

PHY 712 Electrodynamics

11-11:50 AM MWF in Olin 103

Notes for Lecture 16:

Finish reading Chapter 6

- 1. Some details of Liénard-Wiechert results**
- 2. Energy density and flux associated with electromagnetic fields**
- 3. Time harmonic fields**

Course schedule for Spring 2022

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Mon: 01/10/2022	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/14/2022
2	Wed: 01/12/2022	Chap. 1	Electrostatic energy calculations	#2	01/19/2022
3	Fri: 01/14/2022	Chap. 1	Electrostatic energy calculations	#3	01/21/2022
	Mon: 01/17/2022		MLK Holiday -- no class		
4	Wed: 01/19/2022	Chap. 1 & 2	Electrostatic potentials and fields	#4	01/24/2022
5	Fri: 01/21/2022	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#5	01/26/2022
6	Mon: 01/24/2022	Chap. 1 - 3	Brief introduction to numerical methods	#6	01/28/2022
7	Wed: 01/26/2022	Chap. 2 & 3	Image charge constructions	#7	01/31/2022
8	Fri: 01/28/2022	Chap. 2 & 3	Cylindrical and spherical geometries	#8	02/02/2022
9	Mon: 01/31/2022	Chap. 3 & 4	Spherical geometry and multipole moments	#9	02/04/2022
	Wed: 02/02/2022	No class	Fire caution		
	Fri: 02/04/2022	No class	Fire caution		
10	Mon: 02/07/2022	Chap. 4	Dipoles and Dielectrics	#10	02/09/2022
11	Wed: 02/09/2022	Chap. 4	Dipoles and Dielectrics	#11	02/11/2022
12	Fri: 02/11/2022	Chap. 5	Magnetostatics	#12	02/14/2022
13	Mon: 02/14/2022	Chap. 5	Magnetic dipoles and hyperfine interaction	#13	02/16/2022
14	Wed: 02/16/2022	Chap. 5	Magnetic dipoles and dipolar fields	#14	02/18/2022
15	Fri: 02/18/2022	Chap. 6	Maxwell's Equations		
	Mon: 02/21/2022	Chap. 1-6	Homework review & presentations		
16	Wed: 02/23/2022	Chap. 6	Electromagnetic energy and forces	#15	02/25/2022
17	Fri: 02/25/2022	Chap. 7	Electromagnetic plane waves		
18	Mon: 02/28/2022	Chap. 7	Electromagnetic plane waves		

PHY 712 -- Assignment #15

February 23, 2022

Finish reading Chapter 6 in **Jackson** .

1. From the definitions of the Poynting vector **S** and Maxwell stress tensor $T_{\alpha\beta}$ given in Chapter 6, determine their units for each in terms of fundamental SI variables.

Spring break

FEBRUARY

S	M	T	W	T	F	S
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28					

MARCH

S	M	T	W	T	F	S
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30	31		

APRIL

S	M	T	W	T	F	S
						1 2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26	27	28	29	30

Mid term grades due

Prepare presentations

Scheduling details

When would you like to make your presentations – expected to need two double class periods?

- A. Week of March 21?
- B. Week of April 25 (last week of class)?

Remaining topics to be covered in PHY 712

- Chap. 7 – Plane waves and optical properties of materials
- Chap. 8 – Wave guides
- Chap. 9 – Radiation from time harmonic sources
- Chap. 11 – Theory of special relativity
- Chap. 14 – Synchrotron radiation and other special radiation effects

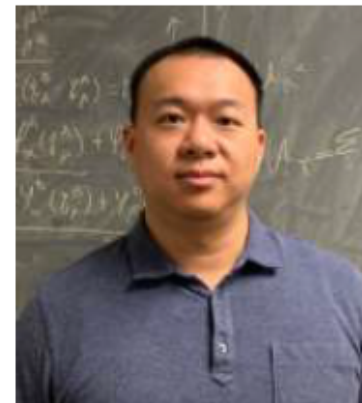
PHYSICS COLLOQUIUM

4 PM Olin 101
WEDNESDAY

•
FEBRUARY 23, 2022

“Plasmon-Coupled Resonance Energy Transfer”

