# PHY 712 Electrodynamics
# 11-11:50 AM MWF Olin 103

## Discussion for Lecture 31:

### Start reading Chap. 15 –

### Radiation from collisions of charged particles

1. Overview
2. X-ray tube
3. Radiation from Rutherford scattering
4. Other collision models

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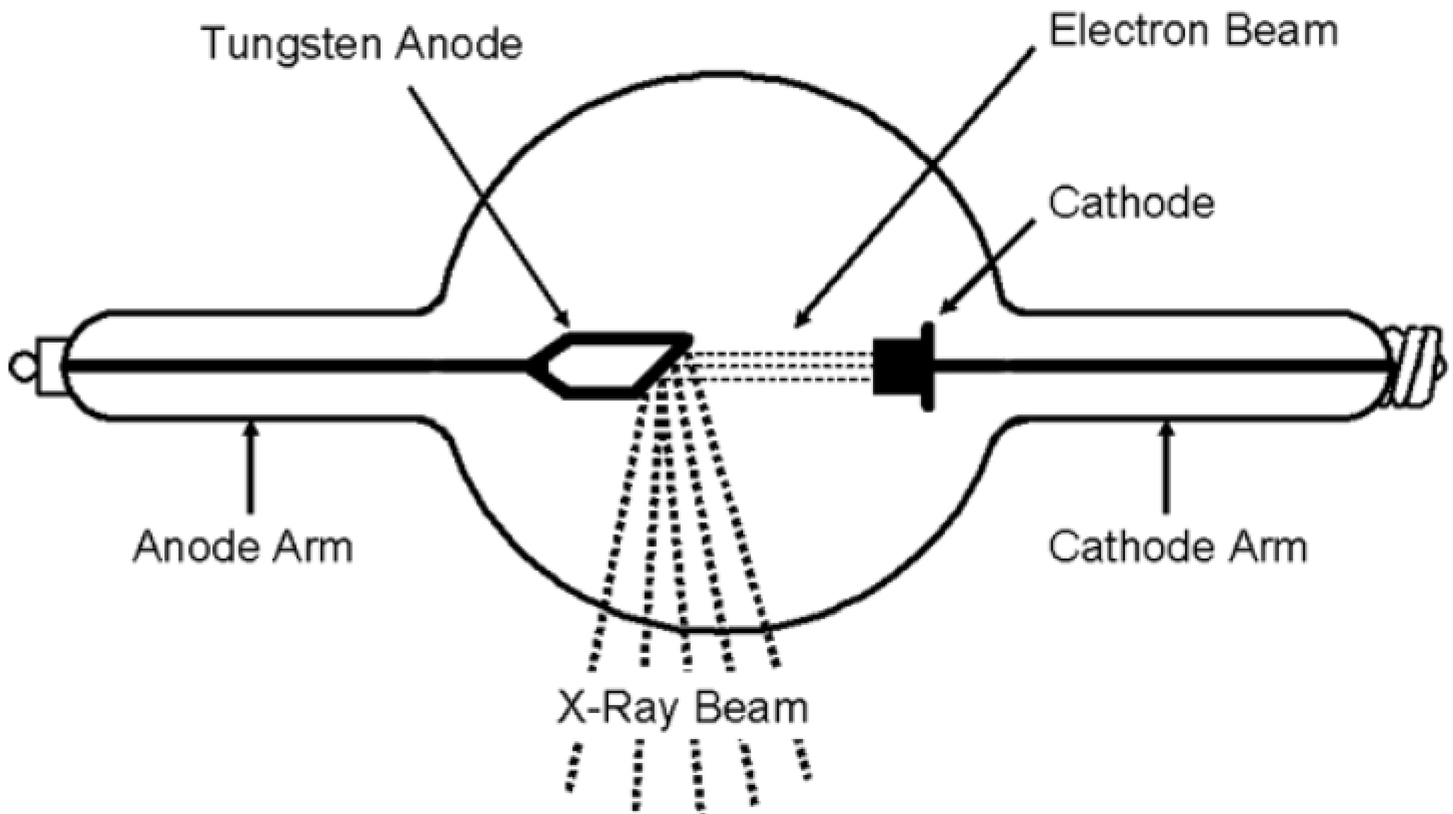
|---|---|---|---|---|---|
| **21** | Fri: 03/25/2022 | Chap. 9 | Radiation from localized oscillating sources | #18 | 03/30/2022 |
| **22** | Mon: 03/28/2022 | Chap. 9 | Radiation from oscillating sources | | |
| **23** | Wed: 03/30/2022 | Chap. 9 & 10 | Radiation and scattering | #19 | 04/01/2022 |
| **24** | Fri: 04/01/2022 | Chap. 11 | Special Theory of Relativity | #20 | 04/04/2022 |
| **25** | Mon: 04/04/2022 | Chap. 11 | Special Theory of Relativity | #21 | 04/06/2022 |
| **26** | Wed: 04/06/2022 | Chap. 11 | Special Theory of Relativity | | |
| **27** | Fri: 04/08/2022 | Chap. 14 | Radiation from moving charges | #22 | 04/11/2022 |
| **28** | Mon: 04/11/2022 | Chap. 14 | Radiation from accelerating charged particles | #23 | 04/18/2022 |
| **29** | Wed: 04/13/2022 | Chap. 14 | Synchrotron radiation | | |
| | Fri: 04/15/2022 | No class | *Holiday* | | |
| **30** | Mon: 04/18/2022 | Chap. 14 & 15 | Thompson and Compton scattering | #24 | 04/20/2022 |
| **31** | Wed: 04/20/2022 | Chap. 15 | Radiation from collisions of charged particles | | |
| **32** | Fri: 04/22/2022 | Chap. 13 | Cherenkov radiation | | |

