PHY 712 Electrodynamics
12-12:50 PM MWF Olin 103

Notes for Lecture 27:

Start reading Chap. 14 –

Radiation by moving charges

1. **Motion in a line**

2. **Motion in a circle**

3. **Spectral analysis of radiation**

PHY 712 -- Assignment #22

April 8, 2022

Start reading Chap. 14 in **Jackson** .

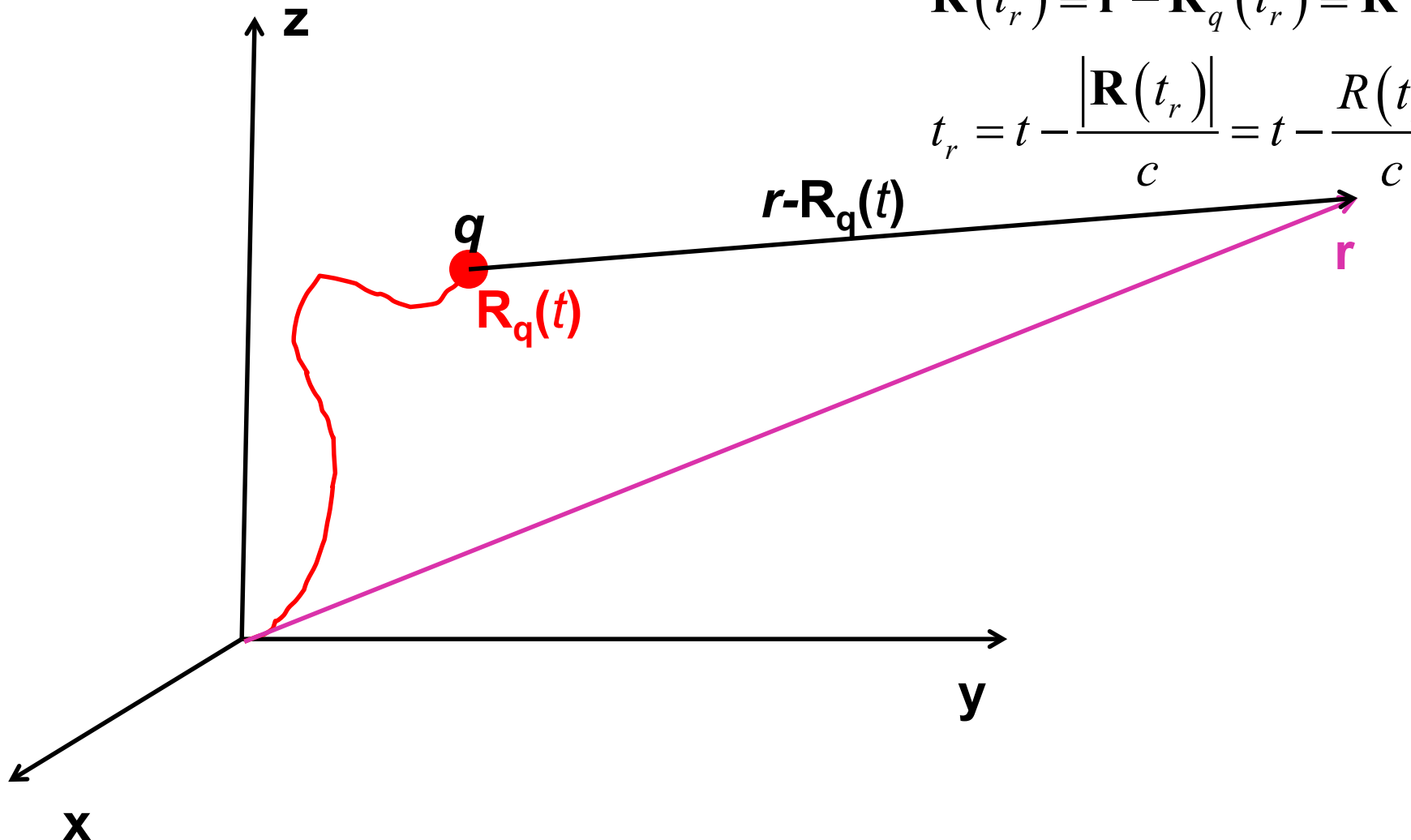1. Consider an electron moving at constant speed $\beta c \approx c$ in a circular trajectory of radius ρ. Its total energy is $E = \gamma m c^2$. Determine the ratio of the energy lost during one full cycle to its total energy. Evaluate the expression for an electron with total energy of 200 GeV in a synchroton of radius $\rho = 10^3$ m. Note if you use, the analysis of **Jackson**, please explain the details.

Radiation from a moving charged particle

Variables (notation):

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$$

$$t_r = t - \frac{|\mathbf{R}(t_r)|}{c} = t - \frac{R(t_r)}{c}$$

r-**R**$_\mathbf{q}$(*t*)

q

R$_\mathbf{q}$(*t*)

z

r

y

x

Liénard-Wiechert fields (cgs Gaussian units):

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R} \times \left\{\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \times \frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right].$$

$$(19)$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{q}{c}\left[\frac{-\mathbf{R} \times \mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\mathbf{R} \times \dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right].$$

$$(20)$$

In this case, the electric and magnetic fields are related according to

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mathbf{R} \times \mathbf{E}(\mathbf{r}, t)}{R}.$$

$$(21)$$

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v} \qquad \mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R} \qquad \dot{\mathbf{v}} \equiv \frac{d^2\mathbf{R}_q(t_r)}{dt_r^{\,2}}$$

Comment --

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R} \times \left\{\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \times \frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right].$$

(19)

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R} \times \mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\mathbf{R} \times \dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right].$$

(20)

In this case, the electric and magnetic fields are related according to

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R} \times \mathbf{E}(\mathbf{r},t)}{R}.$$

(21)

Note that (21) can be demonstrated by evaluating *R x E(r,t)*

Other helpful identities: *ax(bxc)=b(a·c)-c(a·b)*
                               a·(bxc) = b·(cxa) = c·(axb)

Electric field far from source:

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left\{ \mathbf{R} \times \left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c} \right) \times \frac{\dot{\mathbf{v}}}{c^2} \right] \right\}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R} \times \mathbf{E}(\mathbf{r},t)}{R}$$

Note that all of the variables on the right hand side of the equations depend on t_r .