Resonance energy transfer (RET) can be a great tool for various applications, including photosynthesis, probing molecular structure dynamics, bio and chemical sensing, etc. The theoretical models for such process was well studied in homogenous environments, such as in solutions. However, advances in experimental techniques and methods enabled the exploration of RET in complex environments involving nanoparticles and nanocavities, which is beyond the capability of the commonly used Förster theory. I will briefly introduce the recently developed method of plasmon-coupled RET and the new concepts of coupling spectrum, which determines the RET rate and is influenced by the surrounding environment. The behavior and physical interpretation of the coupling spectrum will be discussed as well.



Wendu Ding, Ph.D.

Assistant Professor
Department of Chemistry
Wake Forest University
Winston-Salem, NC

4:00 pm - Olin 101*

*Link provided for those unable to attend in person.

Note: For additional information on the seminar
or to obtain the video conference link, contact

wfduphys@wfu.edu

Colloquium: “Applied Ecological Economics: Integrating Nature in Decision making” — Thursday, February 24, 2022 at 4 PM

Ken Cousins, PhD
Ecological Economist, Research Principal
Earth Economics
Tacoma, WA

A reception will be held outside Olin Building Entrance* at 3:30 PM prior to the colloquium. All interested persons are cordially invited to attend.

*We encourage all to wander outside to the front entrance or up to the Observatory Deck on the 3rd floor to enjoy their refreshments.

The 20th century fundamentally changed the way humans understand the world, from developments in Physics, Ecology, and Economics, to the first images of our planet as seen from space. The challenges of balancing human wellbeing with a livable environment requires that we acknowledge the critical importance of Nature to our ability to survive and thrive as a species. Whether our destiny is in the stars or here on Earth, we need approaches to decision making that sustain the larger material, energetic, and living systems that support our species. A relatively young field, Ecological Economics relies heavily on data about the extent, form, and health of ecosystems throughout the world; while such data are still rather limited in scope, the range of possibilities expands continuously. We envision a future in which high-resolution data—spatial and temporal—about all Earth systems are available to support decisions which allow both communities and nature to thrive together.

Solution of Maxwell's equations in the Lorentz gauge – Review from previous lecture --

Liénard-Wiechert potentials and fields --

Determination of the scalar and vector potentials for a moving point particle (also see Landau and Lifshitz ***The Classical Theory of Fields***, Chapter 8.)

Consider the fields produced by the following source: a point charge q moving on a trajectory $\mathbf{R}_q(t)$.

Charge density: $\rho(\mathbf{r}, t) = q\delta^3(\mathbf{r} - \mathbf{R}_q(t))$

Current density: $\mathbf{J}(\mathbf{r}, t) = q\dot{\mathbf{R}}_q(t)\delta^3(\mathbf{r} - \mathbf{R}_q(t))$, where $\dot{\mathbf{R}}_q(t) \equiv \frac{d\mathbf{R}_q(t)}{dt}$.



Solution of Maxwell's equations in the Lorentz gauge -- continued

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \iint d^3r' dt' \frac{\rho(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \delta(t' - (t - |\mathbf{r} - \mathbf{r}'|/c))$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0 c^2} \iint d^3r' dt' \frac{\mathbf{J}(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \delta(t' - (t - |\mathbf{r} - \mathbf{r}'|/c)).$$

We performing the integrations over first d^3r' and then dt' making use of the fact that for any function of t' ,

