



PHY 712 Electrodynamics

11-11:50 AM MWF Online

Discussion for Lecture 24:

Start reading Chap. 11

- A. Equations in cgs (Gaussian) units**
- B. Special theory of relativity**
- C. Lorentz transformation relations**

	Mon: 03/21/2022		Project presentations I		
	Wed: 03/23/2022		Project presentations II		
21	Fri: 03/25/2022	Chap. 9	Radiation from localized oscillating sources	#18	03/30/2022
22	Mon: 03/28/2022	Chap. 9	Radiation from oscillating sources		
23	Wed: 03/30/2022	Chap. 9 & 10	Radiation and scattering	#19	04/01/2022
24	Fri: 04/01/2022	Chap. 11	Special Theory of Relativity	#20	04/01/2022
25	Mon: 04/04/2022	Chap. 11	Special Theory of Relativity		

PHY 712 -- Assignment #20

April 1, 2022

Start reading Chapters 11 in **Jackson** .

1. In class, we examined measurements of quantities that could be measured in two different reference frames which were related by the Lorentz transformation matrix. In particular, we are interested in the velocity components such as u_x and u'_x . For the geometry used in class, work out the algebraic relationships that show that the 4-components of the vector $(\gamma_u c, \gamma_u u_x, \gamma_u u_y, \gamma_u u_z)$ are related by a Lorentz transformation to the corresponding 4-component vector measured in the moving frame -- $(\gamma_{u'} c, \gamma_{u'} u'_x, \gamma_{u'} u'_y, \gamma_{u'} u'_z)$.

Units - SI vs Gaussian

Below is a table comparing SI and Gaussian unit systems. The fundamental units for each system are so labeled and are used to define the derived units.

Variable	SI		Gaussian		SI/Gaussian
	Unit	Relation	Unit	Relation	
length	m	fundamental	cm	fundamental	100
mass	kg	fundamental	gm	fundamental	1000
time	s	fundamental	s	fundamental	1
force	N	$kg \cdot m / s^2$	$dyne$	$gm \cdot cm / s^2$	10^5
current	A	fundamental	$statampere$	$statcoulomb/s$	$\frac{1}{10c}$
charge	C	$A \cdot s$	$statcoulomb$	$\sqrt{dyne \cdot cm^2}$	$\frac{1}{10c}$

Basic equations of electrodynamics

CGS (Gaussian)	SI
$\nabla \cdot \mathbf{D} = 4\pi\rho$	$\nabla \cdot \mathbf{D} = \rho$
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
$\nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t}$	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$
$\mathbf{F} = q(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})$	$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$
$u = \frac{1}{8\pi} (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$	$u = \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$
$\mathbf{S} = \frac{c}{4\pi} (\mathbf{E} \times \mathbf{H})$	$\mathbf{S} = (\mathbf{E} \times \mathbf{H})$



More relationships

CGS (Gaussian)

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = \epsilon\mathbf{E}$$

$$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} = \frac{1}{\mu}\mathbf{B}$$

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

MKS (SI)

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon\mathbf{E}$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} = \frac{1}{\mu}\mathbf{B}$$

