PHY 712 Electrodynamics
11-11:50 AM MWF Olin 103

Notes for Lecture 22:

Continue reading Chap. 9

A. **Electromagnetic waves due to specific sources**

B. **Dipole radiation examples**

C. **Radiation from antennas**

	Mon: 03/21/2022		Project presentations I		
	Wed: 03/23/2022		Project presentations II		
21	Fri: 03/25/2022	Chap. 9	Radiation from localized oscillating sources	#18	03/30/2022
22	Mon: 03/28/2022	Chap. 9	Radiation from oscillating sources		
23	Wed: 03/30/2022	Chap. 9 & 10	Radiation and scattering	#19	04/01/2022
24	Fri: 04/01/2022	Chap. 11	Special Theory of Relativity		

Important results from last time – EM waves from time harmonic sources – open isotropic boundaries

For scalar potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\rho}(\mathbf{r}',\omega),$$

where $\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{\Phi}_0(\mathbf{r},\omega) = 0$

For vector potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \tilde{\mathbf{A}}_0(\mathbf{r},\omega) + \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega),$$

where $\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{\mathbf{A}}_0(\mathbf{r},\omega) = 0$

Useful expansion :

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik\sum_{lm} j_l(kr_<)h_l(kr_>)Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}')$$

Spherical Bessel function : $j_l(kr)$

Spherical Hankel function : $h_l(kr) = j_l(kr) + in_l(kr)$

$$\Rightarrow \quad \widetilde{\Phi}(\mathbf{r},\omega) = \widetilde{\Phi}_0(\mathbf{r},\omega) + \sum_{lm} \widetilde{\phi}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\widetilde{\phi}_{lm}(r,\omega) = \frac{ik}{\varepsilon_0}\int d^3r'\,\widetilde{\rho}(\mathbf{r}',\omega)\,j_l(kr_<)h_l(kr_>)Y^*_{lm}(\hat{\mathbf{r}}')$$

$$\Rightarrow \quad \widetilde{\mathbf{A}}(\mathbf{r},\omega) = \widetilde{\mathbf{A}}_0(\mathbf{r},\omega) + \sum_{lm} \widetilde{\mathbf{a}}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\widetilde{\mathbf{a}}_{lm}(r,\omega) = ik\mu_0\int d^3r'\,\widetilde{\mathbf{J}}(\mathbf{r}',\omega)\,j_l(kr_<)h_l(kr_>)Y^*_{lm}(\hat{\mathbf{r}}')$$

Example of dipole radiation source

$$\widetilde{\mathbf{J}}(\mathbf{r},\omega)=\hat{\mathbf{z}}J_0 e^{-r/R} \qquad \widetilde{\rho}(\mathbf{r},\omega)=\frac{J_0}{-i\omega R}\cos\theta e^{-r/R}$$

Note that the continuity of charge and current must be satisfied. For the Fourier amplitudes, the relations are as bellow:

Recall continuity condition: $-i\omega\,\tilde{\rho}(\mathbf{r},\omega)+\nabla\cdot\tilde{\mathbf{J}}(\mathbf{r},\omega)=0$

$$-i\omega\mathbf{r}\;\tilde{\rho}(\mathbf{r},\omega)+\mathbf{r}\nabla\cdot\tilde{\mathbf{J}}(\mathbf{r},\omega)$$

Jackson's clever trick!

$$\int d^3r\;\mathbf{r}\;\tilde{\rho}(\mathbf{r},\omega)=\frac{1}{i\omega}\int d^3r\;\mathbf{r}\nabla\cdot\tilde{\mathbf{J}}(\mathbf{r},\omega)$$

$$=-\frac{1}{i\omega}\int d^3r\;\tilde{\mathbf{J}}(\mathbf{r},\omega)=\mathbf{p}(\omega)$$