$$\text{Let} \quad \hat{\mathbf{R}} \equiv \frac{\mathbf{R}}{R} \qquad \boldsymbol{\beta} \equiv \frac{\mathbf{v}}{c} \qquad \dot{\boldsymbol{\beta}} \equiv \frac{\dot{\mathbf{v}}}{c}$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{cR\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^3} \left\{ \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right\}$$

$$\mathbf{B}(\mathbf{r},t) = \hat{\mathbf{R}} \times \mathbf{E}(\mathbf{r},t)$$

Poynting vector:

$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}(\mathbf{E} \times \mathbf{B})$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{cR(1-\boldsymbol{\beta} \cdot \hat{\mathbf{R}})^3}\left\{\hat{\mathbf{R}} \times \left[(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t) = \hat{\mathbf{R}} \times \mathbf{E}(\mathbf{r},t) \qquad \mathbf{E} \times (\hat{\mathbf{R}} \times \mathbf{E}(\mathbf{r},t)) = \hat{\mathbf{R}}|\mathbf{E}|^2 - \mathbf{E}(\hat{\mathbf{R}} \cdot \mathbf{E})$$

$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}\hat{\mathbf{R}}|\mathbf{E}(\mathbf{r},t)|^2 = \frac{q^2}{4\pi c R^2}\hat{\mathbf{R}}\frac{\left|\hat{\mathbf{R}} \times \left[(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right]\right|^2}{(1-\boldsymbol{\beta} \cdot \hat{\mathbf{R}})^6}$$

Note: We have used the fact that

$$\hat{\mathbf{R}} \cdot \mathbf{E}(\mathbf{r},t) = 0 \quad \text{which follows from the vector identities.}$$

Power radiated

$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}\hat{\mathbf{R}}\left|\mathbf{E}(\mathbf{r},t)\right|^2 = \frac{q^2}{4\pi c R^2}\hat{\mathbf{R}}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

$$\frac{dP}{d\Omega} = \mathbf{S}\cdot\hat{\mathbf{R}}R^2 = \frac{q^2}{4\pi c}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

In the non-relativistic limit: $\beta \ll 1$

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c}\left|\hat{\mathbf{R}}\times\left[\hat{\mathbf{R}}\times\dot{\boldsymbol{\beta}}\right]\right|^2 = \frac{q^2}{4\pi c^3}\left|\dot{\mathbf{v}}\right|^2\sin^2\Theta$$

Radiation from a moving charged particle
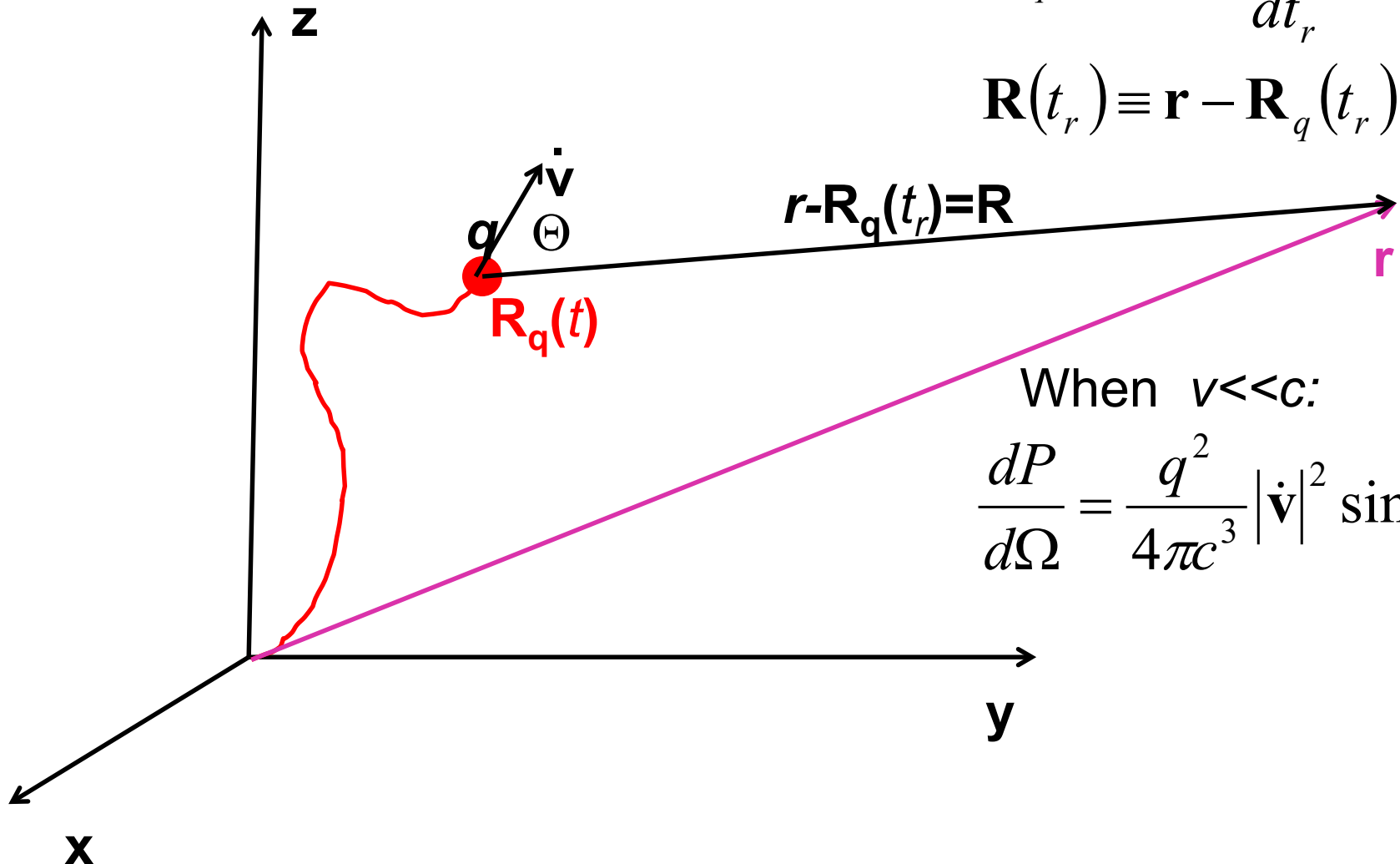
Variables (notation):

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$$

z

$\dot{\mathbf{v}}$

q Θ

$\mathbf{R_q}(t)$

r-$\mathbf{R_q}(t_r)$=$\mathbf{R}$

r

y

x

When *v<<c:*

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}|\dot{\mathbf{v}}|^2 \sin^2\Theta$$

