PHY 712 Electrodynamics
11-11:50 AM MWF in Olin 103

Notes for Lecture 23:

Complete reading of Chap. 9 & 10

A. Superposition of radiation

B. Scattered radiation

	Mon: 03/21/2022		Project presentations I		
	Wed: 03/23/2022		Project presentations II		
21	Fri: 03/25/2022	Chap. 9	Radiation from localized oscillating sources	#18	03/30/2022
22	Mon: 03/28/2022	Chap. 9	Radiation from oscillating sources		
23	Wed: 03/30/2022	Chap. 9 & 10	Radiation and scattering	#19	04/01/2022
24	Fri: 04/01/2022	Chap. 11	Special Theory of Relativity		

PHY 712 -- Assignment #19

March 30, 2022

Finish reading Chapters 9 and 10 in **Jackson** .

1. Work problem 9.16(a) in **Jackson**. Note that you can use an approach similar to that discussed in Section 9.4 of the textbook, replacing the "center-fed" antenna with the given antenna configuration.

PHYSICS COLLOQUIUM

4 PM Olin 101

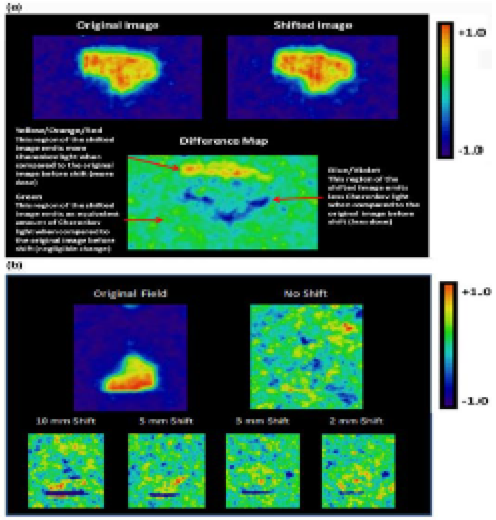
THURSDAY

MARCH 31, 2022

Radiation Treatment Delivery Verification Using Cherenkov Light

About half of all diagnosed cancer patients are treated using controlled exposures of ionizing radiation. In general, patients are given radiation delivered in a prescribed number of fractionated doses. As such, proper delivery of radiation therapy requires accurate patient positioning, to millimeters or even sub-millimeter accuracy. It is essential that positioning to be reproducible for each fraction delivery. This need for accurate and consistent patient positioning becomes even more important as the complexity or the fractional radiation dose increases. Conventionally, patient positioning is verified before radiation delivery, and immobilization devices are used to minimize patient movement after setup. An improvement on this methodology would be positioning verification performed in real-time, concurrent with treatment. Recently, groups have been working towards establishing methods for real time verification of radiation treatment delivery. One specific phenomenon that has been investigated for this purpose is Cherenkov light.

When highly energetic particles travel through matter, one of the emission products is optical light, through a process known as Cherenkov light emission. Cherenkov light is observable on patient skin during radiation treatment delivery. This type of light emission has been shown to correlate with ionizing radiation dose delivery in solid tissue, allowing real-time verification of radiation treatment delivery. We focused our study of Cherenkov light emission on the feasibility of radiation treatment field verification. Specifically, Cherenkov light images were acquired during radiation beam delivery to standard and anthropomorphic phantoms. Two clinical treatment scenarios were tested: 1) Observation of field overlaps or gaps in matched radiation fields and 2) Patient positioning shifts during modulated dose



Paul J. Black, PhD, DABR
Michael T. Munley, PhD, DABR
Department of Radiation Oncology,
Wake Forest Baptist Medical Center

4:00 pm - Olin 101*

Electromagnetic waves from time harmonic sources – review:

For scalar potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

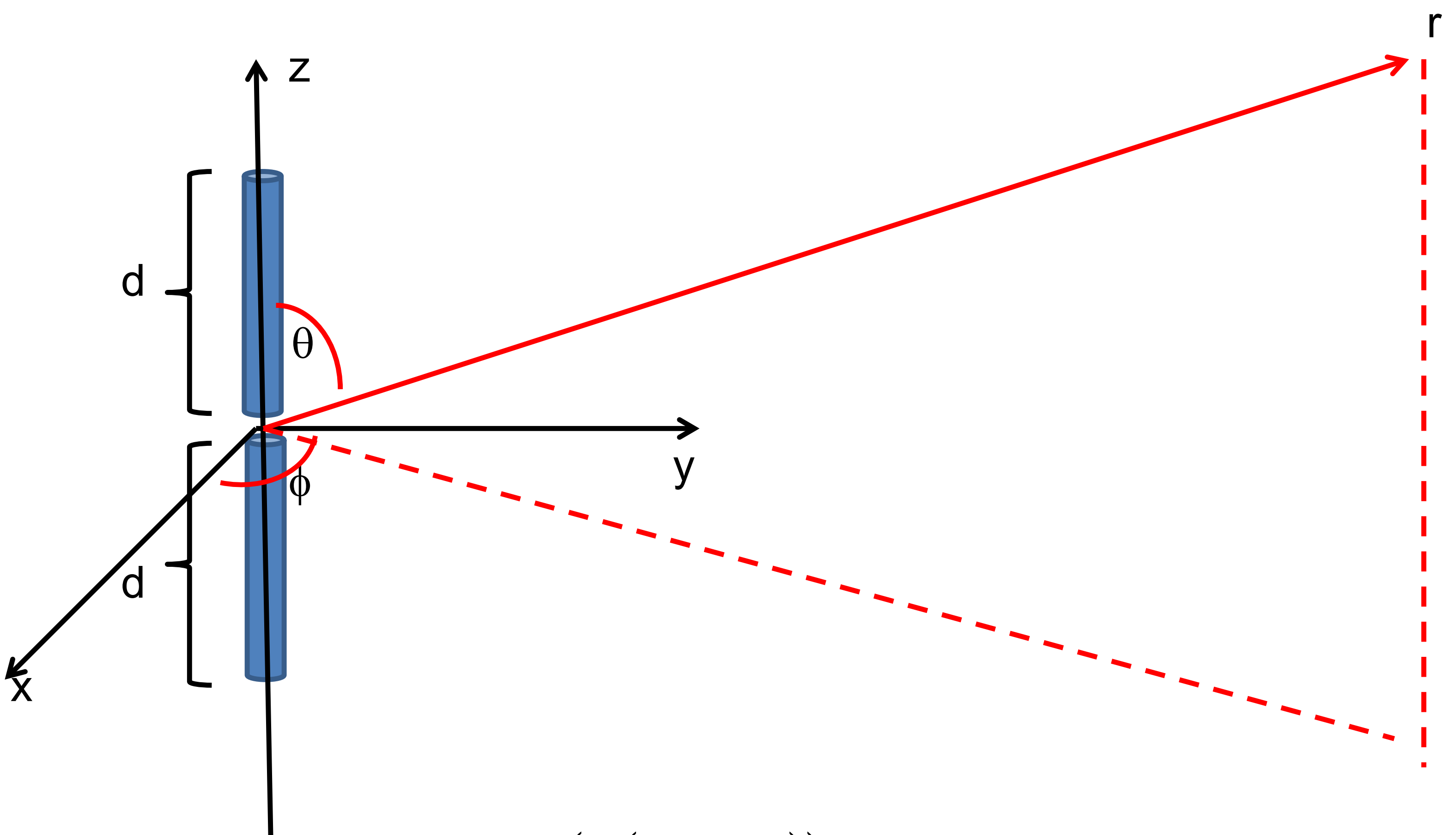
$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\rho}(\mathbf{r}',\omega)$$

For vector potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \tilde{\mathbf{A}}_0(\mathbf{r},\omega) + \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

Consider antenna source (center-fed)