Example of dipole radiation source

$$\tilde{\mathbf{J}}(\mathbf{r},\omega)=\hat{\mathbf{z}}J_0 e^{-r/R} \qquad \tilde{\rho}(\mathbf{r},\omega)=\frac{J_0}{-i\omega R}\cos\theta e^{-r/R}$$

$$\mathbf{p}(\omega)=\hat{\mathbf{z}}\frac{J_0}{-i\omega R}\int d^3r\ r\cos(\theta)\left(\cos(\theta)e^{-r/R}\right)$$

$$=\hat{\mathbf{z}}\frac{J_0}{-i\omega R}\frac{4\pi}{3}\int_0^\infty dr\ r^3 e^{-r/R}=\hat{\mathbf{z}}J_0\frac{8\pi R^3}{-i\omega}$$

$$=\frac{1}{-i\omega}\int d^3r\ \tilde{\mathbf{J}}(\mathbf{r},\omega)$$

From the analysis valid for *kr>>1* and *kR<<1*:

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)=-\frac{i\mu_0\omega}{4\pi}\mathbf{p}(\omega)\frac{e^{ikr}}{r}=\hat{\mathbf{z}}J_0\mu_0 2R^3\frac{e^{ikr}}{r}$$

$$\tilde{\Phi}(\mathbf{r},\omega)=-\frac{ik}{4\pi\varepsilon_0}\mathbf{p}(\omega)\cdot\hat{\mathbf{r}}\left(1+\frac{i}{kr}\right)\frac{e^{ikr}}{r}=\frac{J_0 2R^3}{\varepsilon_0 c}\left(1+\frac{i}{kr}\right)\frac{e^{ikr}}{r}\cos\theta$$

Example of dipole radiation source -- exact results for r>>R:

$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} J_0 e^{-r/R} \qquad \tilde{\rho}(\mathbf{r},\omega) = \frac{J_0}{-i\omega R} \cos\theta e^{-r/R}$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \hat{\mathbf{z}} J_0 \left(ik\mu_0\right) \int_0^\infty r'^2\, dr'\, e^{-r'/R} h_0(kr_>) j_0(kr_<)$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{J_0 k}{\varepsilon_0 \omega R} \cos\theta \int_0^\infty r'^2\, dr'\, e^{-r'/R} h_1(kr_>) j_1(kr_<)$$

Evaluation for $r >> R$:

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \hat{\mathbf{z}} J_0 \mu_0 \frac{e^{ikr}}{r} \frac{2R^3}{\left(1+k^2R^2\right)^2}$$

$$\tilde{\Phi}(\mathbf{r},\omega) = \frac{J_0 k}{\varepsilon_0 \omega} \cos\theta \frac{e^{ikr}}{r} \left(1 + \frac{i}{kr}\right) \frac{2R^3}{\left(1+k^2R^2\right)^2}$$

Agrees with dipole approximation for *kR<<1*.

More details --

$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} J_0 e^{-r/R} \qquad \tilde{\rho}(\mathbf{r},\omega) = \frac{J_0}{-i\omega R}\cos\theta e^{-r/R}$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \hat{\mathbf{z}} J_0 \left(ik\mu_0\right)\int_0^\infty r'^2\, dr' e^{-r'/R} h_0(kr_>) j_0(kr_<)$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{J_0 k}{\varepsilon_0 \omega R}\cos\theta \int_0^\infty r'^2\, dr' e^{-r'/R} h_1(kr_>) j_1(kr_<)$$