Radiation power in non-relativistic case -- continued

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3} \left| \dot{\mathbf{v}} \right|^2 \sin^2 \Theta$$

$$P = \int d\Omega \frac{dP}{d\Omega} = \frac{2}{3} \frac{q^2}{c^3} \left| \dot{\mathbf{v}} \right|^2$$

Radiation distribution in the relativistic case

$$\frac{dP}{d\Omega} = \mathbf{S} \cdot \hat{\mathbf{R}} R^2 = \frac{q^2}{4\pi c} \frac{\left| \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^6} \Bigg|_{t_r = t - R/c}$$

This expression gives us the energy per unit field time t. We are often interested in the power per unit retarded time $t_r = t - R/c$:

$$\frac{dP_r(t)}{d\Omega} = \frac{dP(t)}{d\Omega} \frac{dt}{dt_r} \qquad\qquad \frac{dt}{dt_r} = 1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}$$

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{\left| \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^5} \Bigg|_{t_r = t - R/c}$$

Some details –

The power derived from the Poynting vector in terms of the field times is given by:

$$\frac{dP}{d\Omega} = \mathbf{S} \cdot \hat{\mathbf{R}} R^2 = \frac{q^2}{4\pi c} \left. \frac{\left| \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^6} \right|_{t_r = t - R/c}$$

The integrated power would be given by

$$W = \int dt \frac{dP(t)}{d\Omega} = \int dt_r \frac{dt}{dt_r} \frac{dP(t)}{d\Omega} \longrightarrow \frac{dP_r(t_r)}{d\Omega}$$

More comments

$$t_r = t - \frac{\left|\mathbf{r} - \mathbf{R}_q(t_r)\right|}{c}$$

$$t = t_r + \frac{\left|\mathbf{r} - \mathbf{R}_q(t_r)\right|}{c}$$

$$\frac{dt}{dt_r} = 1 + \left(-\frac{d\mathbf{R}_q(t_r)}{c\,dt_r} \right) \cdot \frac{\mathbf{r} - \mathbf{R}_q(t_r)}{\left|\mathbf{r} - \mathbf{R}_q(t_r)\right|} = 1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}$$

$$\rightarrow \qquad \frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{\left| \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^5} \Bigg|_{t_r = t - R/c}$$

Why do you think it useful to measure the power as energy per unit retarded time P_r?

1. Jackson likes to torture us.
2. There should be no difference.
3. ???

Radiation distribution in the relativistic case -- continued

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{\left| \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^5} \Bigg|_{t_r = t - R/c}$$

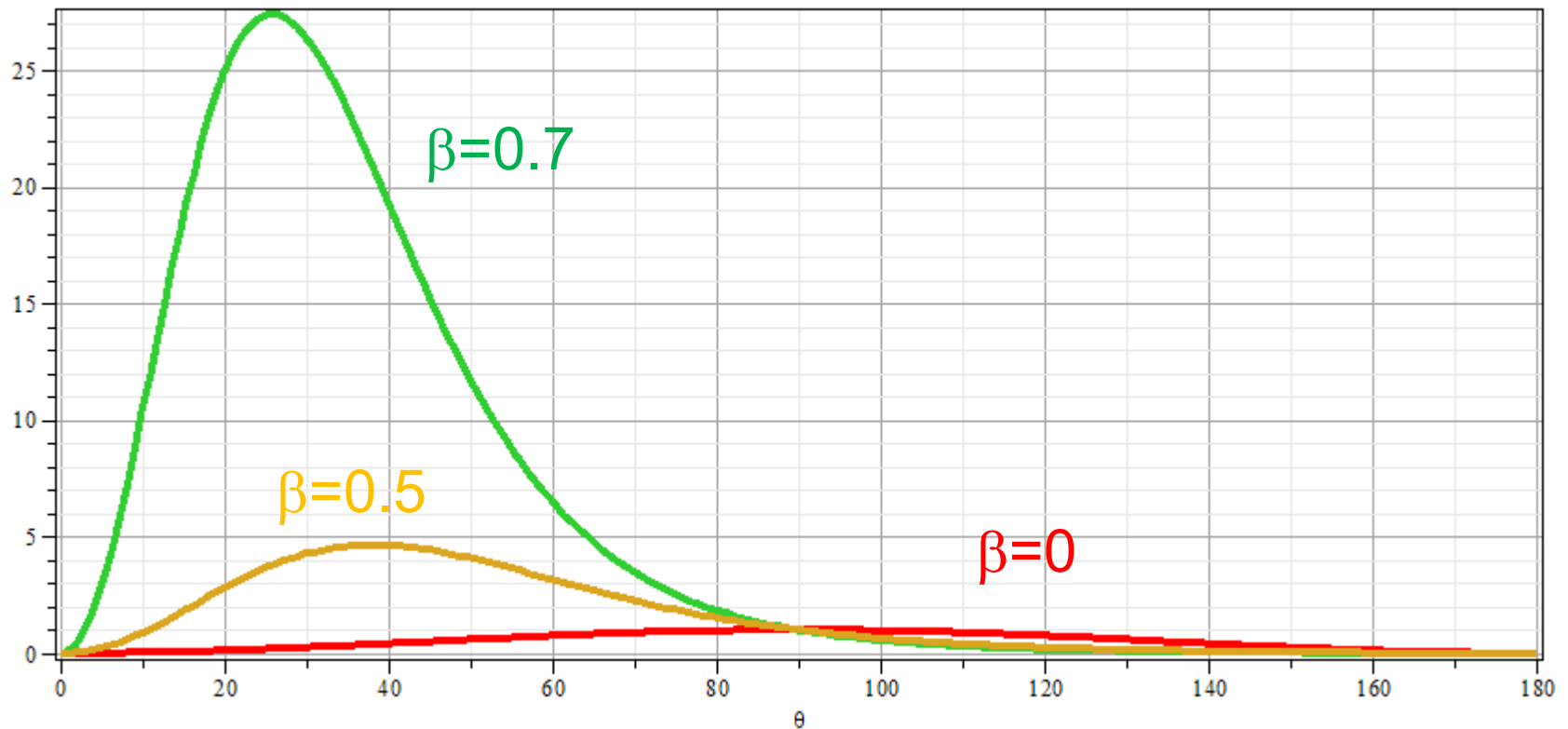For linear acceleration: $\qquad \boldsymbol{\beta} \times \dot{\boldsymbol{\beta}} = 0$