Note – these notes differ from previous formulation d/2 ←→ d



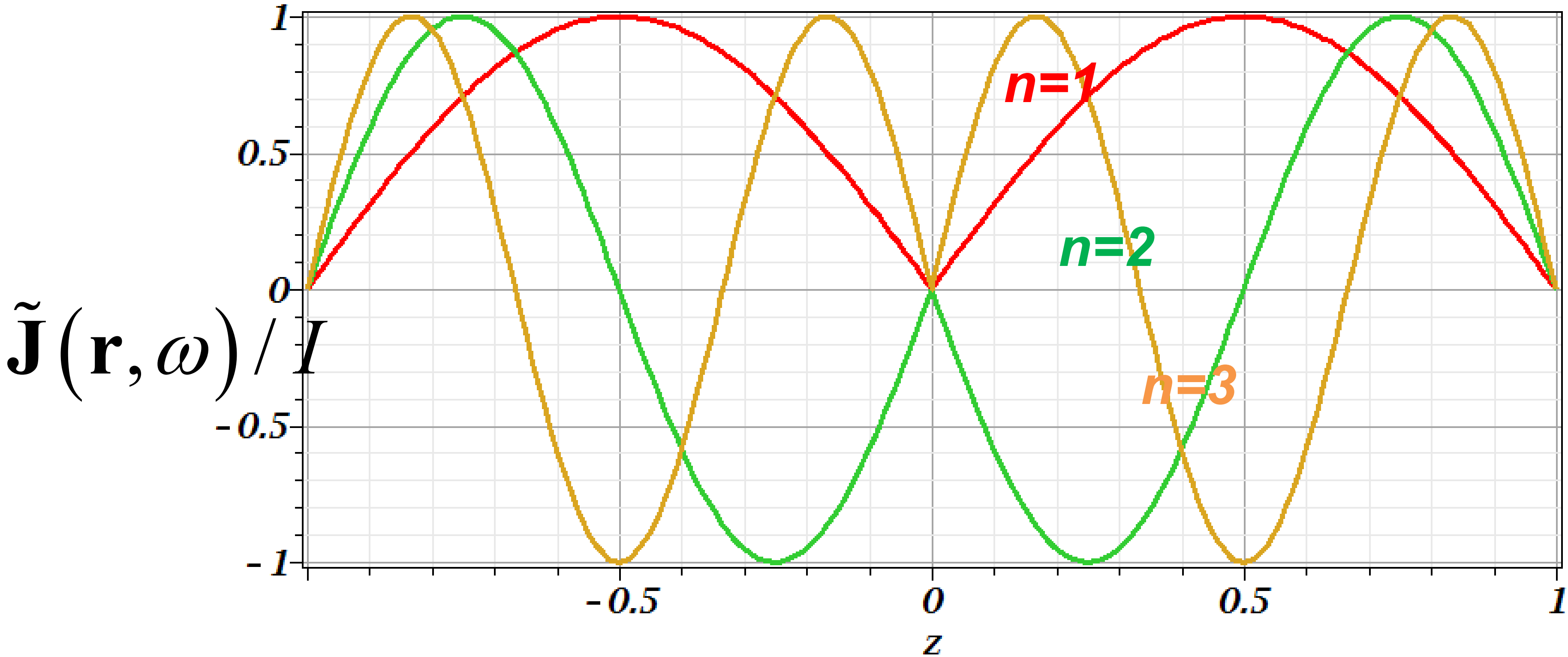
$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} I \sin\left(k\left(d-|z|\right)\right)\delta(x)\delta(y) \quad \text{for } -d \le z \le d$$

$$k \equiv \frac{\omega}{c}$$

Consider antenna source -- continued

$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} I \sin\left(k\left(d - |z|\right)\right)\delta(x)\delta(y) \qquad \text{for} \quad -d \le z \le d$$

$$\text{for} \quad k \equiv \frac{\omega}{c} = \frac{n\pi}{d}; \qquad n = 1, 2, 3....$$



Consider antenna source -- continued

$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} I \sin\left(k\left(d-|z|\right)\right)\delta(x)\delta(y) \quad \text{for} \quad -d \le z \le d$$

$$k \equiv \frac{\omega}{c}$$

Vector potential from source:

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

For $r >> d$ $\quad \tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r' e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r} I \int_{-d}^{d} dz' \, e^{-ikz'\cos\theta} \quad \sin\left(k\left(d-|z'|\right)\right)$$

Consider antenna source -- continued

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r} I \int_{-d}^{d} dz\, e^{-ikz\cos\theta} \sin\left(k\left(d-|z|\right)\right)$$

$$= \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{e^{ikr}}{kr} 2I\left[\frac{\cos(kd\cos\theta)-\cos(kd)}{\sin^2\theta}\right]$$

In the radiation zone:

$$\tilde{\mathbf{B}}(\mathbf{r},\omega) = \nabla\times\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx ik\hat{\mathbf{r}}\times\tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) \approx -ikc\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\tilde{\mathbf{A}}(\mathbf{r},\omega)\right)$$

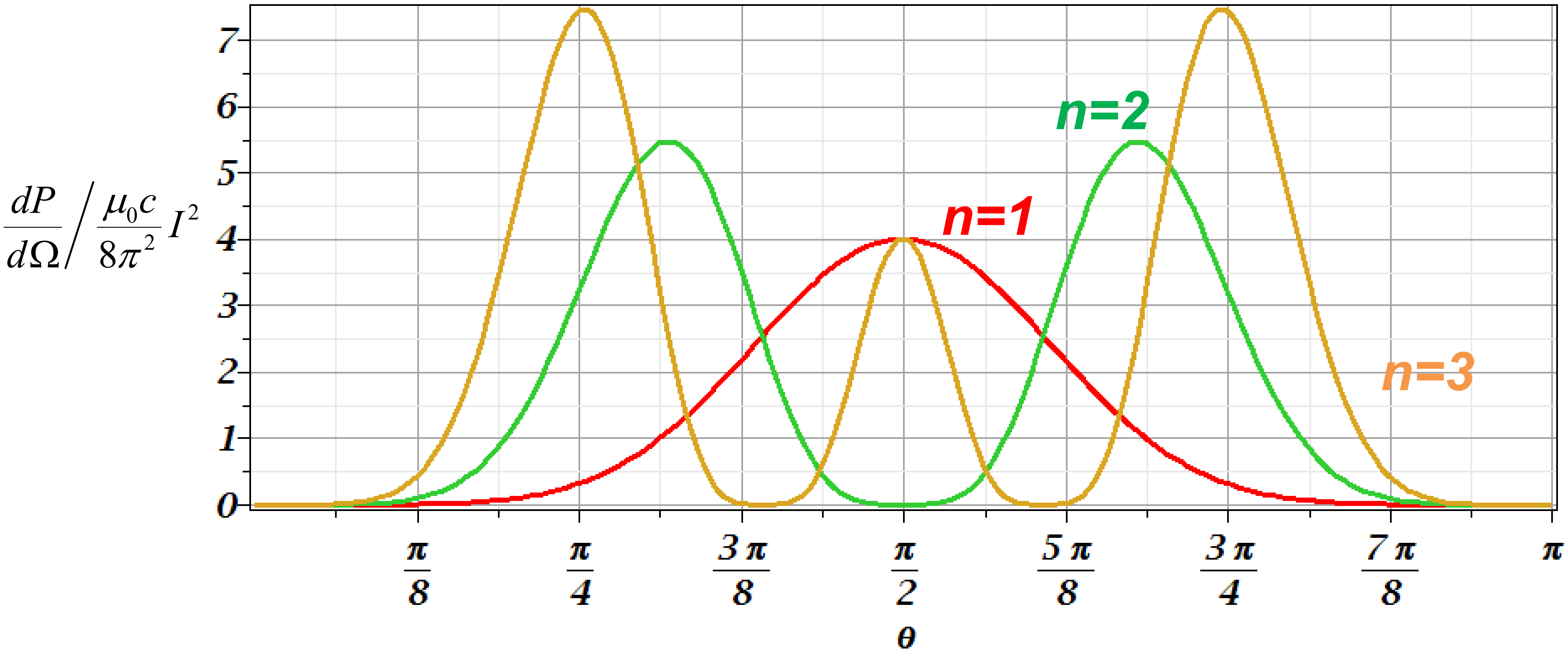
$$\frac{dP}{d\Omega} = \frac{1}{2\mu_0} r^2 \hat{\mathbf{r}}\cdot\Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\times\tilde{\mathbf{B}}^*(\mathbf{r},\omega)\right) = \frac{k^2 c}{2\mu_0} r^2 \left(\left|\tilde{\mathbf{A}}(\mathbf{r},\omega)\right|^2 - \left|\hat{\mathbf{r}}\cdot\tilde{\mathbf{A}}(\mathbf{r},\omega)\right|^2\right)$$

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[\frac{\cos(kd\cos\theta)-\cos(kd)}{\sin\theta}\right]^2$$