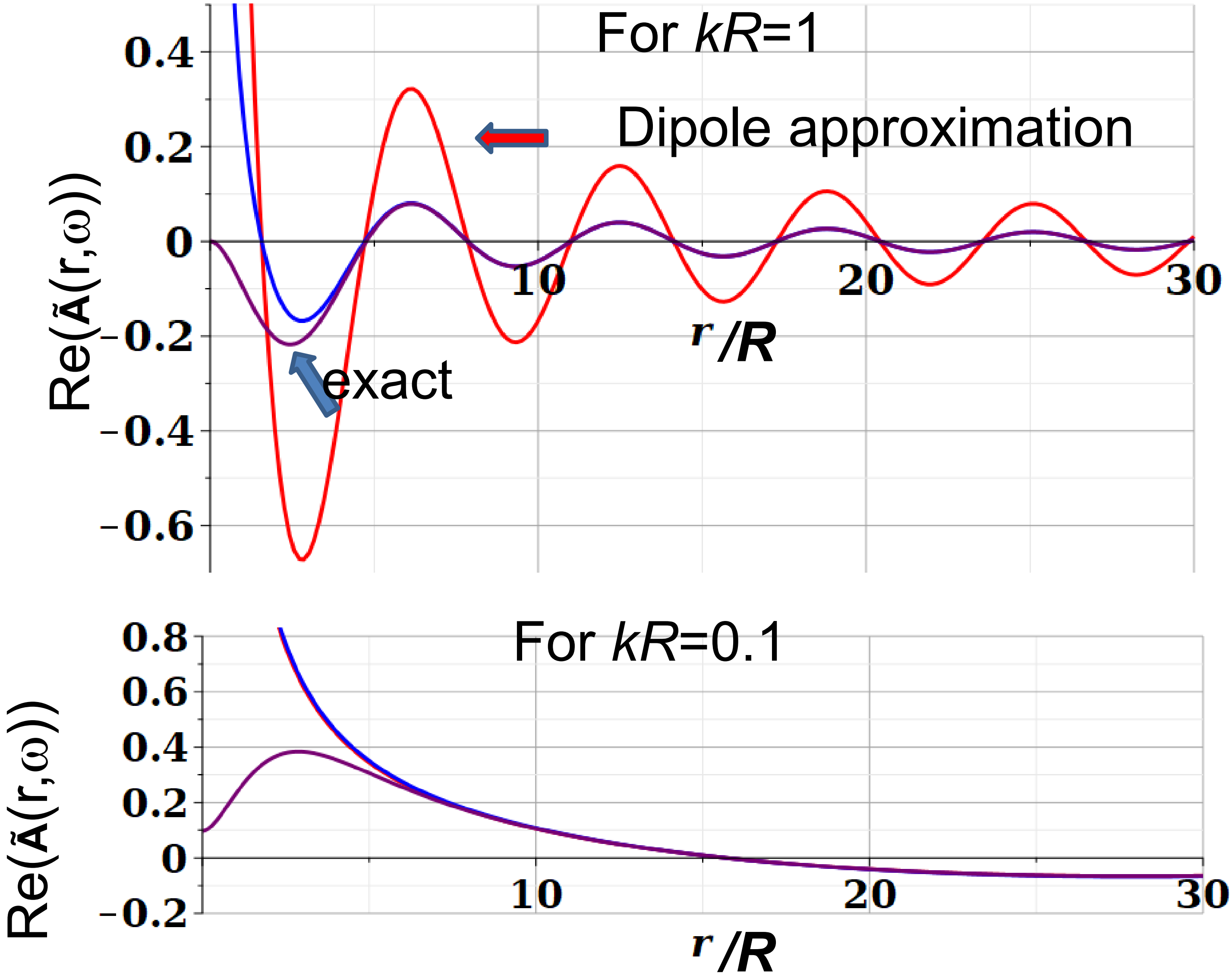
$$\tilde{\mathbf{A}}(r,\omega) = \hat{\mathbf{z}} J_0 \mu_0 \left(\frac{e^{ikr}}{kr}\int_0^r r'dr' e^{-r'/R}\sin(kr') + \frac{\sin(kr)}{kr}\int_r^\infty r'dr' e^{-r'/R+ikr'} \right)$$

$$\underset{r \gg R}{\approx} \hat{\mathbf{z}} J_0 \mu_0 \frac{e^{ikr}}{r} \frac{2R^3}{\left(k^2R^2+1\right)^2}$$

Correct when this term is negligible.

$$\tilde{\mathbf{A}}_{\text{dipole approx}}(r,\omega) = \hat{\mathbf{z}} J_0 \mu_0 \left(\frac{e^{ikr}}{kr}\int_0^\infty r'dr' e^{-r'/R}(kr') \right) = \hat{\mathbf{z}} J_0 \mu_0 \frac{e^{ikr}}{r} 2R^3$$