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{\left| \hat{\mathbf{R}} \times \left(\hat{\mathbf{R}} \times \dot{\boldsymbol{\beta}} \right) \right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^5} \Bigg|_{t_r = t - R/c} = \frac{q^2}{4\pi c^3} \left| \dot{\mathbf{v}} \right|^2 \frac{\sin^2 \theta}{\left(1 - \beta \cos\theta \right)^5}$$
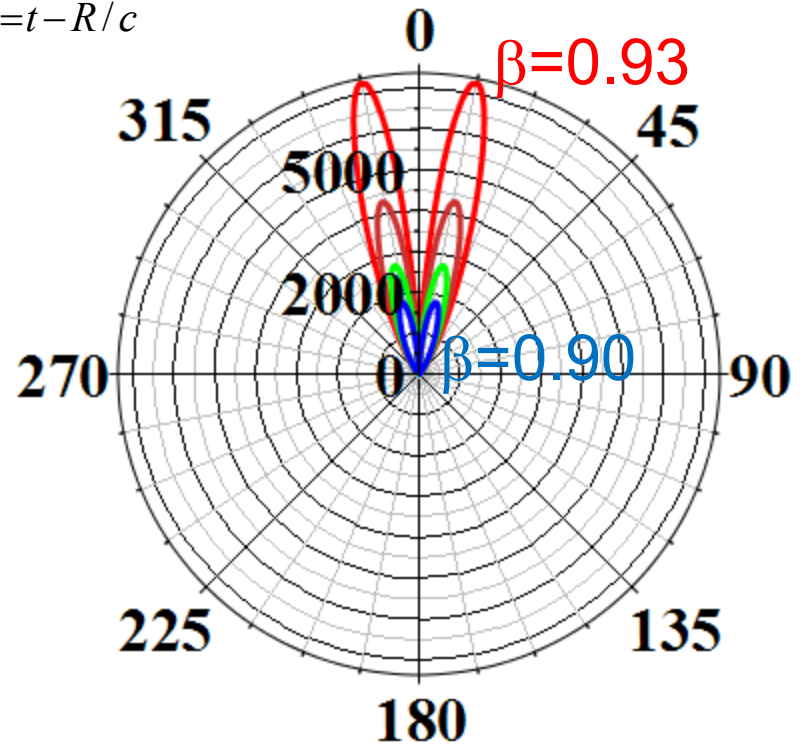
Power from linearly accelerating particle

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c}\frac{\left|\hat{\mathbf{R}}\times\left(\hat{\mathbf{R}}\times\dot{\boldsymbol{\beta}}\right)\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^5}\Bigg|_{t_r=t-R/c} = \frac{q^2}{4\pi c^3}\left|\dot{\mathbf{v}}\right|^2\frac{\sin^2\theta}{\left(1-\beta\cos\theta\right)^5}$$

β=0.7

β=0.5

β=0

Polar plots:

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c}\frac{\left|\hat{\mathbf{R}}\times\left(\hat{\mathbf{R}}\times\dot{\boldsymbol{\beta}}\right)\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^5}\Bigg|_{t_r=t-R/c} = \frac{q^2}{4\pi c^3}\left|\dot{\mathbf{v}}\right|^2\frac{\sin^2\theta}{\left(1-\beta\cos\theta\right)^5}$$

β=0.6
β=0
β=0.93
β=0.90

Note – two separate plots are introduced in order to see the drastic change of scale at values of β close to 1.
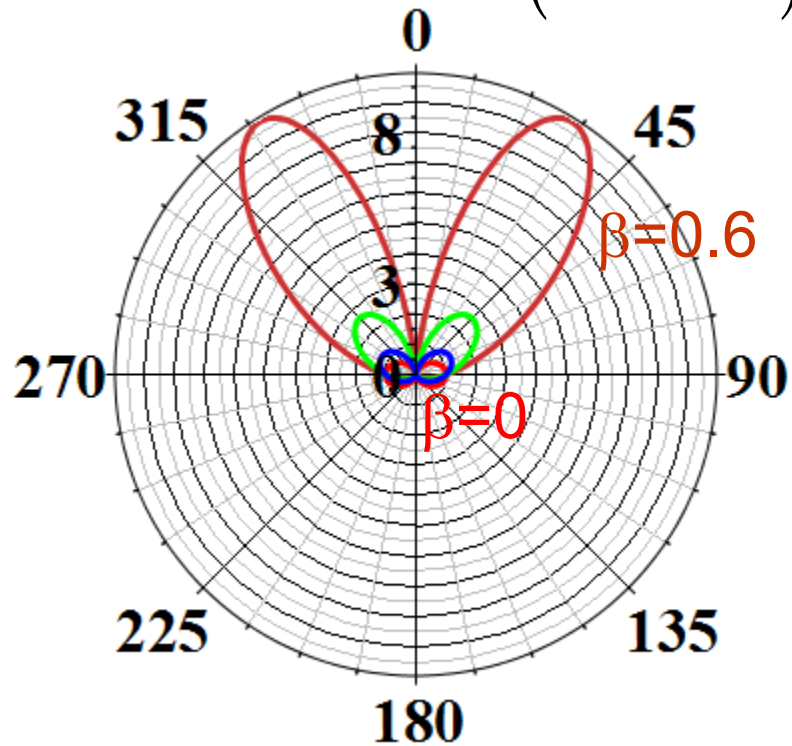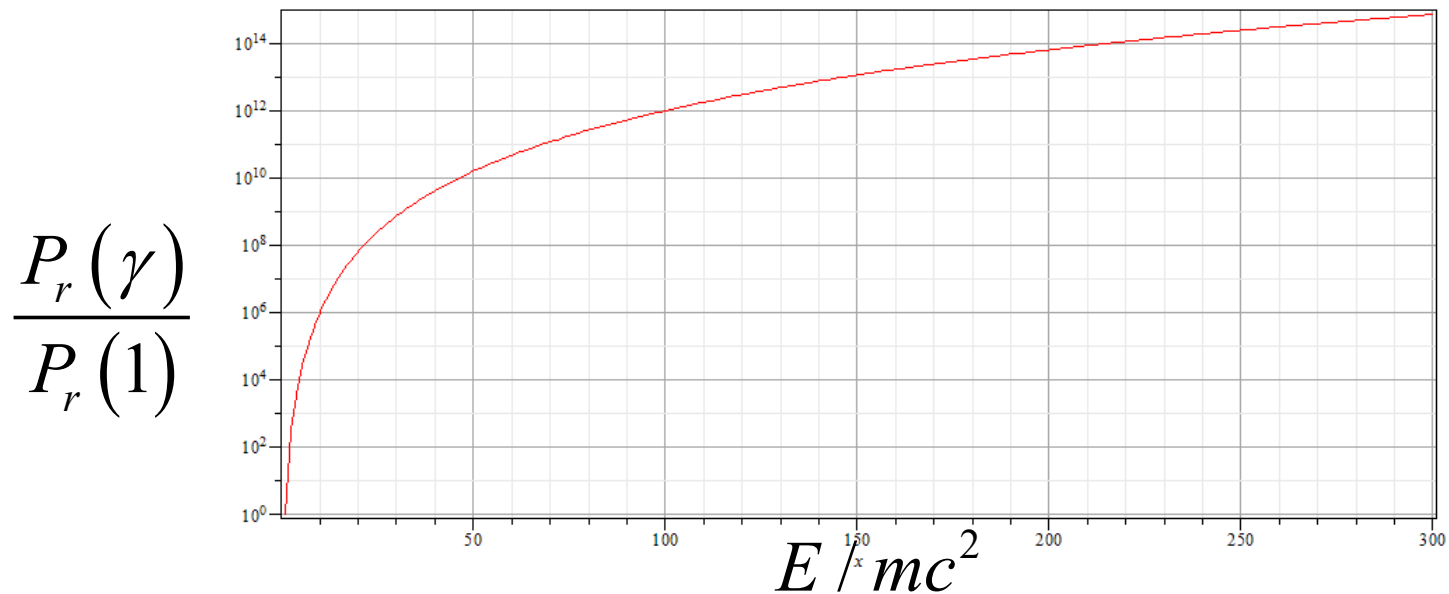
Power from linearly accelerating particle