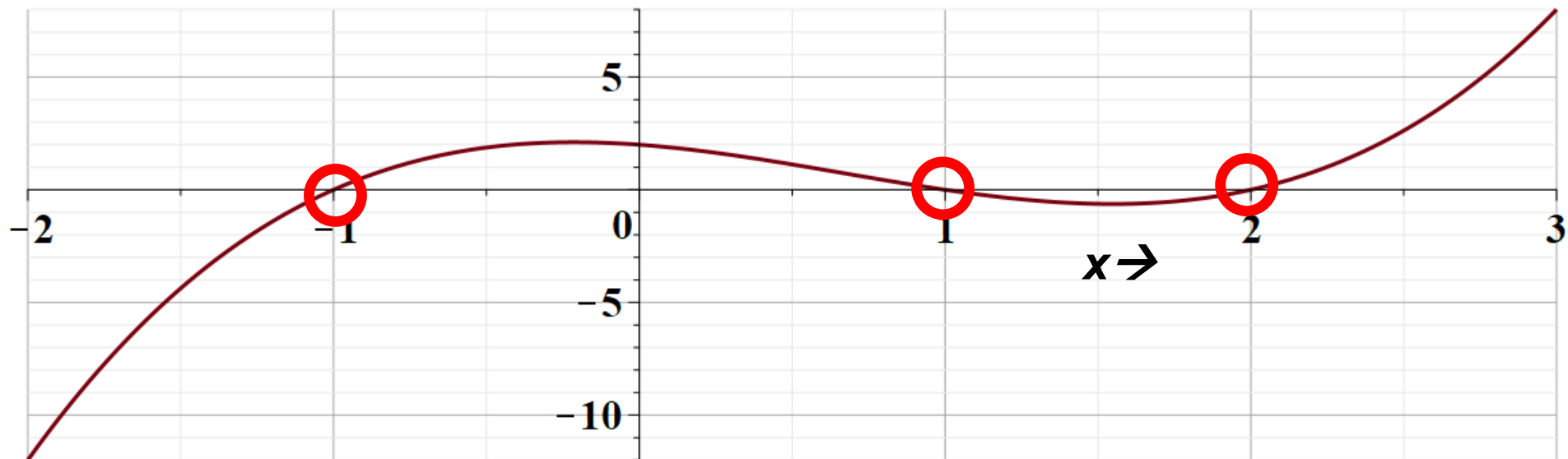
$$\int_{-\infty}^{\infty} dt' f(t') \delta(t' - (t - |\mathbf{r} - \mathbf{R}_q(t')|/c)) = \frac{f(t_r)}{1 - \frac{\dot{\mathbf{R}}_q(t_r) \cdot (\mathbf{r} - \mathbf{R}_q(t_r))}{c |\mathbf{r} - \mathbf{R}_q(t_r)|}},$$

where the "retarded time" is defined to be

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}.$$

Comment about delta functions -- See Pg. 26 in **Jackson**

$$\int_{-\infty}^{\infty} dx \Psi(x) \delta(f(x)) = \sum_i \frac{\Psi(x_i)}{\left| \frac{df(x)}{dx} \right|_{x=x_i}}$$



Solution of Maxwell's equations in the Lorentz gauge -- continued

Resulting scalar and vector potentials:

$$\Phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}},$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 c^2} \frac{\mathbf{v}}{R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}},$$

Notation: $\mathbf{R} \equiv \mathbf{r} - \mathbf{R}_q(t_r)$ $t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}.$

$$\mathbf{v} \equiv \dot{\mathbf{R}}_q(t_r),$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

In order to find the electric and magnetic fields, we need to evaluate

$$\mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \frac{\partial\mathbf{A}(\mathbf{r}, t)}{\partial t}$$

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$$

The trick of evaluating these derivatives is that the retarded time t_r depends on position \mathbf{r} and on itself. We can show the following results using the shorthand notation:

$$\nabla t_r = -\frac{\mathbf{R}}{c\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)} \quad \text{and} \quad \frac{\partial t_r}{\partial t} = \frac{R}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)}.$$

Some details --

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}$$

$$\frac{\partial t_r}{\partial t} = 1 + \frac{(\mathbf{r} - \mathbf{R}_q(t_r)) \cdot \frac{d\mathbf{R}_q(t_r)}{dt_r}}{c |\mathbf{r} - \mathbf{R}_q(t_r)|} \frac{\partial t_r}{\partial t}$$

