap. 5	Magnetic dipoles and hyperfine interaction	#13	02/16/2022
14	Wed: 02/16/2022	Chap. 5	Magnetic dipoles and dipolar fields	#14	02/18/2022
15	Fri: 02/18/2022	Chap. 6	Maxwell's Equations		
	Mon: 02/21/2022	Chap. 1-6	Homework review & and presentations		

Plan for Monday 2/21/2022 --

PHY 712 and PHY 742
Schedule for presentations on Monday 2/21/2022

Time	Name	PHY 712 HW #	PHY 742 #
11:05-11:25	Can	10	11
11:25-11:45	Wells	9	10
11:45-12:05	Mani	12	3
12:10-12:30	Owen	11	7
12:30-12:50	Ramesh	3	6

Full electrodynamics with time varying fields and sources

Maxwell's equations

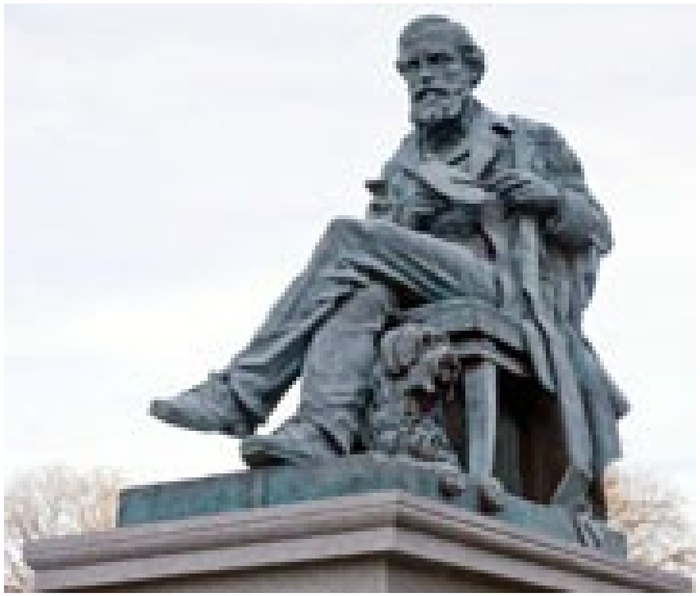


Image of statue of James Clerk-Maxwell (1831-1879) in Edinburgh

"From a long view of the history of mankind - seen from, say, ten thousand years from now - there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics"

Richard P Feynman

http://www.clerkmaxwellfoundation.org/

Maxwell's equations

Coulomb's law : $\nabla \cdot \mathbf{D} = \rho_{free}$

Ampere-Maxwell's law : $\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$

Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles : $\nabla \cdot \mathbf{B} = 0$

Maxwell's equations

Microscopic or vacuum form $(\mathbf{P}=0;\ \mathbf{M}=0)$:

Coulomb's law: $\nabla\cdot\mathbf{E}=\rho/\varepsilon_0$

Ampere-Maxwell's law: $\nabla\times\mathbf{B}-\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}=\mu_0\mathbf{J}$

Faraday's law: $\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0$

No magnetic monopoles: $\nabla\cdot\mathbf{B}=0$

$$\Rightarrow c^2=\frac{1}{\varepsilon_0\mu_0}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials

$$\nabla \cdot \mathbf{B} = 0 \qquad \Rightarrow \mathbf{B} = \nabla \times \mathbf{A}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \Rightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi$$

$$\text{or} \quad \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0 :$$

$$-\nabla^2 \Phi - \frac{\partial \left(\nabla \cdot \mathbf{A} \right)}{\partial t} = \rho / \varepsilon_0$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

$$\nabla \times \left(\nabla \times \mathbf{A} \right) + \frac{1}{c^2} \left(\frac{\partial \left(\nabla \Phi \right)}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) = \mu_0 \mathbf{J}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

General form for the scalar and vector potential equations:

$$-\nabla^2\Phi - \frac{\partial\left(\nabla\cdot\mathbf{A}\right)}{\partial t} = \rho / \varepsilon_0$$

$$\nabla\times\left(\nabla\times\mathbf{A}\right) + \frac{1}{c^2}\left(\frac{\partial\left(\nabla\Phi\right)}{\partial t} + \frac{\partial^2\mathbf{A}}{\partial t^2}\right) = \mu_0\mathbf{J}$$