Consider antenna source -- continued

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta} \right]^2$$

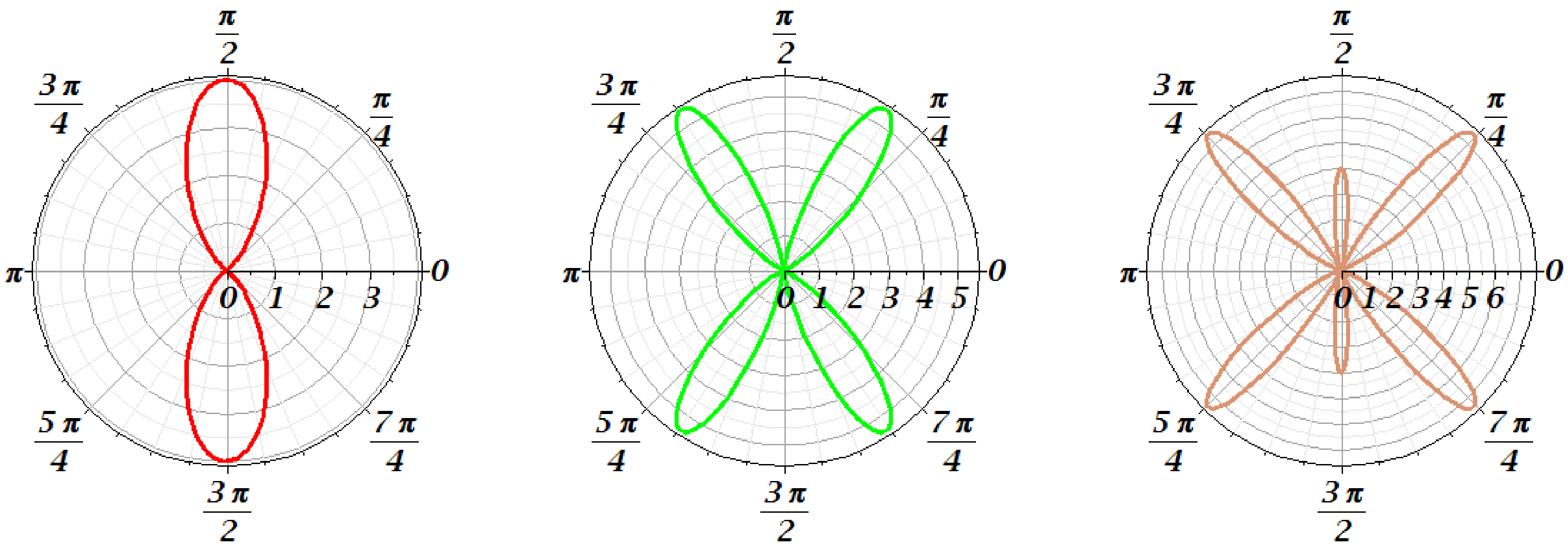
$$\text{for } k \equiv \frac{\omega}{c} = \frac{n\pi}{d}; \qquad n = 1, 2, 3....$$



Consider antenna source -- continued

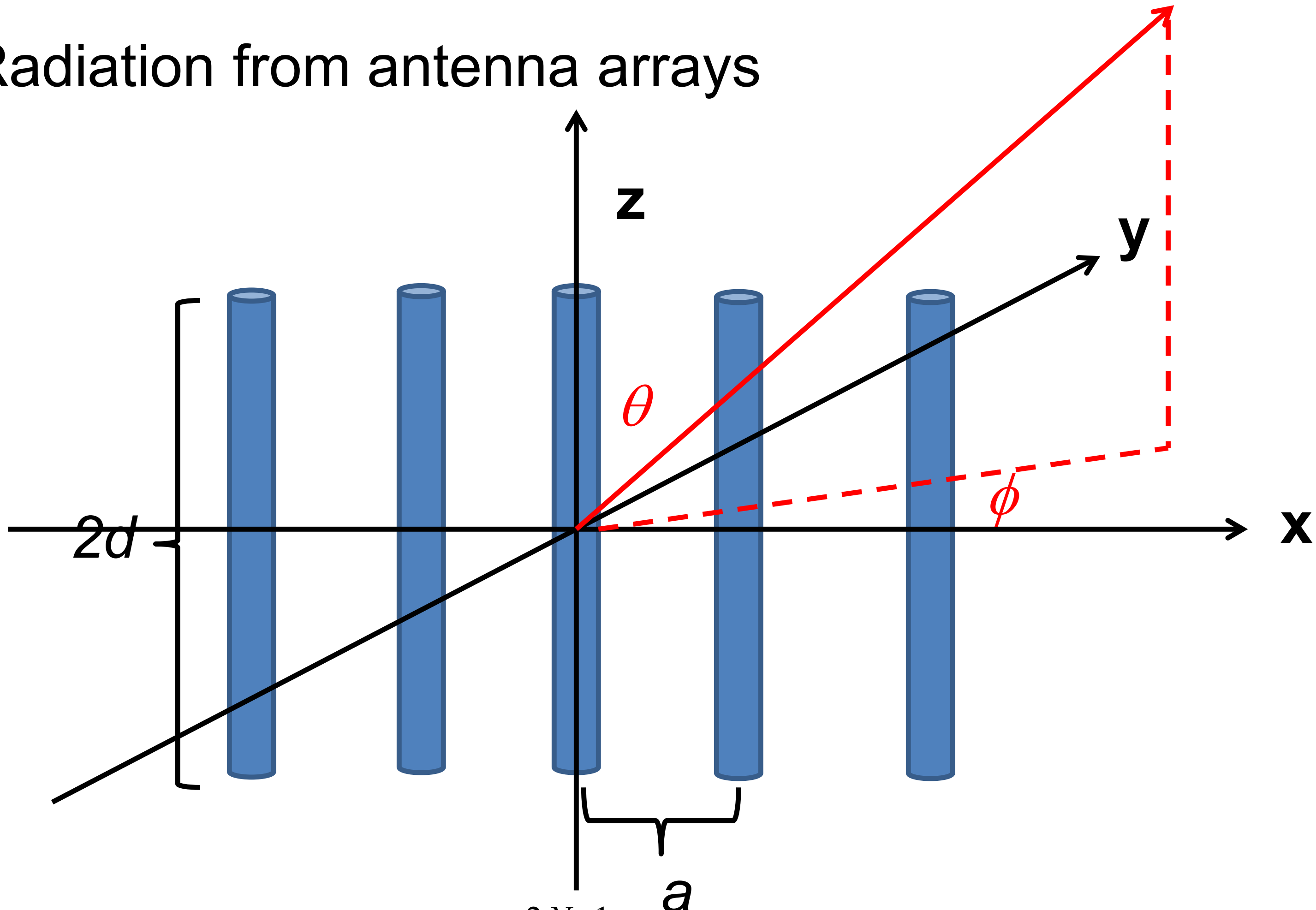
$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta} \right]^2$$

For $kd = n\pi$:



n=1 ***n=2*** ***n=3***

Radiation from antenna arrays



$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} I \sin\left(k\left(d - |z|\right)\right) \sum_{j=1}^{2N+1} \delta\left(x - (N+1-j)a\right)\delta(y) \quad \text{for } -d \le z \le d$$

$$k \equiv \frac{\omega}{c} = \frac{n\pi}{d}; \qquad n = 1,2,3....$$

Radiation from antenna arrays -- continued

Vector potential from array source:

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)=\frac{\mu_0}{4\pi}\int d^3r'\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)\approx\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r' e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{J}}(\mathbf{r},\omega)=\hat{\mathbf{z}}I\sin\left(k\left(d-|z|\right)\right)\sum_{j=1}^{2N+1}\delta\left(x-\left(N+1-j\right)a\right)\delta(y)\quad \text{for } -d\le z\le d$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)\approx\hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\sum_{j=-N}^{N}e^{-ikaj\sin\theta\cos\phi}\right)I\int_{-d}^{d}dz\, e^{-ikz\cos\theta}\ \sin\left(k\left(d-|z|\right)\right)$$

$$\sum_{j=-N}^{N}e^{-ikaj\sin\theta\cos\phi}=\frac{\sin\left(\frac{1}{2}ka(2N+1)\sin\theta\cos\phi\right)}{\sin\left(\frac{1}{2}ka\sin\theta\cos\phi\right)}$$

Digression – summation of a geometric series

$$\sum_{j=-N}^{N} e^{-iAj} = e^{-iA} \sum_{j=-N}^{N} e^{-iAj} + e^{iAN} - e^{-iA(N+1)}$$

$$\sum_{j=-N}^{N} e^{-iAj} = \frac{e^{iAN} - e^{-iA(N+1)}}{1-e^{-iA}} = \frac{e^{iA/2}}{e^{iA/2}} \frac{e^{iAN} - e^{-iA(N+1)}}{1-e^{-iA}}$$

$$= \frac{2i\sin\left(A\left(N+1/2\right)\right)}{2i\sin(A/2)}$$

$$= \frac{\sin\left(A\left(N+1/2\right)\right)}{\sin(A/2)}$$

$$\sum_{j=-N}^{N} e^{-ikaj\sin\theta\cos\varphi} = \frac{\sin\left(\frac{1}{2}ka\left(2N+1\right)\sin\theta\cos\varphi\right)}{\sin\left(\frac{1}{2}ka\sin\theta\cos\varphi\right)}$$