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

ϵ



ϵ / ϵ_0

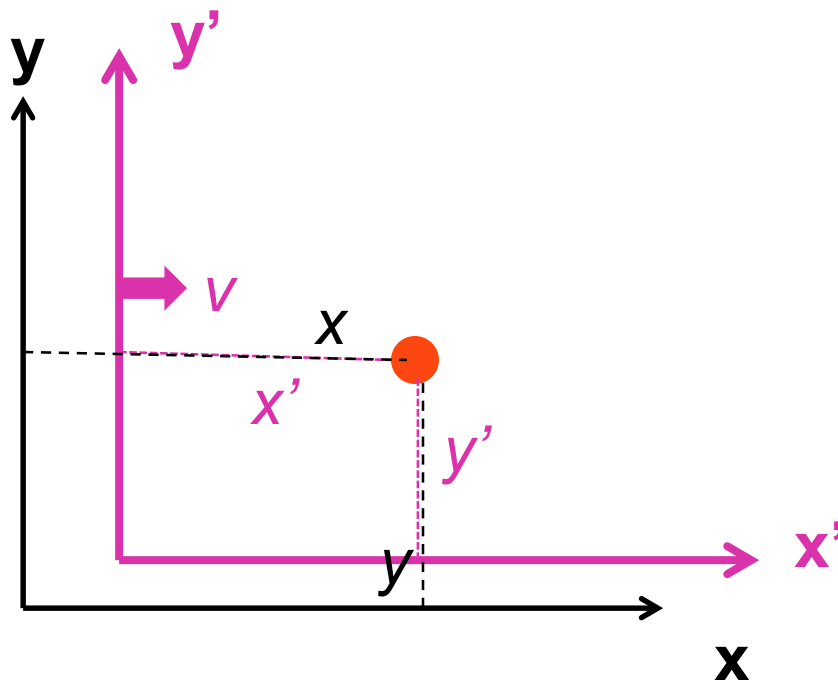
μ



μ / μ_0

Notions of special relativity

- The basic laws of physics are the same in all frames of reference (at rest or moving at constant velocity).
- The speed of light in vacuum c is the same in all frames of reference.

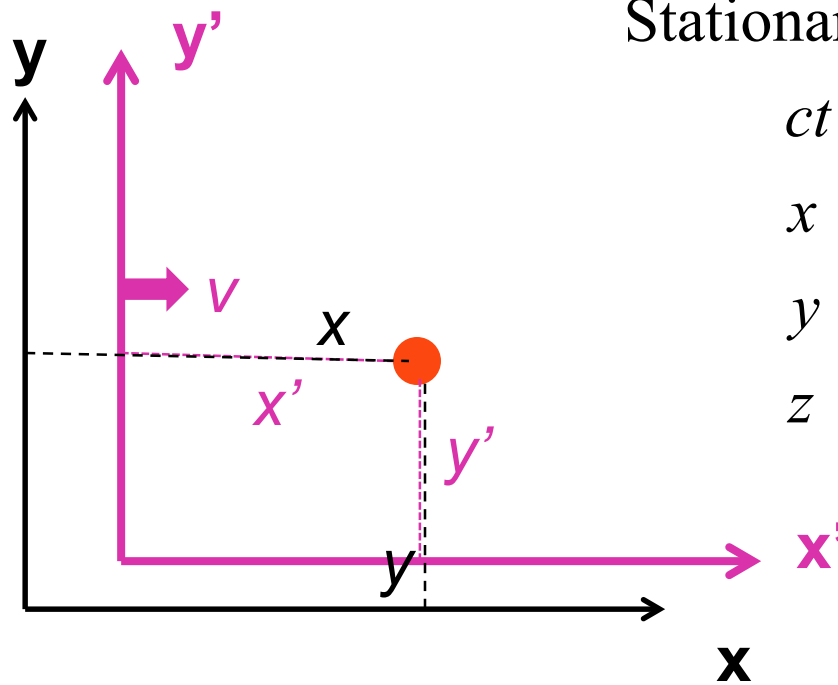


Lorentz transformations

Convenient notation :

$$\beta \equiv \frac{v}{c}$$

$$\gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}$$



Stationary frame

Moving frame

$$\begin{aligned} ct &= \gamma(ct' + \beta x') \\ x &= \gamma(x' + \beta ct') \\ y &= y' \\ z &= z' \end{aligned}$$

Lorentz transformations -- continued

$$\beta \equiv \frac{v}{c} \quad \gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}$$

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\mathcal{L} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{L}^1 = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \mathcal{L} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \mathcal{L}^1 \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Notice:

$$c^2 t^2 - x^2 - y^2 - z^2 = c^2 t'^2 - x'^2 - y'^2 - z'^2$$

Examples of other 4-vectors applicable to the Lorentz transformation:

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\beta \equiv \frac{v}{c} \quad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

$$\mathcal{L} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{L}^{-1} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix} = \mathcal{L} \begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix}$$

$$\begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix}$$

Note: $\omega t - \mathbf{k} \cdot \mathbf{r} = \omega' t' - \mathbf{k}' \cdot \mathbf{r}'$
In free space:

$$\left(\frac{\omega}{c}\right)^2 - k^2 = \left(\frac{\omega'}{c}\right)^2 - k'^2 = 0$$

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \mathcal{L} \begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix}$$

$$\begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix}$$

Note: $E^2 - p^2 c^2 = E'^2 - p'^2 c^2$



The Doppler Effect

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\mathcal{L} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{L}^{-1} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix} = \mathcal{L} \begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix}$$

$$\begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix}$$

Note: $\omega t - \mathbf{k} \cdot \mathbf{r} = \omega' t' - \mathbf{k}' \cdot \mathbf{r}'$

$$\omega'/c = \gamma(\omega/c - \beta k_x)$$

$$k'_x = \gamma(k_x - \beta\omega/c)$$

$$k'_y = k_y$$

$$k'_z = k_z$$

The Doppler Effect -- continued

$$\omega' / c = \gamma(\omega / c - \beta k_x)$$

$$k'_x = \gamma(k_x - \beta \omega / c)$$

$$k'_y = k_y$$

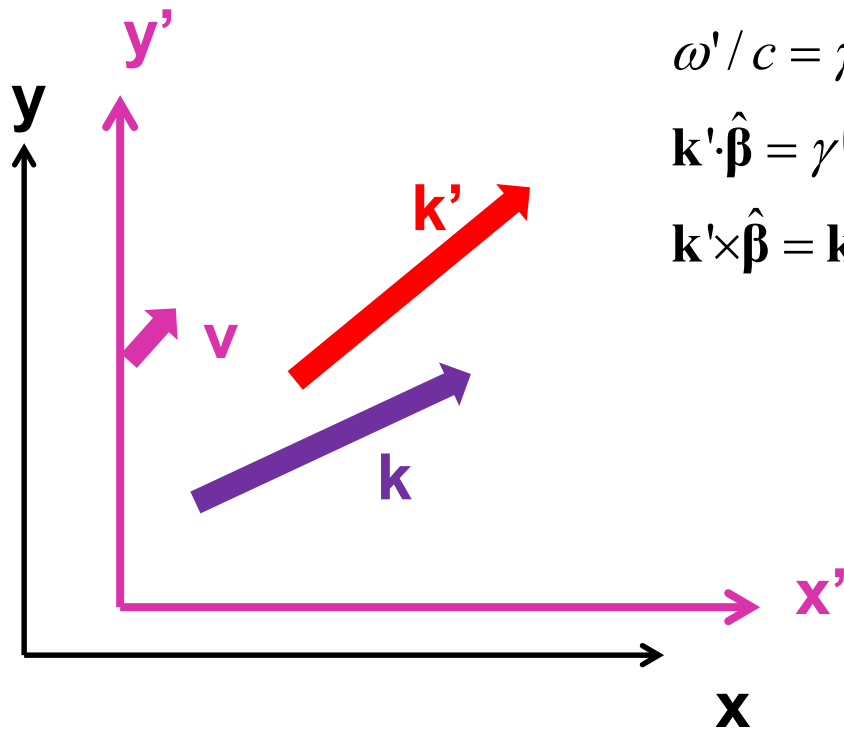
$$k'_z = k_z$$

More generally:

$$\omega' / c = \gamma(\omega / c - \boldsymbol{\beta} \cdot \mathbf{k}) \equiv \gamma(\omega / c - \beta k \cos \theta)$$

$$\mathbf{k}' \cdot \hat{\boldsymbol{\beta}} = \gamma(\mathbf{k} \cdot \hat{\boldsymbol{\beta}} - \beta \omega / c) \equiv k' \cos \theta' = \gamma(k \cos \theta - \beta \omega / c)$$

$$\mathbf{k}' \times \hat{\boldsymbol{\beta}} = \mathbf{k} \times \hat{\boldsymbol{\beta}}$$