Coulomb gauge form -- require $\nabla\cdot\mathbf{A}_C = 0$

$$-\nabla^2\Phi_C = \rho / \varepsilon_0$$

$$-\nabla^2\mathbf{A}_C + \frac{1}{c^2}\frac{\partial^2\mathbf{A}_C}{\partial t^2} + \frac{1}{c^2}\frac{\partial\left(\nabla\Phi_C\right)}{\partial t} = \mu_0\mathbf{J}$$

Note that $\mathbf{J} = \mathbf{J}_l + \mathbf{J}_t$ with $\nabla\times\mathbf{J}_l = 0$ and $\nabla\cdot\mathbf{J}_t = 0$

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

Coulomb gauge form -- require $\nabla \cdot \mathbf{A}_C = 0$

$$-\nabla^2 \Phi_C = \rho / \varepsilon_0$$

$$-\nabla^2 \mathbf{A}_C + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_C}{\partial t^2} + \frac{1}{c^2}\frac{\partial (\nabla \Phi_C)}{\partial t} = \mu_0 \mathbf{J}$$

Note that $\mathbf{J} = \mathbf{J}_l + \mathbf{J}_t$ with $\nabla \times \mathbf{J}_l = 0$ and $\nabla \cdot \mathbf{J}_t = 0$

Continuity equation for charge and current density:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J}_l = 0 \qquad \Rightarrow \frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}_l = -\varepsilon_0 \nabla \cdot \frac{\partial (\nabla \Phi_C)}{\partial t}$$

$$\Rightarrow \frac{1}{c^2}\frac{\partial (\nabla \Phi_C)}{\partial t} = \varepsilon_0 \mu_0 \frac{\partial (\nabla \Phi_C)}{\partial t} = \mu_0 \mathbf{J}_l$$

$$-\nabla^2 \mathbf{A}_C + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_C}{\partial t^2} = \mu_0 \mathbf{J}_t$$

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

Review of the general equations:

$$-\nabla^2\Phi - \frac{\partial(\nabla\cdot\mathbf{A})}{\partial t} = \rho/\varepsilon_0$$

$$\nabla\times(\nabla\times\mathbf{A}) + \frac{1}{c^2}\left(\frac{\partial(\nabla\Phi)}{\partial t} + \frac{\partial^2\mathbf{A}}{\partial t^2}\right) = \mu_0\mathbf{J}$$

Lorentz gauge form -- require $\nabla\cdot\mathbf{A}_L + \frac{1}{c^2}\frac{\partial\Phi_L}{\partial t} = 0$

$$-\nabla^2\Phi_L + \frac{1}{c^2}\frac{\partial^2\Phi_L}{\partial t^2} = \rho/\varepsilon_0$$

$$-\nabla^2\mathbf{A}_L + \frac{1}{c^2}\frac{\partial^2\mathbf{A}_L}{\partial t^2} = \mu_0\mathbf{J}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

Lorentz gauge form -- require $\nabla \cdot \mathbf{A}_L + \frac{1}{c^2}\frac{\partial \Phi_L}{\partial t} = 0$

$$-\nabla^2 \Phi_L + \frac{1}{c^2}\frac{\partial^2 \Phi_L}{\partial t^2} = \rho / \varepsilon_0$$

$$-\nabla^2 \mathbf{A}_L + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_L}{\partial t^2} = \mu_0 \mathbf{J}$$

Alternate potentials: $\mathbf{A}'_L = \mathbf{A}_L + \nabla\Lambda$ and $\Phi'_L = \Phi_L - \frac{\partial \Lambda}{\partial t}$

Yields same physics provided that: $\nabla^2 \Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} = 0$

Solution of Maxwell's equations in the Lorentz gauge

$$\nabla^2 \Phi_L - \frac{1}{c^2}\frac{\partial^2 \Phi_L}{\partial t^2} = -\rho / \varepsilon_0$$

$$\nabla^2 \mathbf{A}_L - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_L}{\partial t^2} = -\mu_0 \mathbf{J}$$

Consider the general form of the 3-dimensional wave equation:

$$\nabla^2 \Psi - \frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2} = -4\pi f$$

$\Psi(\mathbf{r}, t) \Rightarrow$ wave field $\quad f(\mathbf{r}, t) \Rightarrow$ source

Solution of Maxwell's equations in the Lorentz gauge -- continued

Let Ψ represent Φ, A_x, A_y, A_z Let f represent ρ, J_x, J_y, J_z

$$\nabla^2\Psi(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2\Psi(\mathbf{r},t)}{\partial t^2} = -4\pi f(\mathbf{r},t)$$

Green's function :