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{\left|\hat{\mathbf{R}} \times \left(\hat{\mathbf{R}} \times \dot{\boldsymbol{\beta}}\right)\right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^5}\Bigg|_{t_r = t - R/c} = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \frac{\sin^2 \theta}{\left(1 - \beta \cos \theta\right)^5}$$

$$P_r(t_r) = \int \frac{dP_r(t_r)}{d\Omega} d\Omega = \frac{2}{3} \frac{q^2}{c^3} |\dot{\mathbf{v}}|^2 \gamma^6 \quad \text{where} \quad \gamma \equiv \frac{1}{\sqrt{1 - \beta^2}} = \frac{E}{mc^2}$$

$$\frac{P_r(\gamma)}{P_r(1)}$$

E / mc^2

Power distribution for linear acceleration -- continued
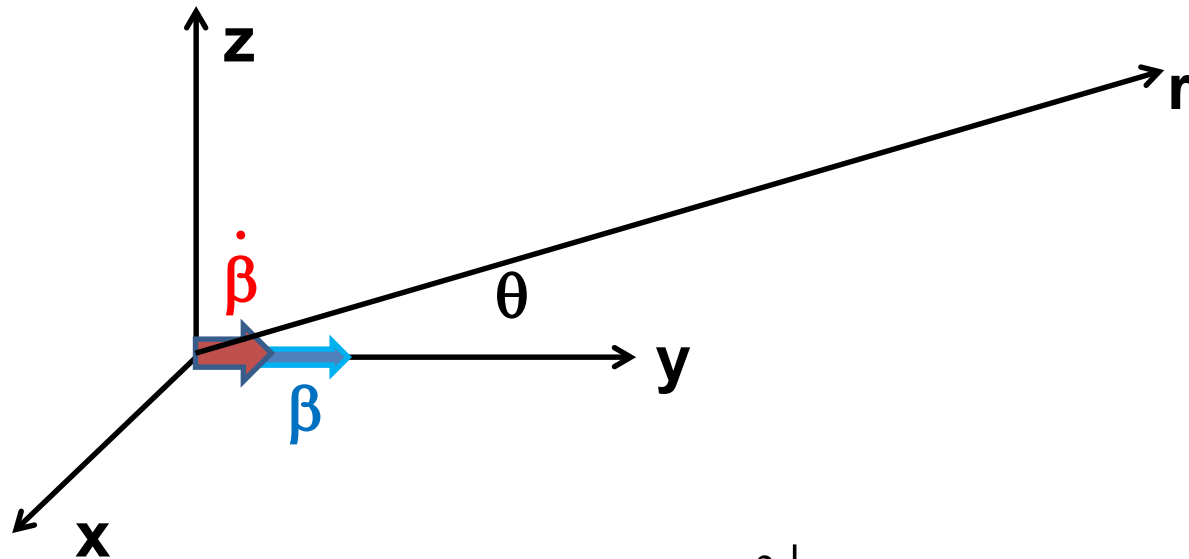
$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{\left|\hat{\mathbf{R}} \times \left(\hat{\mathbf{R}} \times \dot{\boldsymbol{\beta}}\right)\right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^5}\Bigg|_{t_r = t - R/c} = \frac{q^2}{4\pi c^3}\left|\dot{\mathbf{v}}\right|^2 \frac{\sin^2\theta}{\left(1 - \beta\cos\theta\right)^5}$$

$$P_r(t_r) = \int \frac{dP_r(t_r)}{d\Omega} d\Omega = \frac{2}{3}\frac{q^2}{c^3}\left|\dot{\mathbf{v}}\right|^2 \gamma^6 \quad \text{where} \quad \gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}$$
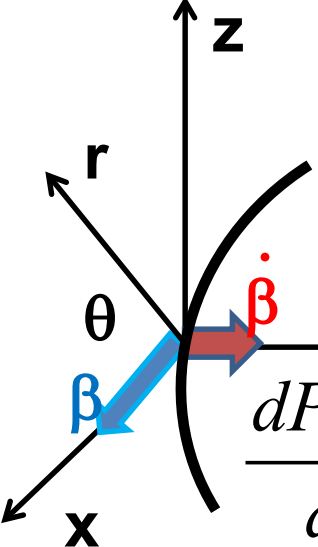
Power distribution for circular acceleration