Example continued



Continued review of dipole results --

Power in the dipole approximation; Section 9.2 of Jackson

Here we use our notation with $\mathbf{n} \rightarrow \hat{\mathbf{r}}$ and $Z_0 \equiv \sqrt{\frac{\mu_0}{\epsilon_0}}$

$$\frac{dP}{d\Omega} = \frac{r^2}{2} \Re \left| \left(\hat{\mathbf{r}} \cdot \left(\mathbf{E} \times \mathbf{H}^* \right) \right) \right|^2$$

Using the expressions for the dipole fields far from the source:

$$\mathbf{H} = \frac{ck^2}{4\pi} \left(\hat{\mathbf{r}} \times \mathbf{p} \right) \frac{e^{ikr}}{r} \qquad \mathbf{E} = Z_0 \mathbf{H} \times \hat{\mathbf{r}}$$

The power can be written $\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 \left| \left(\left(\hat{\mathbf{r}} \times \mathbf{p} \right) \times \hat{\mathbf{r}} \right) \right|^2$

Defining the angle θ by $\mathbf{p} \cdot \hat{\mathbf{r}} = |\mathbf{p}| \cos\theta$,

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 |\mathbf{p}|^2 \sin^2\theta$$ integrating over solid angles $P = \frac{c^2 Z_0}{12\pi} k^4 |\mathbf{p}|^2$

Review:

Electromagnetic waves from time harmonic sources – continued:

Dipole radiation case:

Define dipole moment at frequency ω:

$$\mathbf{p}(\omega) \equiv \int d^3r \ \mathbf{r}\tilde{\rho}(\mathbf{r},\omega) = -\frac{1}{i\omega}\int d^3r \ \tilde{\mathbf{J}}(\mathbf{r},\omega)$$

For fields outside extent of source and $kr' << 1$ within the source:

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = -\frac{i\mu_0\omega}{4\pi}\mathbf{p}(\omega)\frac{e^{ikr}}{r}$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{ik}{4\pi\varepsilon_0}\mathbf{p}(\omega)\cdot\hat{\mathbf{r}}\left(1+\frac{i}{kr}\right)\frac{e^{ikr}}{r}$$

Review:

Electromagnetic waves from time harmonic sources – continued:

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = -\nabla\tilde{\Phi}(\mathbf{r},\omega) + i\omega\tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left(k^2\left(\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\times\hat{\mathbf{r}}\right) + \left(\frac{3\hat{\mathbf{r}}\left(\hat{\mathbf{r}}\cdot\mathbf{p}(\omega)\right)-\mathbf{p}(\omega)}{r^2}\right)(1-ikr)\right)$$

$$\tilde{\mathbf{B}}(\mathbf{r},\omega) = \nabla\times\tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$= \frac{1}{4\pi\varepsilon_0 c}\frac{e^{ikr}}{r}k^2\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\left(1-\frac{1}{ikr}\right)$$

Power radiated for $kr >> 1$:

$$\frac{dP}{d\Omega} = r^2\hat{\mathbf{r}}\cdot\langle\mathbf{S}\rangle_{avg} = \frac{r^2}{2\mu_0}\hat{\mathbf{r}}\cdot\Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\times\tilde{\mathbf{B}}^*(\mathbf{r},\omega)\right)$$

$$= \frac{c^2k^4}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left|\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\times\hat{\mathbf{r}}\right|^2$$

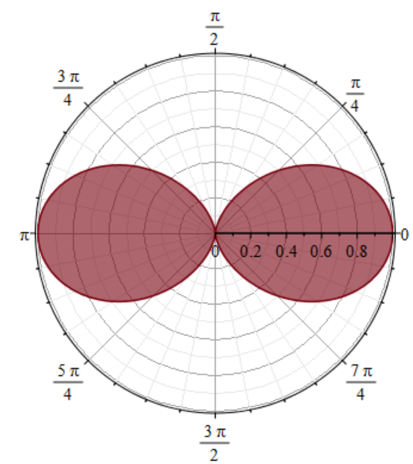
Properties of dipole radiation field for *kr* >>1:

$$\tilde{\mathbf{E}}(\mathbf{r},\omega)=\frac{1}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left(k^2\left(\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\times\hat{\mathbf{r}}\right)\right)$$