Using notation: $\mathbf{R} \equiv \mathbf{r} - \mathbf{R}_q(t_r)$ $\mathbf{v} \equiv \dot{\mathbf{R}}_q(t_r)$,

$$\rightarrow \frac{\partial t_r}{\partial t} = \frac{R}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c} \right)}.$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

$$-\nabla\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0} \frac{1}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left[\mathbf{R} \left(1 - \frac{v^2}{c^2}\right) - \frac{v}{c} \left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right) + \mathbf{R} \frac{\dot{\mathbf{v}} \cdot \mathbf{R}}{c^2} \right],$$

$$-\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} = \frac{q}{4\pi\epsilon_0} \frac{1}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left[\frac{\mathbf{v}R}{c} \left(\frac{v^2}{c^2} - \frac{\mathbf{v} \cdot \mathbf{R}}{Rc} - \frac{\dot{\mathbf{v}} \cdot \mathbf{R}}{c^2}\right) - \frac{\dot{\mathbf{v}}R}{c^2} \left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right) \right].$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0} \frac{1}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c} \right) \left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R} \times \left\{ \left(\mathbf{R} - \frac{\mathbf{v}R}{c} \right) \times \frac{\dot{\mathbf{v}}}{c^2} \right\} \right) \right].$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2} \left[\frac{-\mathbf{R} \times \mathbf{v}}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}} \cdot \mathbf{R}}{c^2}\right) - \frac{\mathbf{R} \times \dot{\mathbf{v}} / c}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^2} \right] = \frac{\mathbf{R} \times \mathbf{E}(\mathbf{r},t)}{cR}$$

Back to general case --

Maxwell's equations

Coulomb's law : $\nabla \cdot \mathbf{D} = \rho_{free}$

Ampere - Maxwell's law : $\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$

Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles : $\nabla \cdot \mathbf{B} = 0$

Energy analysis of electromagnetic fields and sources
Rate of work done on source $\mathbf{J}(\mathbf{r}, t)$ by electromagnetic field:

$$\frac{dW_{mech}}{dt} \equiv \frac{dE_{mech}}{dt} = \int d^3r \quad \mathbf{E} \cdot \mathbf{J}_{free}$$

Expressing source current in terms of fields it produces:

$$\frac{dW_{mech}}{dt} = \int d^3r \quad \mathbf{E} \cdot \left(\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} \right)$$

Energy analysis of electromagnetic fields and sources - continued

$$\begin{aligned}\frac{dW_{mech}}{dt} &= \int d^3r \quad \mathbf{E} \cdot \mathbf{J}_{free} = \int d^3r \quad \mathbf{E} \cdot \left(\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} \right) \\ &= - \int d^3r \quad \left(\nabla \cdot (\mathbf{E} \times \mathbf{H}) + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \right)\end{aligned}$$

Let $\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}$ "Poynting vector"

$$u \equiv \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}) \quad \text{energy density}$$

$$\Rightarrow \frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{J}_{free}$$

Assuming that $\mathbf{D} = \epsilon \mathbf{E}$
and that $\mathbf{B} = \mu \mathbf{H}$

Energy analysis of electromagnetic fields and sources - continued

$$\frac{dE_{mech}}{dt} \equiv \int d^3r \quad \mathbf{E} \cdot \mathbf{J}_{free}$$

Electromagnetic energy density: $u \equiv \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B})$

$$E_{field} \equiv \int d^3r \quad u(\mathbf{r}, t)$$

Poynting vector: $\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}$

From the previous energy analysis: $\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{J}_{free}$

$$\Rightarrow \frac{dE_{mech}}{dt} + \frac{dE_{field}}{dt} = -\int d^3r \quad \nabla \cdot \mathbf{S}(\mathbf{r}, t) = -\oint d^2r \quad \hat{\mathbf{r}} \cdot \mathbf{S}(\mathbf{r}, t)$$