Radiation from antenna arrays -- continued

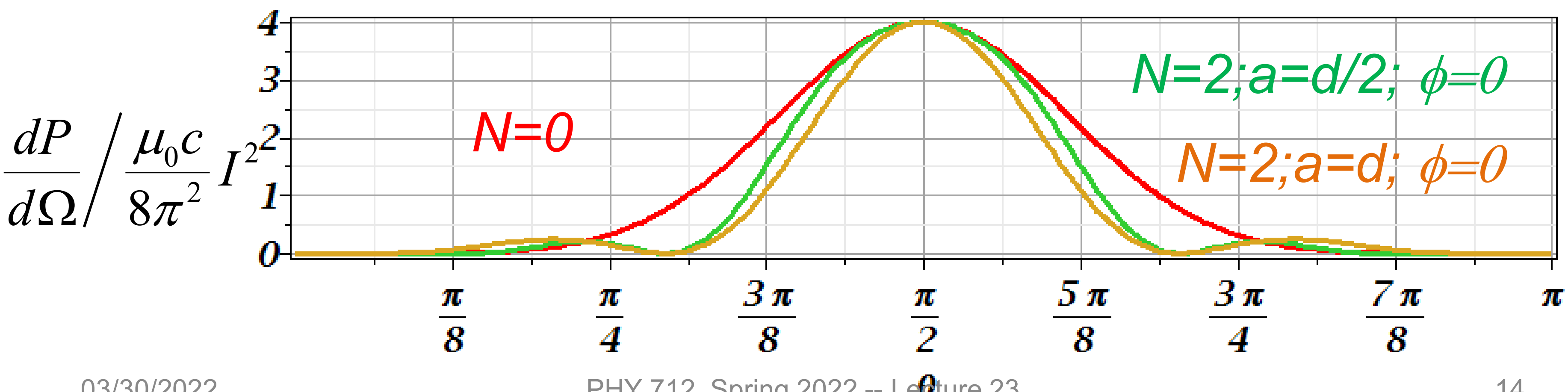
In the radiation zone:

$$\tilde{\mathbf{B}}(\mathbf{r},\omega) = \nabla \times \tilde{\mathbf{A}}(\mathbf{r},\omega) \approx ik\hat{\mathbf{r}} \times \tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) \approx -ikc\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \tilde{\mathbf{A}}(\mathbf{r},\omega)\right)$$

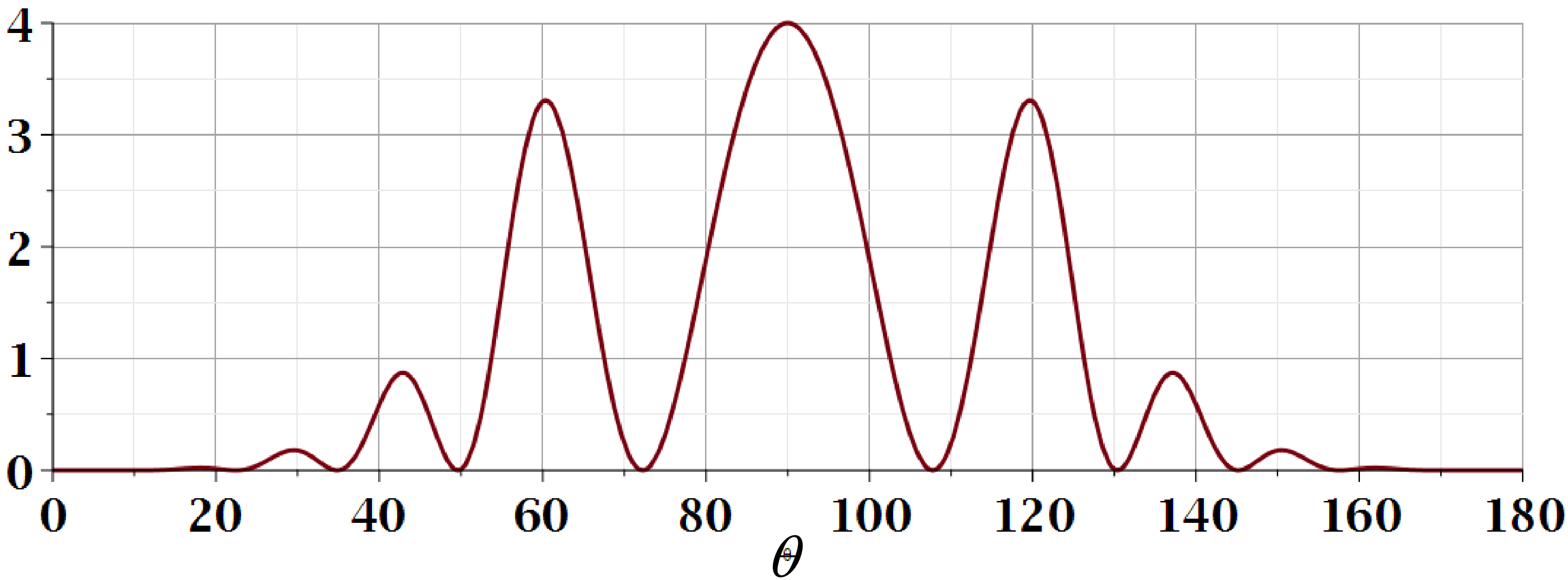
$$\frac{dP}{d\Omega} = \frac{1}{2\mu_0} r^2 \hat{\mathbf{r}} \cdot \Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega) \times \tilde{\mathbf{B}}^*(\mathbf{r},\omega)\right) = \frac{k^2 c r^2}{2\mu_0}\left(\left|\tilde{\mathbf{A}}(\mathbf{r},\omega)\right|^2 - \left|\hat{\mathbf{r}} \cdot \tilde{\mathbf{A}}(\mathbf{r},\omega)\right|^2\right)$$

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta}\right]^2 \left[\frac{\sin\left(\frac{1}{2}ka(2N+1)\sin\theta\cos\phi\right)}{\sin\left(\frac{1}{2}ka\sin\theta\cos\phi\right)}\right]^2$$



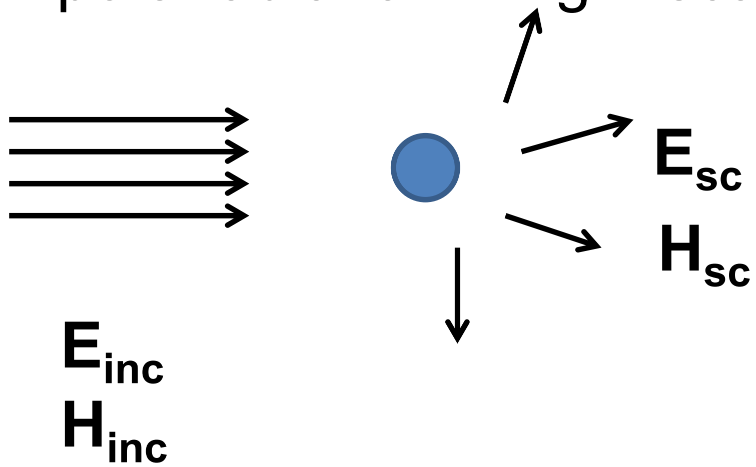
$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta} \right]^2 \left[\frac{\sin\left(\frac{1}{2}ka(2N+1)\sin\theta\cos\varphi\right)}{\sin\left(\frac{1}{2}ka\sin\theta\cos\varphi\right)} \right]^2$$

Example for $\phi = 0, N = 10, \quad kd = \pi = 2ka$



Additional amplitude patterns can be obtained by controlling relative phases of antennas.

Dipole radiation in light scattering by small (dielectric) particles

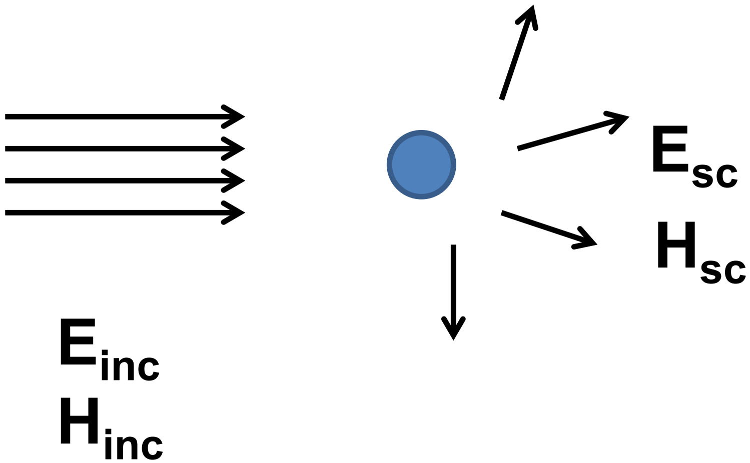


$$\mathbf{E}_{\text{inc}} = \hat{\boldsymbol{\varepsilon}}_0 E_0 e^{ik\hat{\mathbf{k}}_0\cdot\mathbf{r}} \qquad \mathbf{H}_{\text{inc}} = \frac{1}{\mu_0 c}\hat{\mathbf{k}}_0 \times \mathbf{E}_{\text{inc}}$$