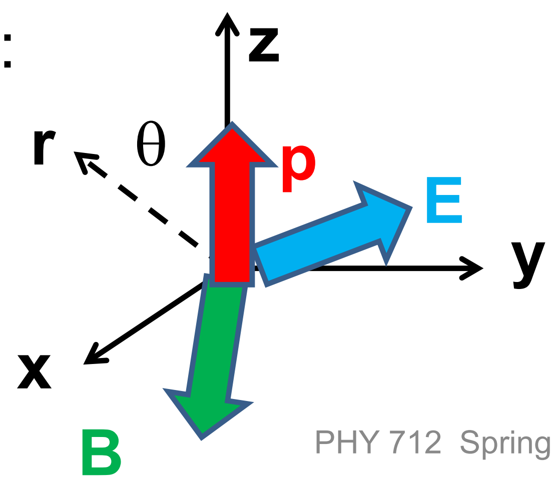
$$\tilde{\mathbf{B}}(\mathbf{r},\omega)=\frac{1}{4\pi\varepsilon_0 c^2}\frac{e^{ikr}}{r}k^2\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)$$

Power radiated for $kr>>1$:

$$\frac{dP}{d\Omega}=r^2\hat{\mathbf{r}}\cdot\langle\mathbf{S}\rangle_{avg}=\frac{c^2k^4}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left|\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\times\hat{\mathbf{r}}\right|^2$$



Example:



Note that vectors **r, E, B** are mutually orthogonal

Alternative approach

Fields from time harmonic source:

$$\tilde{\Phi}(\mathbf{r},\omega) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\rho}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

For $r >> r'$: $\quad |\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}' + \ldots$

$$\tilde{\Phi}(\mathbf{r},\omega) \approx \frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\int d^3r'\, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\,\tilde{\rho}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r'\, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\,\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

For our example:

$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} J_0 e^{-r/R} \qquad \tilde{\rho}(\mathbf{r},\omega) = \frac{J_0}{-i\omega R}\cos\theta e^{-r/R}$$

For $r >> r'$: $\qquad |\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}' + \ldots$

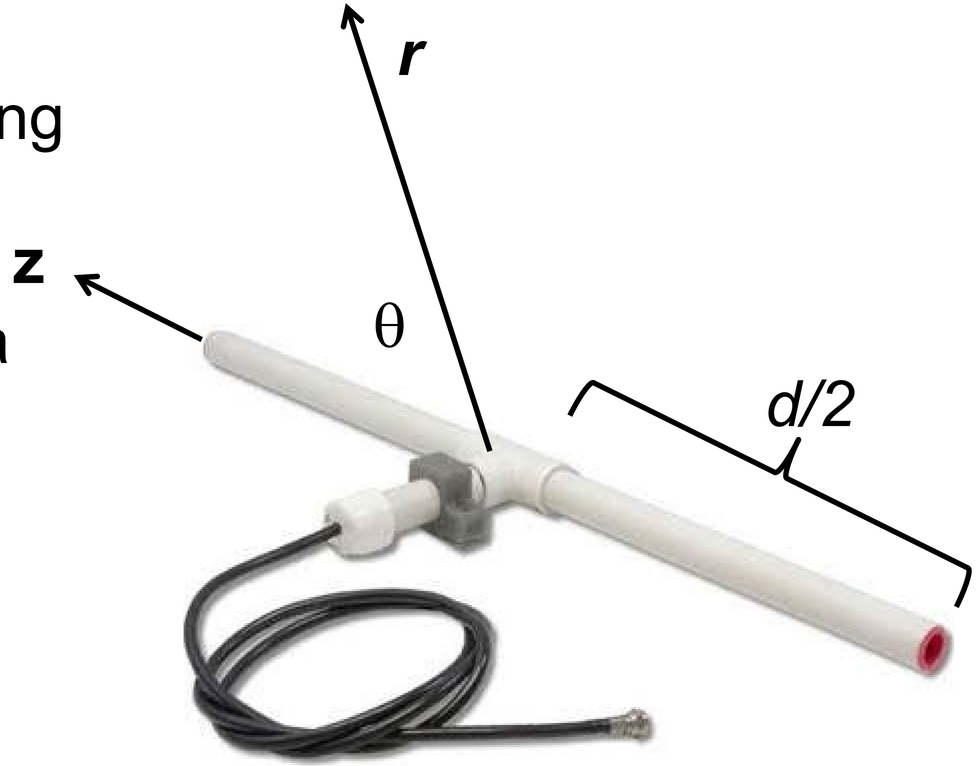
$$\tilde{\Phi}(\mathbf{r},\omega) \approx \frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\int d^3r'\, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\,\tilde{\rho}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r'\, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\,\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

➔Results equivalent to Bessel function expansion in the limit $kr \rightarrow \infty$.