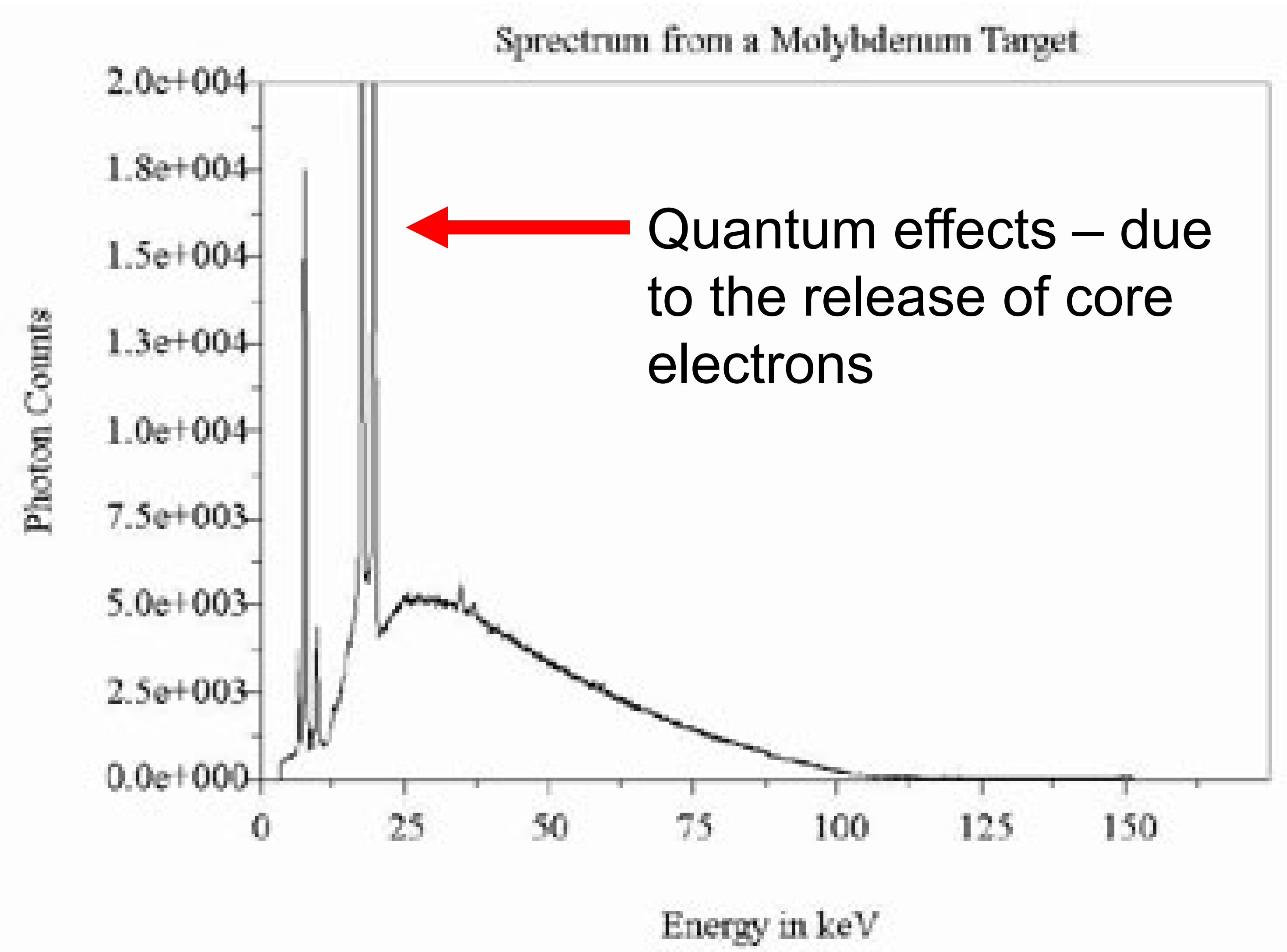
# Generation of X-rays in a Coolidge tube

https://www.orau.org/ptp/collection/xraytubescoolidge/coolidgeinformation.htm



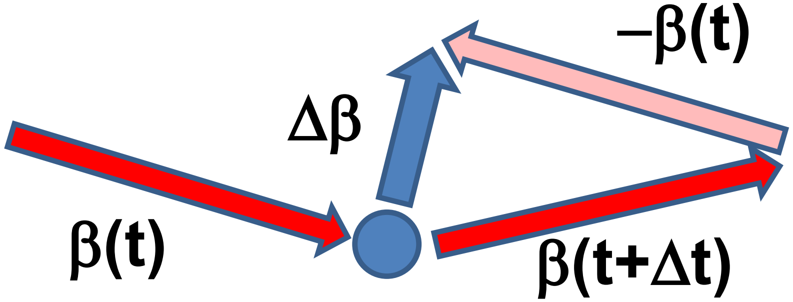

Invented in 1913. Associated with the German word "bremsstrahlung" – meaning breaking radiation.