Momentum analysis of electromagnetic fields and sources

$$\frac{d\mathbf{P}_{mech}}{dt} \equiv \int d^3r \left(\rho \mathbf{E} + \mathbf{J} \times \mathbf{B} \right)$$

Follows by analogy with Lorentz force:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$$\mathbf{P}_{field} = \varepsilon_0 \int d^3r \left(\mathbf{E} \times \mathbf{B} \right)$$

Expression for vacuum fields:

$$\left(\frac{d\mathbf{P}_{mech}}{dt} + \frac{d\mathbf{P}_{field}}{dt} \right)_i = \sum_j \int d^3r \frac{\partial T_{ij}}{\partial r_j}$$

Maxwell stress tensor:

$$T_{ij} \equiv \varepsilon_0 \left(E_i E_j + c^2 B_i B_j - \delta_{ij} \frac{1}{2} (\mathbf{E} \cdot \mathbf{E} + c^2 \mathbf{B} \cdot \mathbf{B}) \right)$$

Summary -- By considering a complete system involving self-contained sources and fields, we examined the energy and force relationships and introduce energy and force equivalents of the electromagnetic fields

Electromagnetic energy density: $u \equiv \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B})$

Poynting vector: $\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}$

Differential relationship: $\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{J}_{free}$

Maxwell stress tensor (for vacuum case):

$$T_{ij} \equiv \epsilon_0 \left(E_i E_j + c^2 B_i B_j - \delta_{ij} \frac{1}{2} (\mathbf{E} \cdot \mathbf{E} + c^2 \mathbf{B} \cdot \mathbf{B}) \right)$$

Integral relationships:

$$\frac{dE_{mech}}{dt} \equiv \int d^3r \quad \mathbf{E} \cdot \mathbf{J}_{free}$$

$$E_{field} \equiv \int d^3r \quad u(\mathbf{r}, t)$$

$$\Rightarrow \frac{dE_{mech}}{dt} + \frac{dE_{field}}{dt} = - \int d^3r \quad \nabla \cdot \mathbf{S}(\mathbf{r}, t) = - \oint d^2r \quad \hat{\mathbf{r}} \cdot \mathbf{S}(\mathbf{r}, t)$$

$$\left(\frac{d\mathbf{P}_{mech}}{dt} + \frac{d\mathbf{P}_{field}}{dt} \right)_i = \sum_j \int d^3r \frac{\partial T_{ij}}{\partial r_j}$$

Comment on treatment of time-harmonic fields
Fourier transformation in time domain :

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \tilde{\mathbf{E}}(\mathbf{r}, \omega) e^{-i\omega t}$$

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} dt \mathbf{E}(\mathbf{r}, t) e^{i\omega t}$$

Note that $\mathbf{E}(\mathbf{r}, t)$ is real $\Rightarrow \tilde{\mathbf{E}}(\mathbf{r}, \omega) = \tilde{\mathbf{E}}^*(\mathbf{r}, -\omega)$

These relations and the notion of the superposition principle,
lead to the common treatment:

$$\mathbf{E}(\mathbf{r}, t) = \Re \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) e^{-i\omega t} \right) \equiv \frac{1}{2} \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) e^{-i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r}, \omega) e^{i\omega t} \right)$$

Comment on treatment of time-harmonic fields -- continued

Equations for time harmonic fields :

$$\mathbf{E}(\mathbf{r}, t) = \Re \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) e^{-i\omega t} \right) \equiv \frac{1}{2} \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) e^{-i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r}, \omega) e^{i\omega t} \right)$$

Equations	in time domain	in frequency domain
Coulomb's law :	$\nabla \cdot \mathbf{D} = \rho_{free}$	$\nabla \cdot \tilde{\mathbf{D}} = \tilde{\rho}_{free}$
Ampere - Maxwell's law :	$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$	$\nabla \times \tilde{\mathbf{H}} + i\omega \tilde{\mathbf{D}} = \tilde{\mathbf{J}}_{free}$
Faraday's law :	$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$	$\nabla \times \tilde{\mathbf{E}} - i\omega \tilde{\mathbf{B}} = 0$
No magnetic monopoles :	$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \tilde{\mathbf{B}} = 0$

Note -- in all of these, the real part is taken at the end of the calculation.