Other radiation sources using ``alternative approach”

Linear center-fed antenna



$$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx$$

$$\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r'\, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\,\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{J}}(\mathbf{r}',\omega) = I_0 \sin\left(\frac{kd}{2} - k\,|\,z\,|\right)\delta(x)\delta(y)\hat{\mathbf{z}}$$

Alternative approach – linear center-fed antenna continued

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \hat{\mathbf{z}} \frac{\mu_0 I_0}{4\pi} \frac{e^{ikr}}{r} \int_{-d/2}^{d/2} dz'\, e^{-ik\cos(\theta) z'} \sin\left(\frac{kd}{2} - k|z'|\right)$$

$$= \hat{\mathbf{z}} \frac{\mu_0 I_0}{2\pi} \frac{e^{ikr}}{kr} \left(\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin^2\theta} \right)$$

Time averaged power:

$$\frac{dP}{d\Omega} = I_0^{\,2} \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{1}{8\pi^2} \left| \frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta} \right|^2$$

Alternative approach – linear center-fed antenna continued

Time averaged power:

$$\frac{dP}{d\Omega} = I_0^{\,2} \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{1}{8\pi^2} \left| \frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta} \right|^2$$

for $kd = \pi$:

$$\frac{dP}{d\Omega} = I_0^{\,2} \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{1}{8\pi^2} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

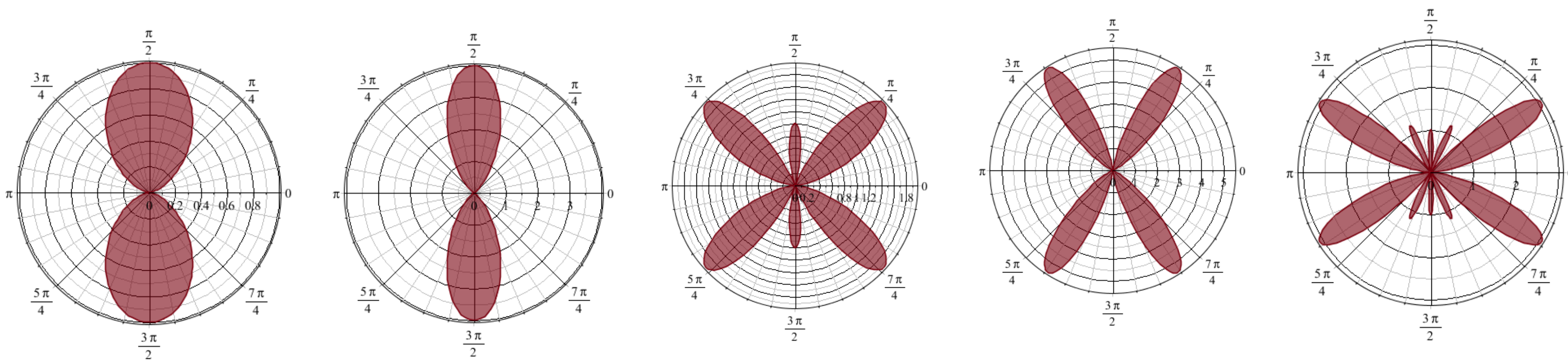
for $kd = 2\pi$:

$$\frac{dP}{d\Omega} = I_0^{\,2} \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{4}{8\pi^2} \frac{\cos^4\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