For $\theta = 0$: ($k = \omega/c$)

$$\omega' = \omega \gamma (1 - \beta) \quad \Rightarrow \quad \omega' = \omega \sqrt{\frac{1 - \beta}{1 + \beta}}$$

For $\theta \neq 0$: ($k = \omega/c$)

$$\tan \theta' = \frac{\sin \theta}{\gamma(\cos \theta - \beta)}$$



Electromagnetic Doppler Effect ($\theta=0$)

$$\omega' = \omega \sqrt{\frac{1 - \beta}{1 + \beta}}$$

$$\beta = \frac{v_{\text{source}} - v_{\text{detector}}}{c \left(1 - \frac{v_{\text{source}} v_{\text{detector}}}{c^2} \right)}$$

(details concerning velocities in the following slides.)

Sound Doppler Effect ($\theta=0$)

$$\omega' = \omega \left(\frac{1 \pm v_{\text{detector}} / c_s}{1 \mp v_{\text{source}} / c_s} \right)$$



Lorentz transformation of the velocity

Stationary frame

Moving frame

$$ct = \gamma(ct' + \beta x')$$

$$x = \gamma(x' + \beta ct')$$

$$y = y'$$

$$z = z'$$

For an infinitesimal increment:

Stationary frame

Moving frame

$$cdt = \gamma(cdt' + \beta dx')$$

$$dx = \gamma(dx' + \beta cdt')$$

$$dy = dy'$$

$$dz = dz'$$

Lorentz transformation of the velocity -- continued

Stationary frame

Moving frame

$$cdt = \gamma(cdt' + \beta dx')$$

$$dx = \gamma(dx' + \beta cdt')$$

$$dy = dy'$$

$$dz = dz'$$

Define :

$$u_x \equiv \frac{dx}{dt} \quad u_y \equiv \frac{dy}{dt} \quad u_z \equiv \frac{dz}{dt}$$
$$u'_x \equiv \frac{dx'}{dt'} \quad u'_y \equiv \frac{dy'}{dt'} \quad u'_z \equiv \frac{dz'}{dt'}$$

$$\frac{dx}{dt} = \frac{\gamma(dx' + \beta cdt')}{\gamma(dt' + \beta dx'/c)} = \frac{u'_x + v}{1 + vu'_x/c^2} = u_x$$

$$\frac{dy}{dt} = \frac{dy'}{\gamma(dt' + \beta dx'/c)} = \frac{u'_y}{\gamma(1 + vu'_x/c^2)} = u_y$$