Comment on treatment of time-harmonic fields -- continued

Equations for time harmonic fields :

$$\mathbf{E}(\mathbf{r}, t) = \Re \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) e^{-i\omega t} \right) \equiv \frac{1}{2} \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) e^{-i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r}, \omega) e^{i\omega t} \right)$$

Poynting vector: $\mathbf{S}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t)$

$$\begin{aligned} \mathbf{S}(\mathbf{r}, t) &= \frac{1}{4} \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) e^{-i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r}, \omega) e^{i\omega t} \right) \times \left(\tilde{\mathbf{H}}(\mathbf{r}, \omega) e^{-i\omega t} + \tilde{\mathbf{H}}^*(\mathbf{r}, \omega) e^{i\omega t} \right) \\ &= \frac{1}{4} \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) \times \tilde{\mathbf{H}}^*(\mathbf{r}, \omega) + \tilde{\mathbf{E}}^*(\mathbf{r}, \omega) \times \tilde{\mathbf{H}}(\mathbf{r}, \omega) \right) \\ &\quad + \frac{1}{4} \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) \times \tilde{\mathbf{H}}(\mathbf{r}, \omega) e^{-2i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r}, \omega) \times \tilde{\mathbf{H}}^*(\mathbf{r}, \omega) e^{2i\omega t} \right) \end{aligned}$$

$$\langle \mathbf{S}(\mathbf{r}, t) \rangle_{t \text{ avg}} = \Re \left(\frac{1}{2} \left(\tilde{\mathbf{E}}(\mathbf{r}, \omega) \times \tilde{\mathbf{H}}^*(\mathbf{r}, \omega) \right) \right)$$

Maxwell's equations

Coulomb's law :

$$\nabla \cdot \mathbf{D} = \rho_{free}$$

Ampere - Maxwell's law :

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$$

Faraday's law :

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

No magnetic monopoles :

$$\nabla \cdot \mathbf{B} = 0$$

Maxwell's equations

For linear isotropic media - - $\mathbf{D} = \epsilon \mathbf{E}$; $\mathbf{B} = \mu \mathbf{H}$
and no sources :

Coulomb's law : $\nabla \cdot \mathbf{E} = 0$

Ampere - Maxwell's law : $\nabla \times \mathbf{B} - \mu \epsilon \frac{\partial \mathbf{E}}{\partial t} = 0$

Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles : $\nabla \cdot \mathbf{B} = 0$

Analysis of Maxwell's equations without sources -- continued:

Coulomb's law : $\nabla \cdot \mathbf{E} = 0$

Ampere - Maxwell's law : $\nabla \times \mathbf{B} - \mu\epsilon \frac{\partial \mathbf{E}}{\partial t} = 0$

Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles : $\nabla \cdot \mathbf{B} = 0$

$$\begin{aligned}\nabla \times \left(\nabla \times \mathbf{B} - \mu\epsilon \frac{\partial \mathbf{E}}{\partial t} \right) &= -\nabla^2 \mathbf{B} - \mu\epsilon \frac{\partial (\nabla \times \mathbf{E})}{\partial t} \\ &= -\nabla^2 \mathbf{B} + \mu\epsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0\end{aligned}$$

$$\begin{aligned}\nabla \times \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) &= -\nabla^2 \mathbf{E} + \frac{\partial (\nabla \times \mathbf{B})}{\partial t} \\ &= -\nabla^2 \mathbf{E} + \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0\end{aligned}$$

Analysis of Maxwell's equations without sources -- continued:
Both \mathbf{E} and \mathbf{B} fields are solutions to a wave equation:

$$\nabla^2 \mathbf{B} - \frac{1}{v^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

$$\nabla^2 \mathbf{E} - \frac{1}{v^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\text{where } v^2 \equiv c^2 \frac{\mu_0 \epsilon_0}{\mu \epsilon} \equiv \frac{c^2}{n^2}$$