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^5}\Bigg|_{t_r = t - R/c}$$

$$= \frac{q^2}{4\pi c} \frac{\left|\dot{\boldsymbol{\beta}}\right|^2 \left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^2 - \left(\hat{\mathbf{R}} \cdot \dot{\boldsymbol{\beta}}\right)^2 \left(1 - \beta^2\right)}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^5}\Bigg|_{t_r = t - R/c}$$

$$P_r(t_r) = \int d\Omega \frac{dP_r(t_r)}{d\Omega} = \frac{2}{3} \frac{q^2}{c^3} \left|\dot{\mathbf{v}}\right|^2 \gamma^4$$

Summary of results --For linear acceleration --
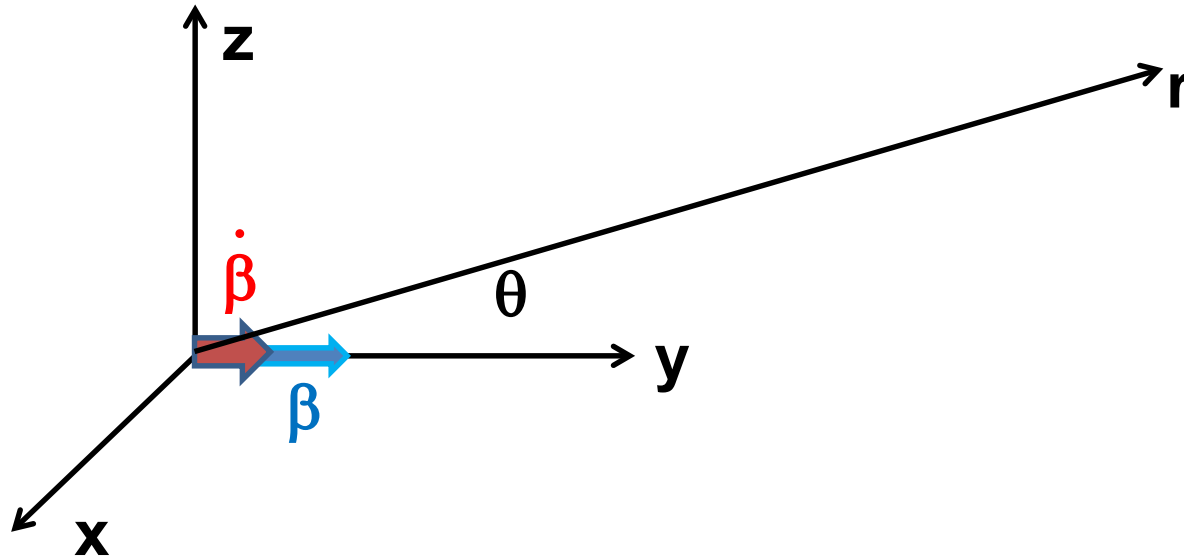
$$\frac{dP_r\left(t_r\right)}{d\Omega} = \frac{q^2}{4\pi c}\frac{\left|\hat{\mathbf{R}}\times\left(\hat{\mathbf{R}}\times\dot{\boldsymbol{\beta}}\right)\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^5}\Bigg|_{t_r=t-R/c} = \frac{q^2}{4\pi c^3}\left|\dot{\mathbf{v}}\right|^2\frac{\sin^2\theta}{\left(1-\beta\cos\theta\right)^5}$$

Power distribution for circular acceleration
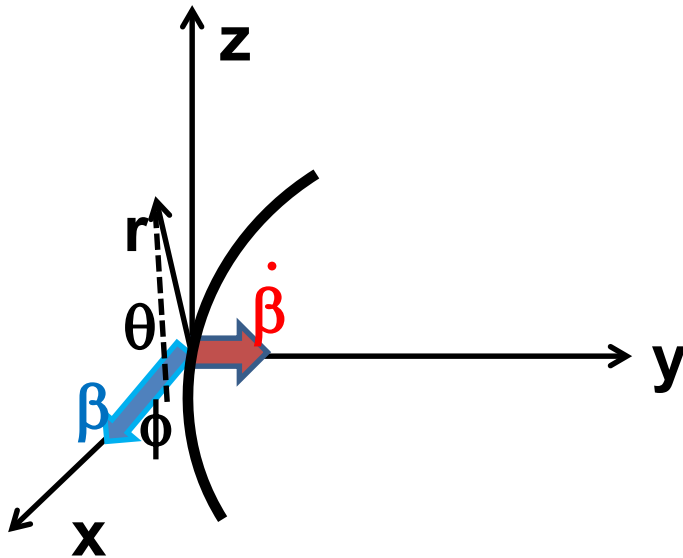
$$\frac{dP_r\left(t_r\right)}{d\Omega} = \frac{q^2}{4\pi c}\frac{\left|\dot{\boldsymbol{\beta}}\right|^2\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^2 - \left(\hat{\mathbf{R}}\cdot\dot{\boldsymbol{\beta}}\right)^2\left(1-\beta^2\right)}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^5}\Bigg|_{t_r=t-R/c}$$

$$= \frac{q^2}{4\pi c^3}\frac{\left|\dot{\mathbf{v}}\right|^2}{\left(1-\beta\cos(\theta)\right)^3}\left(1-\frac{\cos^2\theta\sin^2\phi}{\gamma^2\left(1-\beta\cos(\theta)\right)^2}\right)$$

Angular integrals for the two cases –

Linear acceleration

$$P_r(t_r) = \int \frac{dP_r(t_r)}{d\Omega} d\Omega = 2\pi \int \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \frac{\sin^2\theta \, d\sin\theta}{(1 - \beta\cos\theta)^5} = \frac{2}{3} \frac{q^2}{c^3} |\dot{\mathbf{v}}|^2 \gamma^6$$