Summary of velocity relationships

$$u_x = \frac{u'_x + v}{1 + vu'_x / c^2}$$

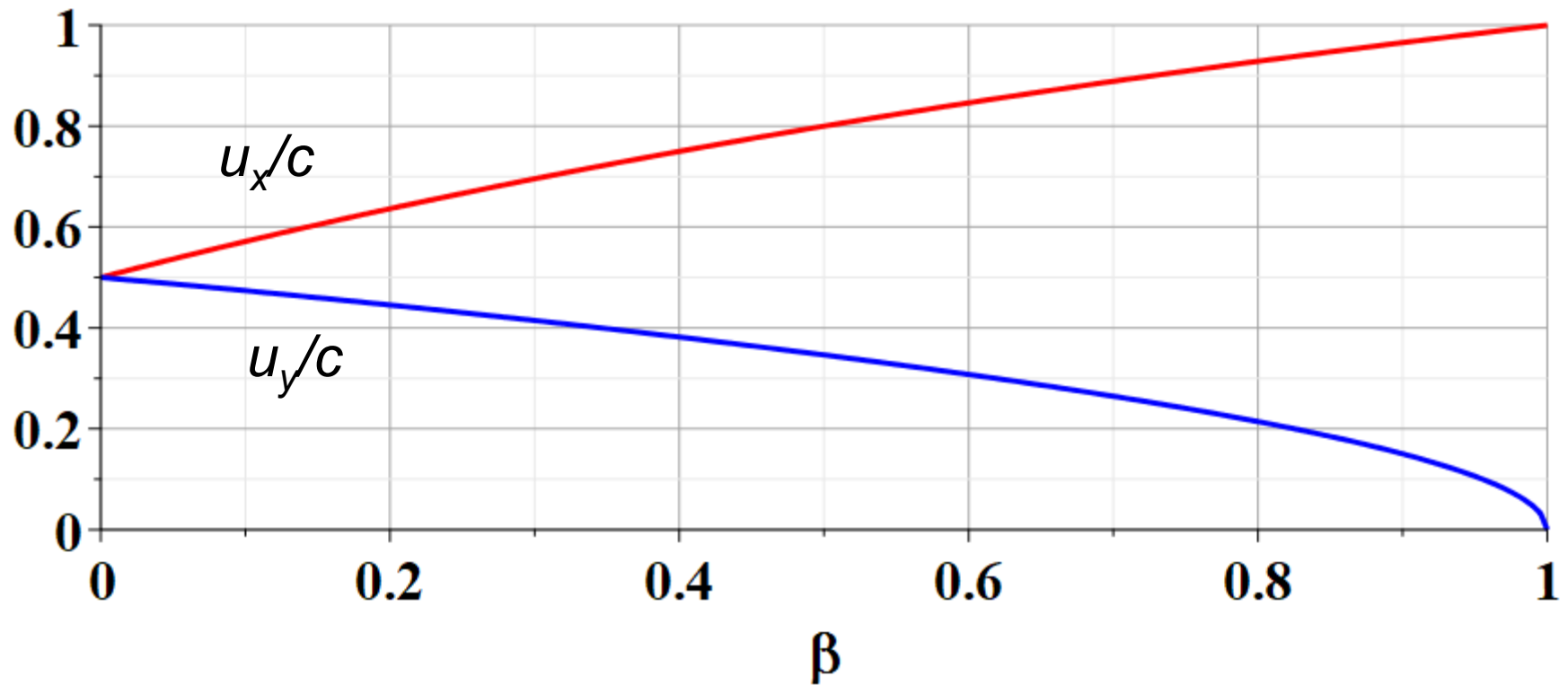
$$u_y = \frac{u'_y}{\gamma \left(1 + vu'_x / c^2\right)} \equiv \frac{u'_y}{\gamma_v \left(1 + vu'_x / c^2\right)}$$

$$u_z = \frac{u'_z}{\gamma \left(1 + vu'_x / c^2\right)} \equiv \frac{u'_z}{\gamma_v \left(1 + vu'_x / c^2\right)}$$



Example of velocity variation with β :

$$(u'_x/c = u'_y/c = 0.5)$$





Extention to tranformation of acceleration

$$\mathbf{a}_{\parallel} = \frac{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}{\left(1 + \frac{\mathbf{v} \cdot \mathbf{u}'}{c^2}\right)^3} \mathbf{a}'_{\parallel}$$

$$\mathbf{a}_{\perp} = \frac{\left(1 - \frac{v^2}{c^2}\right)}{\left(1 + \frac{\mathbf{v} \cdot \mathbf{u}'}{c^2}\right)^3} \left(\mathbf{a}'_{\perp} + \frac{\mathbf{v}}{c^2} \times (\mathbf{a}' \times \mathbf{u}') \right)$$

Comment –

The acceleration equations are obtained by taking the infinitesimal derivative of the velocity relationships and simplifying the expressions. (See Jackson Problem 11.5.)