Alternative approach – linear center-fed antenna continued

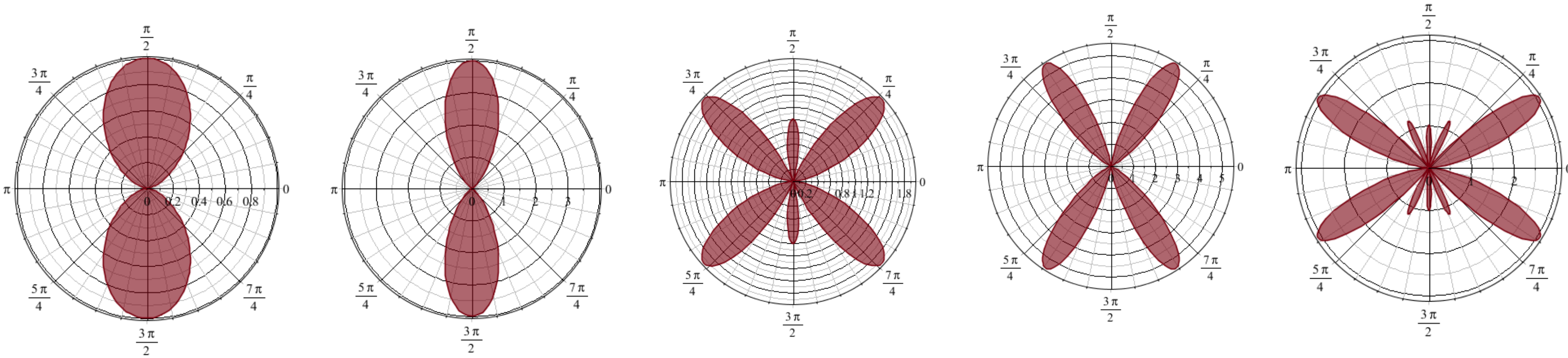
Time averaged power:

$$\frac{dP}{d\Omega} = I_0^{\,2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{1}{8\pi^2}\left|\frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\left(\frac{kd}{2}\right)}{\sin\theta}\right|^2$$

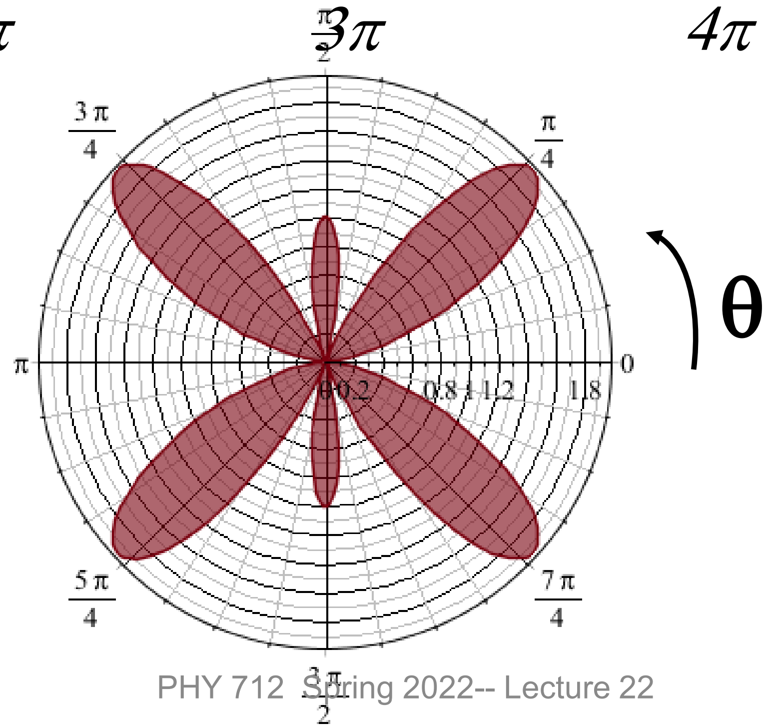


kd= π $\quad 2\pi \quad 3\pi \quad 4\pi \quad 5\pi$

Some details --



kd=π 2π 3π 4π 5π



Polar plot:
Angle indicates values of theta

Radius indicates value scaled to 1.

Next time – we will consider the effects of multiple antennas (antenna arrays including interference effects) and radiation due to light scattering (from Chapter 10 of your textbook).