Circular acceleration

$$P_r(t_r) = \int \frac{dP_r(t_r)}{d\Omega} d\Omega = \int d\phi \, d\sin\theta \frac{q^2}{4\pi c^3} \frac{|\dot{\mathbf{v}}|^2}{(1 - \beta\cos(\theta))^3} \left(1 - \frac{\cos^2\theta \sin^2\phi}{\gamma^2 (1 - \beta\cos(\theta))^2} \right)$$

$$= \frac{2}{3} \frac{q^2}{c^3} |\dot{\mathbf{v}}|^2 \gamma^4$$

Power distribution for circular acceleration
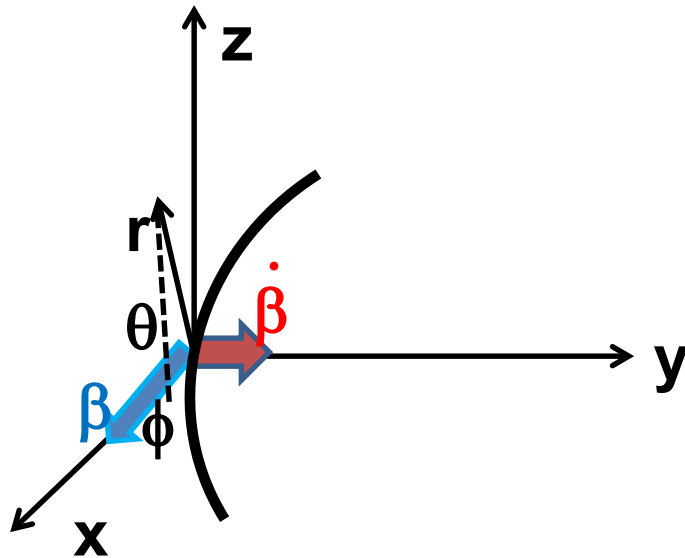
$$\frac{dP_r\left(t_r\right)}{d\Omega} = \frac{q^2}{4\pi c} \frac{\left|\dot{\boldsymbol{\beta}}\right|^2 \left(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^2 - \left(\hat{\mathbf{R}}\cdot\dot{\boldsymbol{\beta}}\right)^2 \left(1 - \beta^2\right)}{\left(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^5}\Bigg|_{t_r = t - R/c}$$

$$= \frac{q^2}{4\pi c^3} \frac{\left|\dot{\mathbf{v}}\right|^2}{\left(1 - \beta\cos(\theta)\right)^3} \left(1 - \frac{\cos^2\theta\sin^2\phi}{\gamma^2\left(1 - \beta\cos(\theta)\right)^2}\right)$$

Spectral composition of electromagnetic radiation

Previously we determined the power distribution from a charged particle:

$$\frac{dP(t)}{d\Omega} = \mathbf{S} \cdot \hat{\mathbf{R}} R^2 = \frac{q^2}{4\pi c} \left. \frac{\left| \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^6} \right|_{t_r = t - R/c}$$

$$\equiv \left| \boldsymbol{a}(t) \right|^2$$

where

$$\boldsymbol{a}(t) \equiv \sqrt{\frac{q^2}{4\pi c}} \left. \frac{\left| \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^3} \right|_{t_r = t - R/c}$$

Time integrated power per solid angle:

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt \frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt \left| \boldsymbol{a}(t) \right|^2 = \int_{-\infty}^{\infty} d\omega \left| \tilde{\boldsymbol{a}}(\omega) \right|^2$$

Spectral composition of electromagnetic radiation -- continued

Time integrated power per solid angle :

$$\frac{dW}{d\Omega} = \int\limits_{-\infty}^{\infty} dt \, \frac{dP(t)}{d\Omega} = \int\limits_{-\infty}^{\infty} dt \, |\boldsymbol{a}(t)|^2 = \int\limits_{-\infty}^{\infty} d\omega \, |\widetilde{\boldsymbol{a}}(\omega)|^2$$

Fourier amplitude :

$$\widetilde{\boldsymbol{a}}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} dt \, \boldsymbol{a}(t) e^{i\omega t} \qquad \boldsymbol{a}(t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} d\omega \, \widetilde{\boldsymbol{a}}(\omega) e^{-i\omega t}$$

Parseval's theorem

Marc-Antoine Parseval des Chênes 1755-1836

http://www-history.mcs.st-andrews.ac.uk/Biographies/Parseval.html

Spectral composition of electromagnetic radiation -- continued

Consequences of Parseval's analysis:

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt \frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt \left| \boldsymbol{a}(t) \right|^2 = \int_{-\infty}^{\infty} d\omega \left| \tilde{\boldsymbol{a}}(\omega) \right|^2$$

Note that : $\tilde{\boldsymbol{a}}(\omega) = \tilde{\boldsymbol{a}}^*(-\omega)$

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} d\omega \left| \tilde{\boldsymbol{a}}(\omega) \right|^2 = \int_0^{\infty} d\omega \left(\left| \tilde{\boldsymbol{a}}(\omega) \right|^2 + \left| \tilde{\boldsymbol{a}}(-\omega) \right|^2 \right) \equiv \int_0^{\infty} d\omega \frac{\partial^2 I}{\partial\Omega\partial\omega}$$

$$\frac{\partial^2 I}{\partial\Omega\partial\omega} \equiv 2 \left| \tilde{\boldsymbol{a}}(\omega) \right|^2$$

What is the significance of $\dfrac{\partial^2 I}{\partial\Omega\partial\omega}$?

1. It is purely a mathematical construct
2. It can be measured

Spectral composition of electromagnetic radiation -- continued

For our case:
$$a(t) \equiv \sqrt{\frac{q^2}{4\pi c}} \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^3}\Bigg|_{t_r = t - R/c}$$

Fourier amplitude:

$$\tilde{a}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} dt \; e^{i\omega t} \; a(t)$$

$$= \sqrt{\frac{q^2}{8\pi^2 c}} \int\limits_{-\infty}^{\infty} dt \; e^{i\omega t} \; \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^3}\Bigg|_{t_r = t - R/c}$$

Spectral composition of electromagnetic radiation -- continued

Fourier amplitude :

$$\widetilde{\boldsymbol{a}}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} dt \, \boldsymbol{a}(t) e^{i\omega t}$$

$$= \sqrt{\frac{q^2}{8\pi^2 c}} \int\limits_{-\infty}^{\infty} dt \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^3}\Bigg|_{t_r = t - R/c} e^{i\omega t}$$

$$= \sqrt{\frac{q^2}{8\pi^2 c}} \int\limits_{-\infty}^{\infty} dt_r \, \frac{dt}{dt_r} \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^3}\Bigg|_{t_r = t - R/c} e^{i\omega\left(t_r + R(t_r)/c\right)}$$

$$= \sqrt{\frac{q^2}{8\pi^2 c}} \int\limits_{-\infty}^{\infty} dt_r \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^2}\Bigg|_{t_r = t - R/c} e^{i\omega\left(t_r + R(t_r)/c\right)}$$

Exact expression :

$$\widetilde{\boldsymbol{a}}(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^2} \Bigg|_{t_r = t - R/c} e^{i\omega\left(t_r + R(t_r)/c\right)}$$