In electric dipole approximation :

$$\mathbf{E}_{\text{sc}} = \frac{1}{4\pi\varepsilon_0}k^2\frac{e^{ikr}}{r}\left(\left(\hat{\mathbf{r}}\times\mathbf{p}\right)\times\hat{\mathbf{r}}\right) \qquad \mathbf{H}_{\text{sc}} = \frac{1}{\mu_0 c}\hat{\mathbf{r}}\times\mathbf{E}_{\text{sc}}$$

Dipole radiation in light scattering by small (dielectric) particles



$$\mathbf{E}_{\text{inc}} = \hat{\boldsymbol{\varepsilon}}_0 E_0 e^{ik\hat{\mathbf{k}}_0\cdot\mathbf{r}}$$

$$\mathbf{H}_{\text{inc}} = \frac{1}{\mu_0 c}\hat{\mathbf{k}}_0 \times \mathbf{E}_{\text{inc}}$$

In electric dipole approximation:

$$\mathbf{E}_{\text{sc}} = \frac{1}{4\pi\varepsilon_0} k^2 \frac{e^{ikr}}{r}\left(\left(\hat{\mathbf{r}}\times\mathbf{p}\right)\times\hat{\mathbf{r}}\right)$$

$$\mathbf{H}_{\text{sc}} = \frac{1}{\mu_0 c}\hat{\mathbf{r}}\times\mathbf{E}_{\text{sc}}$$

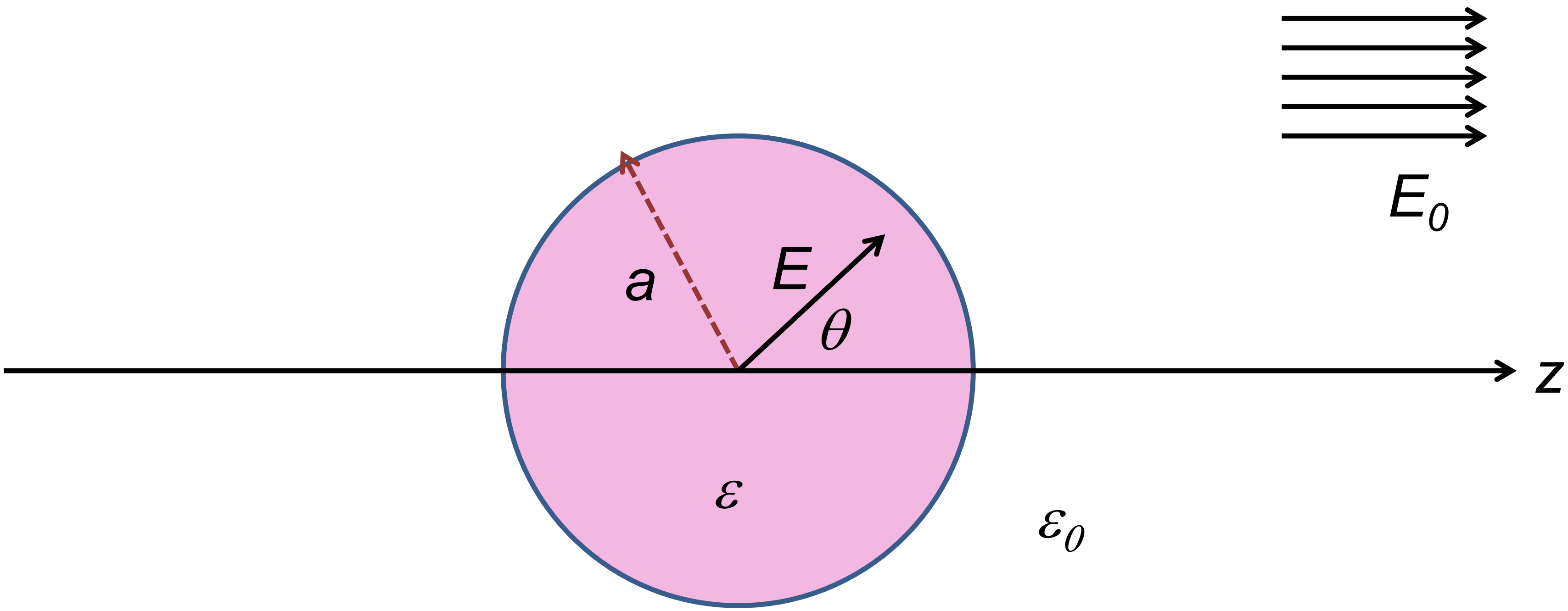
Scattering cross section :

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\boldsymbol{\varepsilon}};\hat{\mathbf{k}}_0,\hat{\boldsymbol{\varepsilon}}_0\right) = \frac{r^2\hat{\mathbf{r}}\cdot\left\langle\mathbf{S}_{sc}\right\rangle_{avg}}{\hat{\mathbf{k}}_0\cdot\left\langle\mathbf{S}_{inc}\right\rangle_{avg}}$$

$$= \frac{r^2\left|\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{E}_{sc}\right|^2}{\left|\hat{\boldsymbol{\varepsilon}}_0\cdot\mathbf{E}_{inc}\right|^2} = \frac{k^4}{\left(4\pi\varepsilon_0 E_0\right)^2}\left|\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{p}\right|^2$$

Recall previous analysis for electrostatic case:

Boundary value problems in the presence of dielectrics – example:



$$\text{At } r = a: \quad \varepsilon \frac{\partial \Phi_<(\mathbf{r})}{\partial r} = \varepsilon_0 \frac{\partial \Phi_>(\mathbf{r})}{\partial r}$$

$$\frac{\partial \Phi_<(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_>(\mathbf{r})}{\partial \theta}$$

Boundary value problems in the presence of dielectrics – example -- continued:

$$\Phi_<(\mathbf{r}) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$\Phi_>(\mathbf{r}) = \sum_{l=0}^{\infty} \left(B_l r^l + \frac{C_l}{r^{l+1}} \right) P_l(\cos\theta)$$

At $r = a$: $\quad \varepsilon \frac{\partial \Phi_<(\mathbf{r})}{\partial r} = \varepsilon_0 \frac{\partial \Phi_>(\mathbf{r})}{\partial r}$

$$\frac{\partial \Phi_<(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_>(\mathbf{r})}{\partial \theta}$$

For $r \to \infty \quad \Phi_>(\mathbf{r}) = -E_0 r \cos\theta$

Solution -- only $l = 1$ contributes

$$B_1 = -E_0$$

$$A_1 = -\left(\frac{3}{2 + \varepsilon/\varepsilon_0} \right) E_0 \qquad C_1 = \left(\frac{\varepsilon/\varepsilon_0 - 1}{2 + \varepsilon/\varepsilon_0} \right) a^3 E_0$$

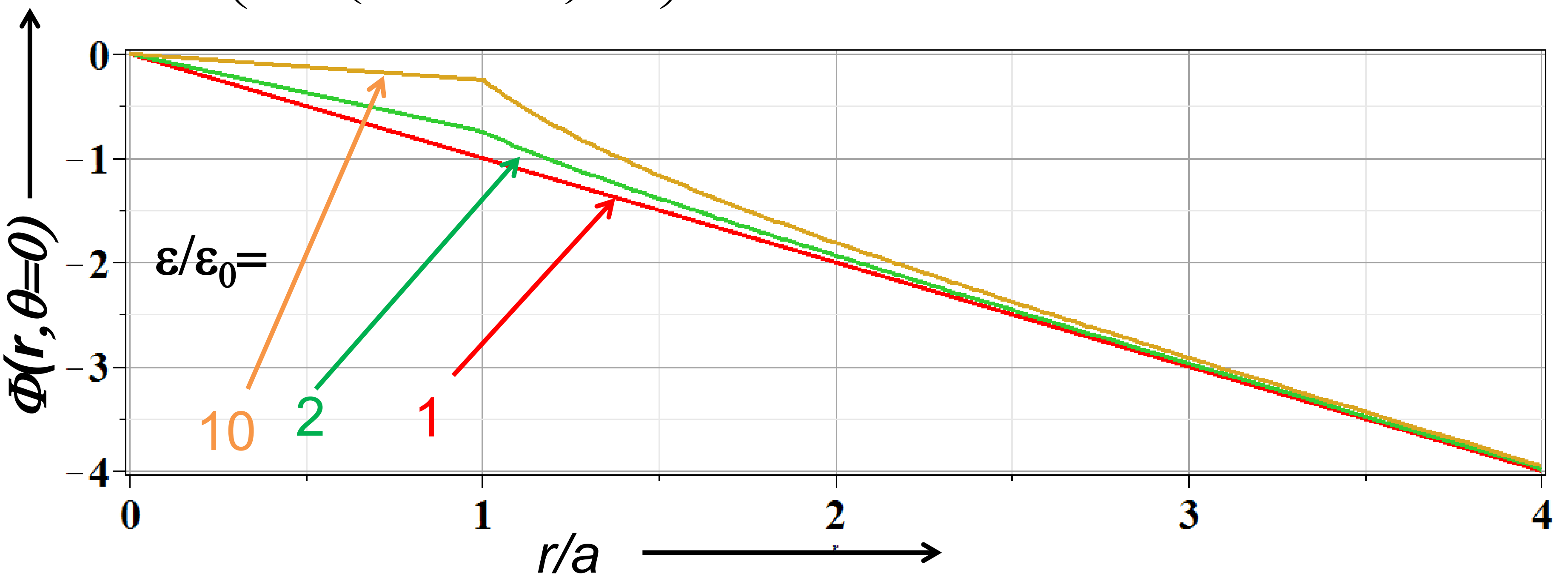
Boundary value problems in the presence of dielectrics – example -- continued:

$$\Phi_<(\mathbf{r}) = -\left(\frac{3}{2+\varepsilon/\varepsilon_0}\right) E_0\, r\cos\theta$$

$$\Phi_>(\mathbf{r}) = -\left(r - \left(\frac{\varepsilon/\varepsilon_0 - 1}{2+\varepsilon/\varepsilon_0}\right)\frac{a^3}{r^2}\right) E_0 \cos\theta$$

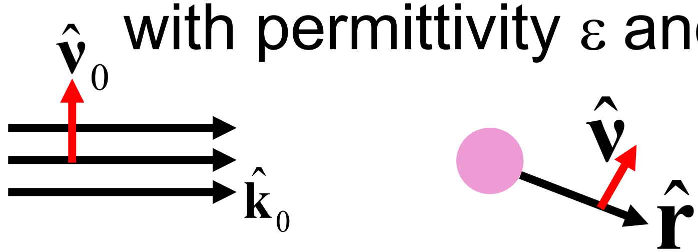
Induced dipole moment:

$$\mathbf{p} = 4\pi a^3 \varepsilon_0 \left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right)\mathbf{E}_0$$



Estimation of scattering dipole moment:

Suppose the scattering particle is a dielectric sphere with permittivity ε and radius *a*:



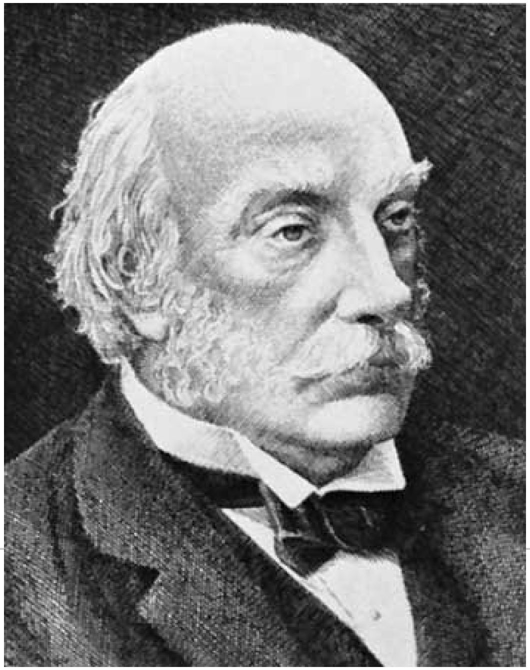
$$\mathbf{p} = 4\pi a^3 \varepsilon_0 \left(\frac{\varepsilon / \varepsilon_0 - 1}{\varepsilon / \varepsilon_0 + 2} \right) \mathbf{E}_{inc} \qquad \mathbf{E}_{\text{inc}} = \hat{\mathbf{v}}_0 E_0 e^{ik\hat{\mathbf{k}}_0 \cdot \mathbf{r}}$$

Scattering cross section:

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}}, \hat{\mathbf{v}}; \hat{\mathbf{k}}_0, \hat{\mathbf{v}}_0\right) = \frac{r^2 \left|\hat{\mathbf{v}} \cdot \mathbf{E}_{sc}\right|^2}{\left|\hat{\mathbf{v}}_0 \cdot \mathbf{E}_{inc}\right|^2} = \frac{k^4}{\left(4\pi\varepsilon_0 E_0\right)^2} \left|\hat{\mathbf{v}} \cdot \mathbf{p}\right|^2$$

$$= k^4 a^6 \left| \frac{\varepsilon / \varepsilon_0 - 1}{\varepsilon / \varepsilon_0 + 2} \right|^2 \left|\hat{\mathbf{v}} \cdot \hat{\mathbf{v}}_0\right|^2$$

https://www.britannica.com/biography/John-William-Strutt-3rd-Baron-Rayleigh



WRITTEN BY: R. Bruce Lindsay

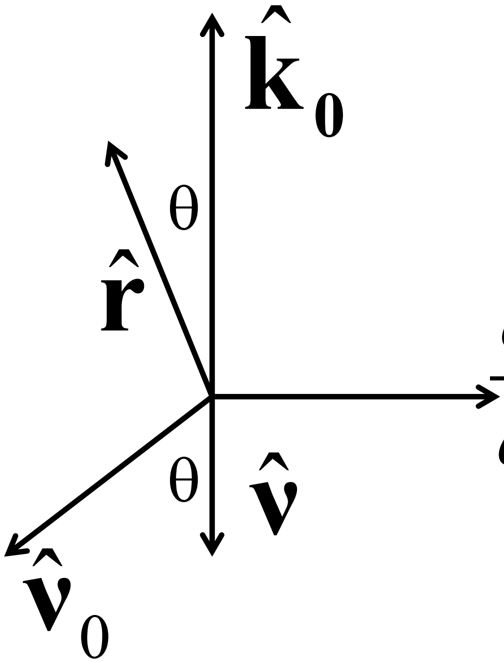
See Article History

Alternative Titles: John William Strutt, 3rd Baron Rayleigh of Terling Place

Lord Rayleigh, in full **John William Strutt, 3rd Baron Rayleigh of Terling Place**, (born November 12, 1842, Langford Grove, Maldon, Essex, England—died June 30, 1919, Terling Place, Witham, Essex), English physical scientist who made fundamental discoveries in the fields of acoustics and optics that are basic to the theory of wave propagation in fluids. He received the Nobel Prize for Physics in 1904 for his successful isolation of argon, an inert atmospheric gas.

Scattering by dielectric sphere with permittivity ε and radius *a*:

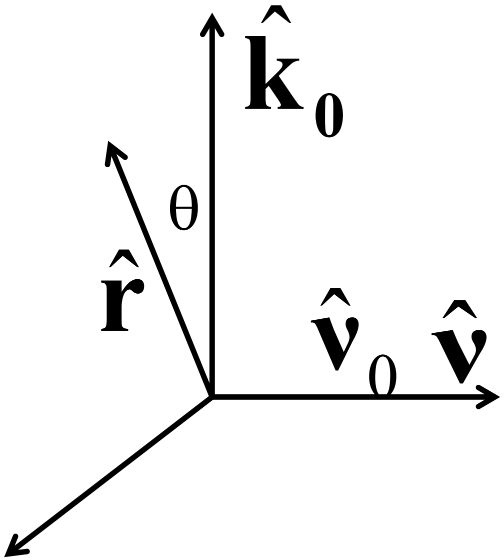
For $\mathbf{E}_{\mathbf{inc}}$ polarized in scattering plane:



$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\mathbf{v}};\hat{\mathbf{k}}_0,\hat{\mathbf{v}}_0\right) == k^4 a^6 \left|\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right|^2 \left|\hat{\mathbf{v}}\cdot\hat{\mathbf{v}}_0\right|^2$$

$$= k^4 a^6 \left|\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right|^2 \cos^2\theta$$

Scattering by dielectric sphere with permittivity ε and radius *a*:



For $\mathbf{E}_{\text{inc}}$ polarized perpendicular to scattering plane:

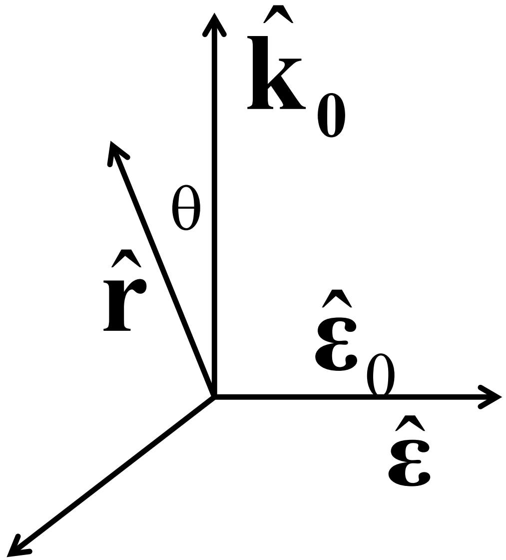
$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\mathbf{v}};\hat{\mathbf{k}}_{\mathbf{0}},\hat{\mathbf{v}}_0\right)=k^4a^6\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2\left|\hat{\mathbf{v}}\cdot\hat{\mathbf{v}}_0\right|^2$$

$$=k^4a^6\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2$$

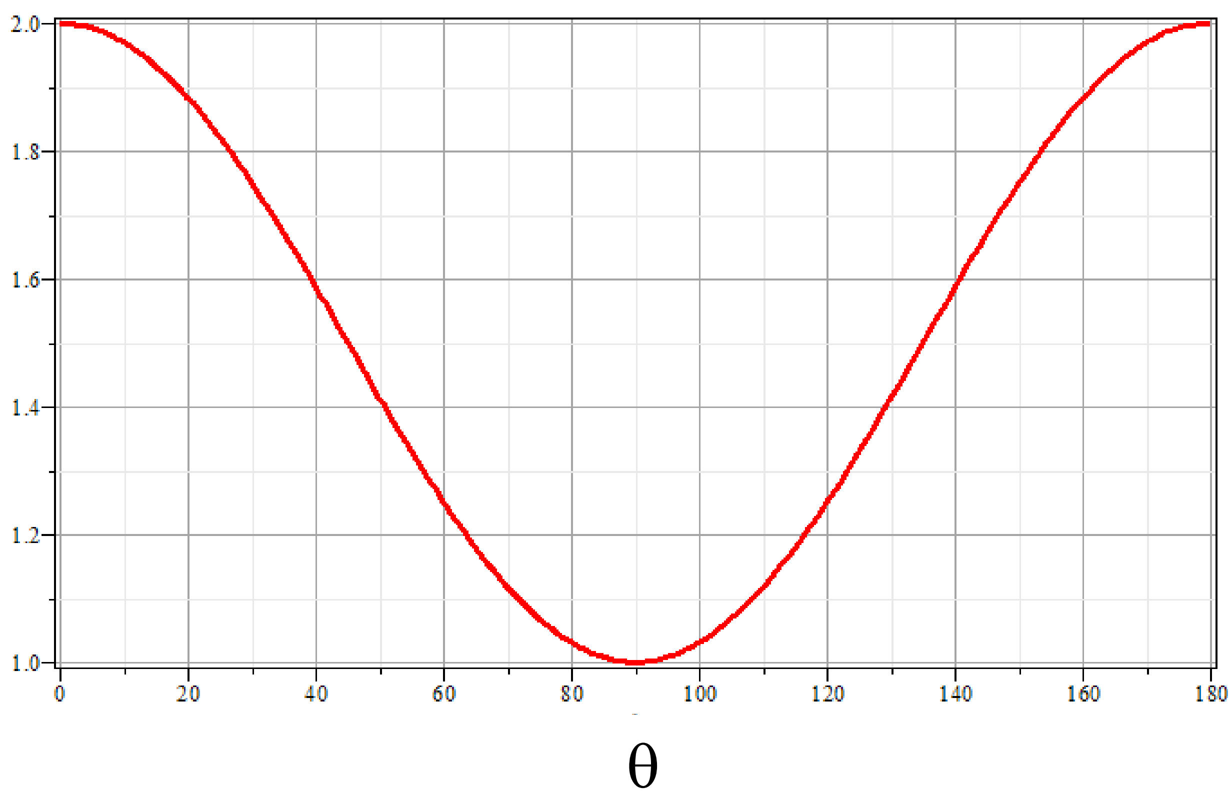
Assuming both incident polarizations are equally likely, average cross section is given by:

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\mathbf{v}};\hat{\mathbf{k}}_{\mathbf{0}},\hat{\mathbf{v}}_0\right)=\frac{k^4a^6}{2}\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2\left(\cos^2\theta+1\right)$$

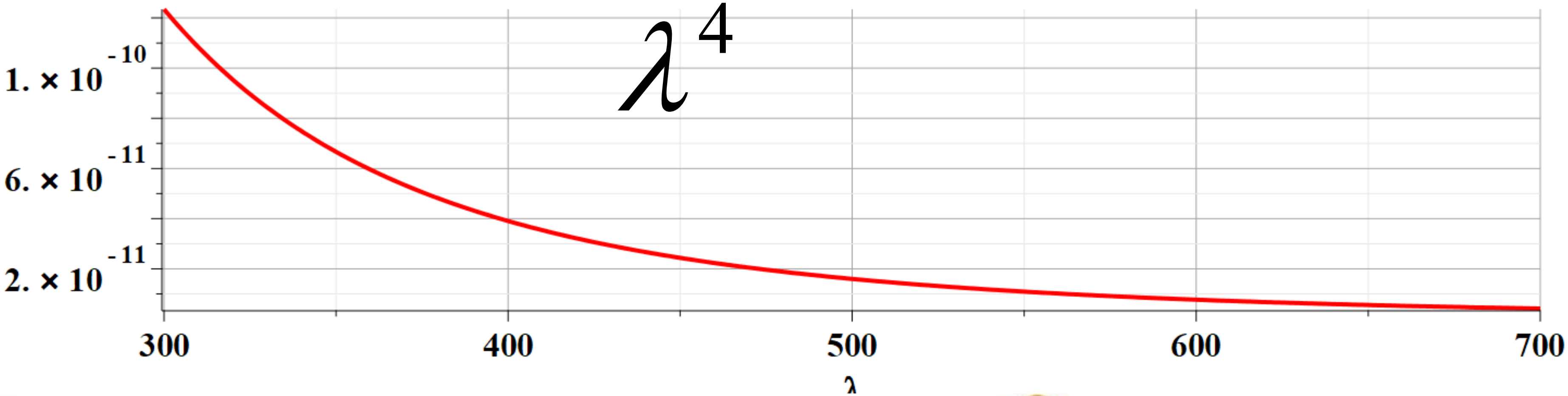
Scattering by dielectric sphere with permittivity ε and radius *a*:

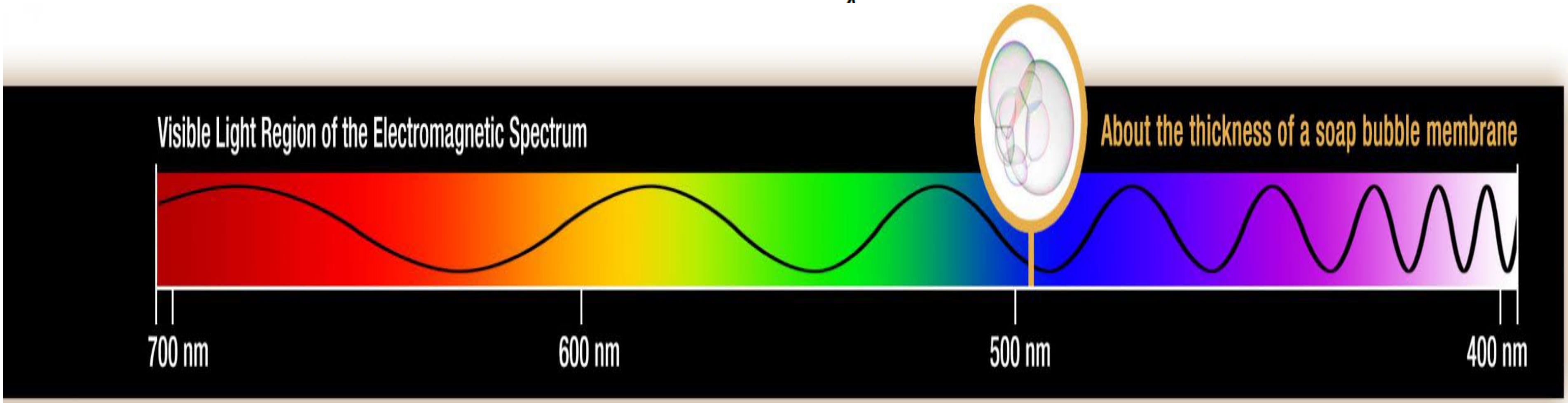


$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\boldsymbol{\varepsilon}};\hat{\mathbf{k}}_0,\hat{\boldsymbol{\varepsilon}}_0\right) = \frac{k^4a^6}{2}\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2\left(\cos^2\theta+1\right)$$



$$\frac{1}{\lambda^4}$$





Brief introduction to multipole expansion of electromagnetic fields (Chap. 9.7)