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r},t;\mathbf{r}',t') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Formal solution for field $\Psi(\mathbf{r},t)$:

$$\Psi(\mathbf{r},t) = \Psi_{f=0}(\mathbf{r},t) + \int d^3r' \int dt' G(\mathbf{r},t;\mathbf{r}',t') f(\mathbf{r}',t')$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

Determination of the form for the Green's function:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r},t;\mathbf{r}',t') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

For the case of isotropic boundary values at infinity:

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right)$$

Formal solution for field $\Psi(\mathbf{r},t)$:

$$\Psi(\mathbf{r},t) = \Psi_{f=0}(\mathbf{r},t) + \int d^3r' \int dt' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right) f(\mathbf{r}',t')$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

Analysis of the Green's function:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r},t;\mathbf{r}',t') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

"Proof" -- Fourier analysis in the time domain -- note that

$$\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\; e^{i\omega(t-t')}$$

Define:

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\; e^{i\omega(t-t')}\;\tilde{G}(\mathbf{r},\mathbf{r}',\omega)$$

$$\Rightarrow \quad \left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{G}(\mathbf{r},\mathbf{r}',\omega) = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

Analysis of the Green's function (continued):

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{G}(\mathbf{r},\mathbf{r}',\omega) = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$$

For the case of isotropic boundary values at infinity:

$$\tilde{G}(\mathbf{r},\mathbf{r}',\omega) = \tilde{G}(\mathbf{r}-\mathbf{r}',\omega)$$

Further assuming that $\tilde{G}(\mathbf{r}-\mathbf{r}',\omega)$ is isotropic in $|\mathbf{r}-\mathbf{r}'| \equiv R$:

$$\left(\frac{1}{R}\frac{d^2}{dR^2}R + \frac{\omega^2}{c^2}\right)\tilde{G}(\mathbf{r},\mathbf{r}',\omega) = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$$

Solution: $\tilde{G}(\mathbf{r},\mathbf{r}',\omega) = \frac{1}{R}e^{\pm i\omega R/c}$

Solution of Maxwell's equations in the Lorentz gauge -- continued

Analysis of the Green's function (continued):

$$\tilde{G}(\mathbf{r},\mathbf{r}',\omega) = \frac{1}{|\mathbf{r}-\mathbf{r}'|} e^{\pm i\omega|\mathbf{r}-\mathbf{r}'|/c}$$

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega(t-t')}\,\tilde{G}(\mathbf{r},\mathbf{r}',\omega)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega(t-t')}\,\frac{1}{|\mathbf{r}-\mathbf{r}'|} e^{\pm i\omega|\mathbf{r}-\mathbf{r}'|/c}$$

$$= \frac{1}{|\mathbf{r}-\mathbf{r}'|}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega\left(t-t'\pm|\mathbf{r}-\mathbf{r}'|/c\right)}\right)$$

$$= \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t-t'\pm|\mathbf{r}-\mathbf{r}'|/c\right) = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-t\mp|\mathbf{r}-\mathbf{r}'|/c\right)$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\,\delta\big(t' - (t \pm |\mathbf{r}-\mathbf{r}'|/c)\big)$$

Solution for field $\Psi(\mathbf{r},t)$:

$$\Psi(\mathbf{r},t) = \Psi_{f=0}(\mathbf{r},t) +$$

$$\int d^3r' \int dt' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\,\delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right) f(\mathbf{r}',t')$$

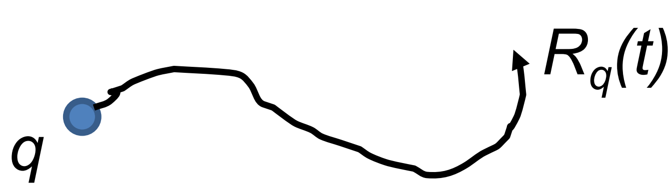
Solution of Maxwell's equations in the Lorentz gauge -- continued

Liènard-Wiechert potentials and fields --
Determination of the scalar and vector potentials for a moving point particle (also see Landau and Lifshitz ***The Classical Theory of Fields***, Chapter 8.)

Consider the fields produced by the following source: a point charge q moving on a trajectory $R_q(t)$.

Charge density: $\rho(\mathbf{r},t) = q\delta^3(\mathbf{r}-\mathbf{R}_q(t))$

Current density: $\mathbf{J}(\mathbf{r},t) = q\,\dot{\mathbf{R}}_q(t)\delta^3(\mathbf{r}-\mathbf{R}_q(t))$, where $\dot{\mathbf{R}}_q(t) \equiv \dfrac{d\mathbf{R}_q(t)}{dt}$.