Recall: $\quad \dot{\mathbf{R}}_q\left(t_r\right) \equiv \dfrac{d\mathbf{R}_q\left(t_r\right)}{dt_r} \equiv \mathbf{v} \qquad \mathbf{R}\left(t_r\right) \equiv \mathbf{r} - \mathbf{R}_q\left(t_r\right) \equiv \mathbf{R}$

For $\quad r \gg R_q\left(t_r\right) \qquad\qquad R\left(t_r\right) \approx r - \hat{\mathbf{r}} \cdot \mathbf{R}_q\left(t_r\right) \quad$ where $\quad \hat{\mathbf{r}} \equiv \dfrac{\mathbf{r}}{r}$

At the same level of approximation: $\quad \hat{\mathbf{R}} \approx \hat{\mathbf{r}}$

Spectral composition of electromagnetic radiation -- continued

Exact expression:

$$\tilde{\mathbf{a}}(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^2}\Bigg|_{t_r = t - R/c} e^{i\omega\left(t_r + R(t_r)/c\right)}$$

Approximate expression:

$$\tilde{\mathbf{a}}(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} e^{i\omega(r/c)} \int_{-\infty}^{\infty} dt_r \frac{\left|\hat{\mathbf{r}} \times \left[\left(\hat{\mathbf{r}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}}\right)^2}\Bigg|_{t_r = t - R/c} e^{i\omega\left(t_r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)/c\right)}$$

Resulting spectral intensity expression:

$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2}{4\pi^2 c} \left|\int_{-\infty}^{\infty} dt_r \frac{\left|\hat{\mathbf{r}} \times \left[\left(\hat{\mathbf{r}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}}\right)^2}\Bigg|_{t_r = t - R/c} e^{i\omega\left(t_r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)/c\right)}\right|^2$$

Example – radiation from a collinear acceleration burst

$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{q^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} dt_r \frac{\left| \hat{\mathbf{r}} \times \left[(\hat{\mathbf{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right] \right|}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^2} \Bigg|_{t_r = t - R/c} e^{i\omega \left(t_r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)/c \right)} \right|^2$$

Suppose that $\dot{\boldsymbol{\beta}} = \begin{cases} \dfrac{\hat{\boldsymbol{\beta}} \Delta v}{c\tau} & 0 < t_r < \tau \\ \\ 0 & \text{otherwise} \end{cases}$

$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{q^2}{4\pi^2 c^3} \left| \frac{\left| \hat{\mathbf{r}} \times \left[\hat{\mathbf{r}} \times \hat{\boldsymbol{\beta}} \right] \right| \Delta v}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^2 \tau} \right|^2 \left| \int_0^{\tau} dt_r e^{i\omega(t_r - \hat{\mathbf{r}} \cdot \boldsymbol{\beta} t_r)} \right|^2 \qquad \text{Let } \boldsymbol{\beta} \cdot \hat{\mathbf{r}} = \beta \cos\theta$$
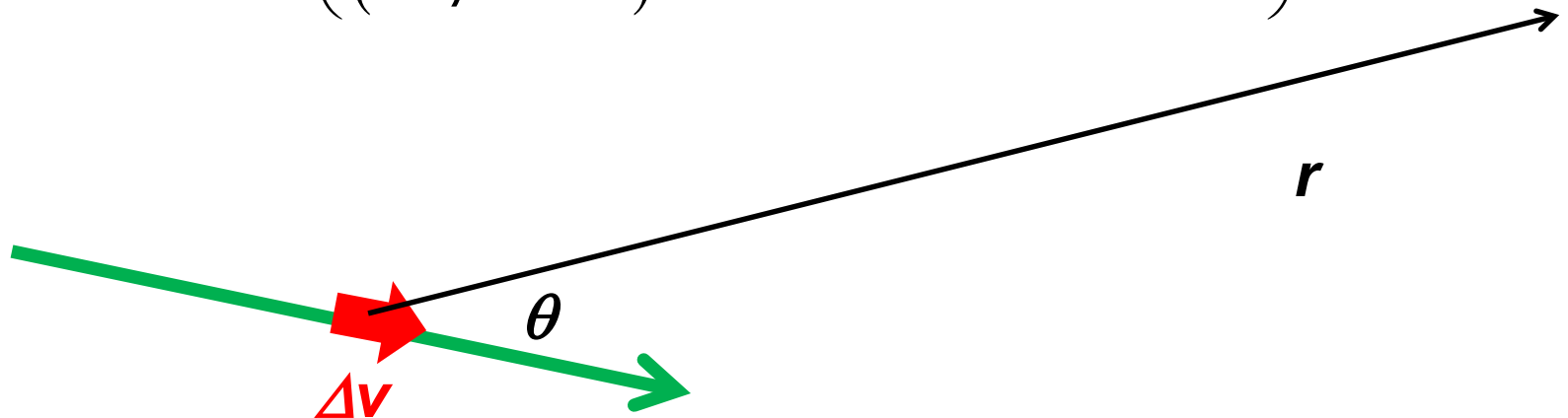
$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{q^2}{4\pi^2 c^3} \left(\frac{\Delta v \sin\theta}{(1 - \beta \cos\theta)^2} \frac{\sin(\omega\tau(1 - \beta\cos\theta)/2)}{(\omega\tau(1 - \beta\cos\theta)/2)} \right)^2$$

Example:

Suppose that $\dot{\mathbf{\beta}} = \begin{cases} \dfrac{\hat{\mathbf{\beta}}\Delta v}{c\tau} & 0 < t_r < \tau \\ \\ 0 & \text{otherwise} \end{cases}$

$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2}{4\pi^2 c^3}\left(\frac{\Delta v \sin\theta}{\left(1-\beta\cos\theta\right)^2}\frac{\sin(\omega\tau(1-\beta\cos\theta)/2)}{(\omega\tau(1-\beta\cos\theta)/2)}\right)^2$$

r

θ

Δv

Example: "Bremsstrahlung" radiation

Spectral composition of electromagnetic radiation -- continued

Alternative expression --

It can be shown that:

$$\frac{\hat{\mathbf{r}} \times \left[(\hat{\mathbf{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^2} = \frac{d}{dt_r} \left(\frac{\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \boldsymbol{\beta})}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})} \right)$$

Integration by parts and assumptions about the integration limit behaviors shows that the spectral intensity depends on the following integral:

$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{q^2 \omega^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} dt_r \left[\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \boldsymbol{\beta}(t_r) \right) \right] e^{i\omega \left(t_r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)/c \right)} \right|^2$$