Sourceless Maxwell's equations

in terms of $\mathbf{E}$ and $\mathbf{H}$ fields with time dependence $e^{-i\omega t}$:

$$\nabla \times \mathbf{E} = ikZ_0\mathbf{H} \qquad \nabla \times \mathbf{H} = -ik\mathbf{E} / Z_0$$

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{H} = 0$$

where $k \equiv \omega / c$ and $Z_0 \equiv \sqrt{\mu_0 / \epsilon_0}$

Decoupled equations:

$$\left(\nabla^2 + k^2\right)\mathbf{E} = 0 \qquad \left(\nabla^2 + k^2\right)\mathbf{H} = 0$$

$$\mathbf{H} = -\frac{i}{kZ_0}\nabla \times \mathbf{E} \qquad \mathbf{E} = \frac{iZ_0}{k}\nabla \times \mathbf{H}$$

Multipole expansion of electromagnetic fields -- continued

Note that:

$$\left(\nabla^2 + k^2\right)\left(\mathbf{r}\cdot\mathbf{E}\right) = 0 \qquad \left(\nabla^2 + k^2\right)\left(\mathbf{r}\cdot\mathbf{H}\right) = 0$$

Convenient operators for angular momentum analysis

Define: $\mathbf{L} \equiv \frac{1}{i}\left(\mathbf{r}\times\nabla\right)$

Note that $\mathbf{r} \cdot \mathbf{L} = 0$

$$\nabla^2 = \frac{1}{r}\frac{\partial^2 r}{\partial r^2} - \frac{L^2}{r^2}$$

Eigenfunctions:

$$L^2 Y_{lm}(\theta,\phi) = -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]Y_{lm}(\theta,\phi) = l(l+1)Y_{lm}(\theta,\phi)$$

Multipole expansion of electromagnetic fields -- continued

Magnetic multipole field:

$$\mathbf{r} \cdot \mathbf{H}_{lm}^{M} \equiv \frac{l(l+1)}{k} g_l(kr) Y_{lm}(\theta,\phi)$$

$$\mathbf{r} \cdot \mathbf{E}_{lm}^{M} = 0$$

$$\mathbf{L} \cdot \mathbf{E}_{lm}^{M} = l(l+1) Z_0 g_l(kr) Y_{lm}(\theta,\phi)$$

spherical Bessel function

Electric multipole field:

$$\mathbf{r} \cdot \mathbf{E}_{lm}^{E} \equiv -Z_0 \frac{l(l+1)}{k} f_l(kr) Y_{lm}(\theta,\phi)$$

$$\mathbf{r} \cdot \mathbf{H}_{lm}^{E} = 0$$

$$\mathbf{L} \cdot \mathbf{H}_{lm}^{E} = l(l+1) f_l(kr) Y_{lm}(\theta,\phi)$$

spherical Bessel function

Multipole expansion of electromagnetic fields -- continued

Vector spherical harmonics: (for $l > 0$)

$$\mathbf{X}_{lm}(\theta,\phi) = \frac{1}{\sqrt{l(l+1)}}\mathbf{L}Y_{lm}(\theta,\phi)$$

Orthogonality conditions:

$$\int d\Omega\ \mathbf{X}_{l'm'}{}^{*}(\theta,\phi)\cdot\mathbf{X}_{lm}(\theta,\phi) = \delta_{ll'}\delta_{mm'}$$

$$\int d\Omega\ \mathbf{X}_{l'm'}{}^{*}(\theta,\phi)\cdot\left(\mathbf{r}\times\mathbf{X}_{lm}(\theta,\phi)\right) = 0$$

General expansion of fields:

$$\mathbf{H} = \sum_{lm}\left[a_{lm}^{E} f_l(kr)\mathbf{X}_{lm}(\theta,\phi) - \frac{i}{k}a_{lm}^{M}\nabla\times\left(g_l(kr)\mathbf{X}_{lm}(\theta,\phi)\right)\right]$$

$$\mathbf{E} = \sum_{lm}\left[\frac{i}{k}a_{lm}^{E}\nabla\times\left(f_l(kr)\mathbf{X}_{lm}(\theta,\phi)\right) + a_{lm}^{M} g_l(kr)\mathbf{X}_{lm}(\theta,\phi)\right]$$

Multipole expansion of electromagnetic fields -- continued

Time averaged power distribution of radiation far from source:

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}\left|\sum_{lm}(-i)^{l+1}\left[a_{lm}^E \mathbf{X}_{lm}(\theta,\phi)\times\hat{\mathbf{r}} + a_{lm}^M \mathbf{X}_{lm}(\theta,\phi)\right]\right|^2$$

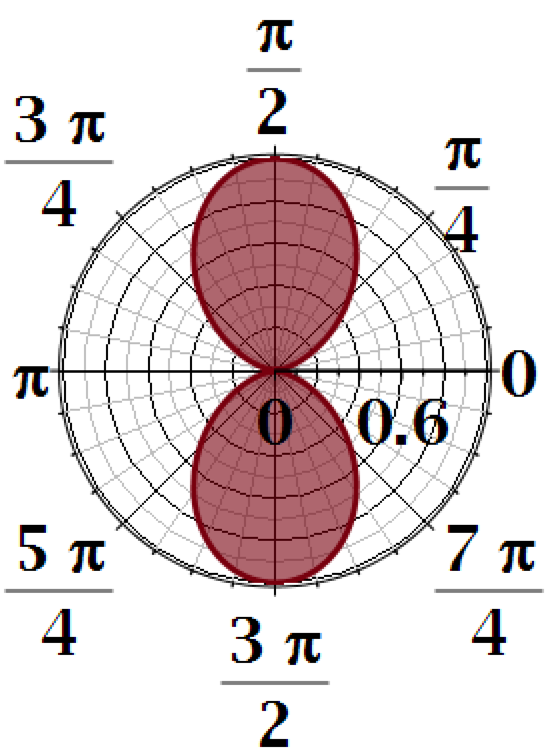
For a pure multipole radiation with either a_{lm}^E or a_{lm}^M :

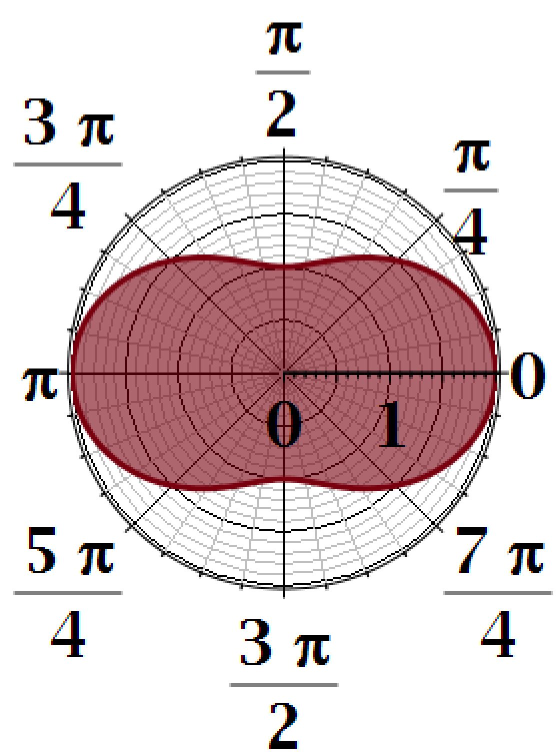
$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}\left|a_{lm}\right|^2 \left|\mathbf{X}_{lm}(\theta,\phi)\right|^2$$

$$\left|\mathbf{X}_{lm}(\theta,\phi)\right|^2 = \frac{1}{2l(l+1)}\left(2m^2\left|Y_{lm}\right|^2 + (l+m)(l-m+1)\left|Y_{l(m-1)}\right|^2 + (l-m)(l+m+1)\left|Y_{l(m+1)}\right|^2\right)$$

For example: $l=1$

$$\left|\mathbf{X}_{10}(\theta,\phi)\right|^2=\frac{3}{8\pi}\sin^2\theta \qquad \left|\mathbf{X}_{11}(\theta,\phi)\right|^2=\left|\mathbf{X}_{1-1}(\theta,\phi)\right|^2=\frac{3}{16\pi}\left(1+\cos^2\theta\right)$$





For example: $l=2$

$$|\mathbf{X}_{20}(\theta,\phi)|^2 = \frac{15}{8\pi}\sin^2\theta\,\cos^2\theta \qquad |\mathbf{X}_{21}(\theta,\phi)|^2 = \frac{5}{16\pi}\left(1-3\cos^2\theta+4\cos^4\theta\right) \qquad |\mathbf{X}_{22}(\theta,\phi)|^2 = \frac{5}{16\pi}\left(1-\cos^4\theta\right)$$