Plane wave solutions to wave equation :

$$\mathbf{B}(\mathbf{r}, t) = \Re(\mathbf{B}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t}) \quad \mathbf{E}(\mathbf{r}, t) = \Re(\mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t})$$

Analysis of Maxwell's equations without sources -- continued:
Plane wave solutions to wave equation :

$$\mathbf{B}(\mathbf{r}, t) = \Re(\mathbf{B}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t}) \quad \mathbf{E}(\mathbf{r}, t) = \Re(\mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t})$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}$$

Note: ϵ, μ, n, k can all be complex; for the moment we will assume that they are all real (no dissipation).

Note that \mathbf{E}_0 and \mathbf{B}_0 are not independent;

$$\text{from Faraday's law : } \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\Rightarrow \mathbf{B}_0 = \frac{\mathbf{k} \times \mathbf{E}_0}{\omega} = \frac{n\hat{\mathbf{k}} \times \mathbf{E}_0}{c}$$

$$\text{also note : } \hat{\mathbf{k}} \cdot \mathbf{E}_0 = 0 \quad \text{and} \quad \hat{\mathbf{k}} \cdot \mathbf{B}_0 = 0$$

Analysis of Maxwell's equations without sources -- continued:

Summary of plane electromagnetic waves:

$$\mathbf{B}(\mathbf{r}, t) = \Re\left(\frac{n\hat{\mathbf{k}} \times \mathbf{E}_0}{c} e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t}\right) \quad \mathbf{E}(\mathbf{r}, t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} \quad \text{and } \hat{\mathbf{k}} \cdot \mathbf{E}_0 = 0$$

Poynting vector for plane electromagnetic waves:

$$\langle \mathbf{S} \rangle_{avg} = \frac{1}{2} \Re\left(\mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} \times \frac{1}{\mu} \left(\frac{n\hat{\mathbf{k}} \times \mathbf{E}_0}{c} e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t}\right)^*\right)$$

$$= \frac{n|\mathbf{E}_0|^2}{2\mu c} \hat{\mathbf{k}} = \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}} |\mathbf{E}_0|^2 \hat{\mathbf{k}}$$

Note that:

$$\begin{aligned} \mathbf{E}_0 \times (\hat{\mathbf{k}} \times \mathbf{E}_0) &= \hat{\mathbf{k}} (\mathbf{E}_0 \cdot \mathbf{E}_0) - \mathbf{E}_0 (\hat{\mathbf{k}} \cdot \mathbf{E}_0) \\ &= \hat{\mathbf{k}} |\mathbf{E}_0|^2 \end{aligned}$$

Analysis of Maxwell's equations without sources -- continued:

Transverse Electric and Magnetic (TEM) waves

Summary of plane electromagnetic waves :

$$\mathbf{B}(\mathbf{r}, t) = \Re \left(\frac{n \hat{\mathbf{k}} \times \mathbf{E}_0}{c} e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} \right) \quad \mathbf{E}(\mathbf{r}, t) = \Re \left(\mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} \right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v} \right)^2 = \left(\frac{n\omega}{c} \right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} \quad \text{and } \hat{\mathbf{k}} \cdot \mathbf{E}_0 = 0$$

Energy density for plane electromagnetic waves :

$$\begin{aligned} \langle u \rangle_{avg} &= \frac{1}{4} \Re \left(\epsilon \mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} \cdot \left(\mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} \right)^* \right) + \\ &\quad \frac{1}{4} \Re \left(\frac{1}{\mu} \frac{n \hat{\mathbf{k}} \times \mathbf{E}_0}{c} e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} \cdot \left(\frac{n \hat{\mathbf{k}} \times \mathbf{E}_0}{c} e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} \right)^* \right) \\ &= \frac{1}{2} \epsilon |\mathbf{E}_0|^2 \end{aligned}$$