Velocity transformations continued:

Consider: $u_x = \frac{u'_x + v}{1 + vu'_x / c^2}$ $u_y = \frac{u'_y}{\gamma_v (1 + vu'_x / c^2)}$ $u_z = \frac{u'_z}{\gamma_v (1 + vu'_x / c^2)}$.

Note that $\gamma_u \equiv \frac{1}{\sqrt{1 - (u/c)^2}} = \frac{1 + vu'_x / c^2}{\sqrt{1 - (u'/c)^2} \sqrt{1 - (v/c)^2}} = \gamma_v \gamma_{u'} (1 + vu'_x / c^2)$

$$\Rightarrow \gamma_u c = \gamma_v (\gamma_{u'} c + \beta_v \gamma_{u'} u'_x)$$

$$\Rightarrow \gamma_u u_x = \gamma_v (\gamma_{u'} u'_x + \gamma_{u'} v) = \gamma_v (\gamma_{u'} u'_x + \beta_v \gamma_{u'} c)$$

$$\Rightarrow \gamma_u u_y = \gamma_{u'} u'_y \quad \gamma_u u_z = \gamma_{u'} u'_z$$

Velocity 4-vector:

$$\begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix} = \mathcal{L}_v \begin{pmatrix} \gamma_{u'} c \\ \gamma_{u'} u'_x \\ \gamma_{u'} u'_y \\ \gamma_{u'} u'_z \end{pmatrix}$$



Some details:

$$\gamma_u = \gamma_v \gamma_{u'} \left(1 + v u'_x / c^2\right) \Rightarrow \left(1 - \frac{v^2}{c^2}\right) \left(1 - \frac{u'^2}{c^2}\right) = \left(1 - \frac{u^2}{c^2}\right) \left(1 + \frac{u_x v}{c^2}\right)^2$$

$$\text{where } u_x = \frac{u'_x + v}{1 + v u'_x / c^2} \quad u_y = \frac{u'_y}{\gamma_v (1 + v u'_x / c^2)} \quad u_z = \frac{u'_z}{\gamma_v (1 + v u'_x / c^2)}.$$

$$\left(\frac{u_x^2}{c^2} + \frac{u_y^2}{c^2} + \frac{u_z^2}{c^2}\right) \left(1 + \frac{u'_x v}{c^2}\right)^2 = \left(\frac{u'_x}{c} + \frac{v}{c}\right)^2 + \left(\frac{u'^2_y}{c^2} + \frac{u'^2_z}{c^2}\right) \left(1 - \frac{v^2}{c^2}\right)$$

$$\frac{u^2}{c^2} \left(1 + \frac{u'_x v}{c^2}\right)^2 = \frac{u'^2}{c^2} \left(1 - \frac{v^2}{c^2}\right) + \left(1 + \frac{u'_x v}{c^2}\right)^2 - \left(1 - \frac{v^2}{c^2}\right)$$

$$\Rightarrow \left(1 - \frac{u^2}{c^2}\right) \left(1 + \frac{u'_x v}{c^2}\right)^2 = \left(1 - \frac{u'^2}{c^2}\right) \left(1 - \frac{v^2}{c^2}\right)$$



Significance of 4-velocity vector:

$$\begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix}$$

Introduce the factor mc where m is the “rest” mass of the particle characterized by velocity $\mathbf{u}$:

$$mc \begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix} = \begin{pmatrix} \gamma_u mc^2 \\ \gamma_u mu_x c \\ \gamma_u mu_y c \\ \gamma_u mu_z c \end{pmatrix} = \begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix}$$

Properties of energy-moment 4-vector:

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \mathcal{L} \begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix}$$

$$\begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix}$$

$$\text{Note: } E^2 - p^2 c^2 = E'^2 - p'^2 c^2$$



Properties of Energy-momentum 4-vector -- continued

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \begin{pmatrix} \gamma_u m c^2 \\ \gamma_u m u_x c \\ \gamma_u m u_y c \\ \gamma_u m u_z c \end{pmatrix}$$

Note: $E^2 - p^2 c^2 = \frac{(m c^2)^2}{1 - \beta_u^2} \left(1 - \left(\frac{u_x}{c} \right)^2 - \left(\frac{u_y}{c} \right)^2 - \left(\frac{u_z}{c} \right)^2 \right) = (m c^2)^2 = E'^2 - p'^2 c^2$

Notion of "rest energy": For $\mathbf{p} \equiv 0$, $E = m c^2$

Define kinetic energy: $E_K \equiv E - m c^2 = \sqrt{p^2 c^2 + m^2 c^4} - m c^2$

Non-relativistic limit: If $\frac{p}{m c} \ll 1$, $E_K = m c^2 \left(\sqrt{1 + \left(\frac{p}{m c} \right)^2} - 1 \right)$
 $\approx \frac{p^2}{2m}$

Summary of relativistic energy relationships

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \begin{pmatrix} \gamma_u m c^2 \\ \gamma_u m u_x c \\ \gamma_u m u_y c \\ \gamma_u m u_z c \end{pmatrix}$$

$$E = \sqrt{p^2 c^2 + m^2 c^4} = \gamma_u m c^2$$

$$\text{Check: } \sqrt{p^2 c^2 + m^2 c^4} = m c^2 \sqrt{\gamma_u^2 \beta_u^2 + 1} = \gamma_u m c^2$$

Example: for an electron $m c^2 = 0.5 \text{ MeV}$

for $E = 200 \text{ GeV}$

$$\gamma_u = \frac{E}{m c^2} = 4 \times 10^5$$

$$\beta_u = \sqrt{1 - \frac{1}{\gamma_u^2}} \approx 1 - \frac{1}{2\gamma_u^2} \approx 1 - 3 \times 10^{-12}$$



Special theory of relativity and Maxwell's equations

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

Lorenz gauge condition:

$$\frac{1}{c} \frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{A} = 0$$

Potential equations:

$$\frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = 4\pi\rho$$

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \frac{4\pi}{c} \mathbf{J}$$

Field relations:

$$\mathbf{E} = -\nabla \Phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$



More 4-vectors:

Time and position :

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \Rightarrow x^\alpha$$

Charge and current :

$$\begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix} \Rightarrow J^\alpha$$

Vector and scalar potentials :

$$\begin{pmatrix} \Phi \\ A_x \\ A_y \\ A_z \end{pmatrix} \Rightarrow A^\alpha$$



Lorentz transformations

$$\mathcal{L}_v = \begin{pmatrix} \gamma_v & \gamma_v \beta_v & 0 & 0 \\ \gamma_v \beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Time and space :

$$x^\alpha = \mathcal{L}_v x'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} x'^\beta$$

Charge and current :

$$J^\alpha = \mathcal{L}_v J'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} J'^\beta$$

Vector and scalar potential :

$$A^\alpha = \mathcal{L}_v A'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} A'^\beta$$

Summary of results --

Time and space :

$$x^\alpha = \mathcal{L}_v x'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} x'^\beta$$

Charge and current :

$$J^\alpha = \mathcal{L}_v J'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} J'^\beta$$

Vector and scalar potential : $A^\alpha = \mathcal{L}_v A'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} A'^\beta$

Here, the notation varies among the textbooks.

In general, it is convenient to use the matrix multiplication conventions to multiply our 4×4 matrices and 4 vectors

$$\text{For example: } \mathcal{L}_v^{\alpha\beta} x'^\beta = \sum_{\beta=1}^4 \mathcal{L}_v^{\alpha\beta} x'^\beta = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$