Solution of Maxwell's equations in the Lorentz gauge -- continued

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int\int d^3r' dt' \frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}']|}\delta\left(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\right)$$

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0 c^2}\int\int d^3r' dt' \frac{\mathbf{J}(\mathbf{r}',\mathrm{t'})}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\right).$$

We performing the integrations over first d^3r' and then dt' making use of the fact that for any function of t',

$$\int_{-\infty}^{\infty} dt' \, f(t')\delta\left(t'-(t-|\mathbf{r}-\mathbf{R}_q(t')|/c)\right) = \frac{f(t_r)}{1-\frac{\dot{\mathbf{R}}_q(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c|\mathbf{r}-\mathbf{R}_q(t_r)|}},$$

where the ``retarded time'' is defined to be

$$t_r \equiv t - \frac{|\mathbf{r}- \ \mathbf{R}_q(t_r)|}{c}.$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

Resulting scalar and vector potentials:

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}},$$

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2}\frac{\mathbf{v}}{R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}},$$

Notation: $\mathbf{R} \equiv \mathbf{r} - \mathbf{R}_q(t_r)$

$$\mathbf{v} \equiv \dot{\mathbf{R}}_q(t_r),$$

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}.$$

Comment on Lienard-Wiechert potential results

$$\int_{-\infty}^{\infty} dt' \; f(t')\delta\left(t' - (t - |\mathbf{r} - \mathbf{R}_q(t')|/c)\right) = \frac{f(t_r)}{1 - \frac{\dot{\mathbf{R}}_q(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c|\mathbf{r}-\mathbf{R}_q(t_r)|}},$$

where the "retarded time" is defined to be

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}.$$

Note that for any function $F(x)$:

$$\int_{-\infty}^{\infty} F(x)\delta(x - x_0)dx = F(x_0)$$

Now consider a function $p(x)$, for which $p(x_i) = 0$ for $i = 1,2,\cdots$

$$\int_{-\infty}^{\infty} F(x)\delta(p(x))dx = \int_{-\infty}^{\infty} F(x)\left(\sum_i \delta\left((x - x_i)\left.\frac{dp}{dx}\right|_{x_i}\right)\right)dx$$

$$= \sum_i \frac{F(x_i)}{\left|\frac{dp}{dx}\right|_{x_i}}$$

Comment on Lienard-Wiechert potential results -- continued

$$\int_{-\infty}^{\infty} dt' \; f(t')\delta\left(t' - (t - |\mathbf{r} - \mathbf{R}_q(t')|/c)\right) = \frac{f(t_r)}{1 - \frac{\dot{\mathbf{R}}_q(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c|\mathbf{r}-\mathbf{R}_q(t_r)|}},$$

where the "retarded time" is defined to be

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}.$$

In this case we have: $$\int_{-\infty}^{\infty} f(t')\delta\left(p(t')\right)dt' = \frac{f(t_r)}{\left|1 - \frac{\dot{\mathbf{R}}_q(t_r)\cdot\left(\mathbf{r}-\mathbf{R}_q(t')\right)}{c\left|\mathbf{r}-\mathbf{R}_q(t_r)\right|}\right|}$$

where: $$p(t') \equiv t' - \left(t - \frac{\left|\mathbf{r}-\mathbf{R}_q(t')\right|}{c}\right)$$

$$\frac{dp(t')}{dt'} = 1 - \frac{\frac{d\mathbf{R}_q(t')}{dt'}\cdot\left(\mathbf{r}-\mathbf{R}_q(t')\right)}{c\left|\mathbf{r}-\mathbf{R}_q(t')\right|} \equiv 1 - \frac{\dot{\mathbf{R}}_q(t')\cdot\left(\mathbf{r}-\mathbf{R}_q(t')\right)}{c\left|\mathbf{r}-\mathbf{R}_q(t')\right|}$$

Summary of results for fields due to moving charge – Liénard Wiechert potentials

Resulting scalar and vector potentials:

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0} \frac{1}{R - \frac{\mathbf{v}\cdot\,\mathbf{R}}{c}},$$

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2} \frac{\mathbf{v}}{R - \frac{\mathbf{v}\cdot\,\mathbf{R}}{c}},$$

Notation: $\mathbf{R} \equiv \mathbf{r} - \mathbf{R}_q(t_r)$

$$\mathbf{v} \equiv \dot{\mathbf{R}}_q(t_r),$$

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}.$$