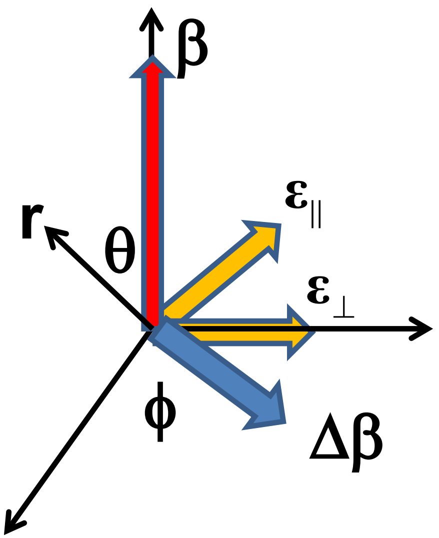
http://www.ndt-ed.org/EducationResources/CommunityCollege/Radiography/Physics/xrays.htm




Quantum effects – due to the release of core electrons

# Radiation during collisions of charged particles




$\boldsymbol{\varepsilon}_{\|}$ is in the plane of **β** and **r**

$\boldsymbol{\varepsilon}_{\perp}$ is perpendicular to the plane of **β** and **r**

Results from previous analyses:

Spectral intensity of radiation from accelerating charged particle :

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\int dt\ e^{i\omega\left(t-\hat{\mathbf{r}}\cdot\mathbf{R}_q(t)/c\right)}\frac{d}{dt}\left[\frac{\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}\right)}{1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta}}\right]\right|^2$$

Note that in the following slides we are taking the limit $\omega \to 0$ but keeping the notation of the differential intensity….

For a collision of duration $\tau$ emitting radiation with polarization $\boldsymbol{\varepsilon}$ and frequency $\omega \to 0;$

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\boldsymbol{\varepsilon}\cdot\left(\frac{\boldsymbol{\beta}(t+\tau)}{1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta}(t+\tau)} - \frac{\boldsymbol{\beta}(t)}{1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta}(t)}\right)\right|^2$$

Note that $\boldsymbol{\varepsilon}$ is perpendicular to **r**.

# Radiation during collisions -- continued

For a collision of duration $\tau$ emitting radiation with polarization $\boldsymbol{\varepsilon}$ and frequency $\omega \to 0$:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\boldsymbol{\varepsilon}\cdot\left(\frac{\boldsymbol{\beta}(t+\tau)}{1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta}(t+\tau)} - \frac{\boldsymbol{\beta}(t)}{1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta}(t)}\right)\right|^2$$

We will evaluate this expression for two cases:

Non-relativistic limit:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\boldsymbol{\varepsilon}\cdot(\Delta\boldsymbol{\beta})\right|^2 \qquad \Delta\boldsymbol{\beta} \equiv \boldsymbol{\beta}(t+\tau) - \boldsymbol{\beta}(t)$$

Relativistic collision with small $|\Delta\boldsymbol{\beta}| \equiv \boldsymbol{\beta}(t+\tau) - \boldsymbol{\beta}(t)$ :

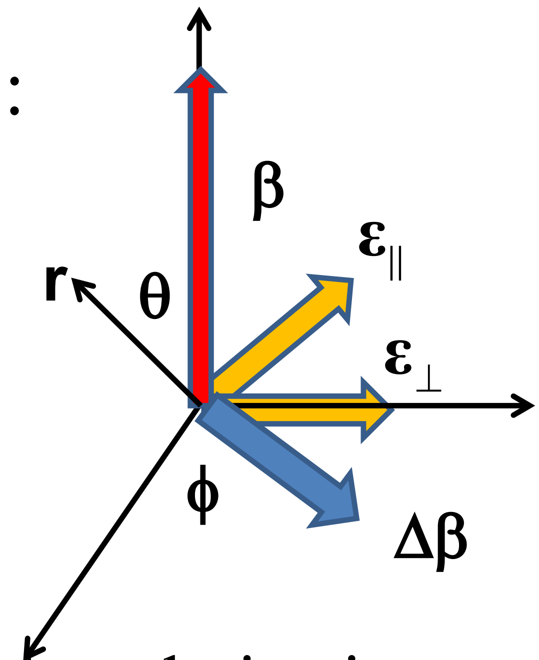
$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\boldsymbol{\varepsilon}\cdot\left(\frac{\Delta\boldsymbol{\beta} + \hat{\mathbf{r}}\times(\boldsymbol{\beta}\times\Delta\boldsymbol{\beta})}{(1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta})^2}\right)\right|^2$$

In the limit $\beta \to 0$, this is the same as the non-relativistic case.

## Radiation during collisions -- continued

Relativistic collision with small $|\Delta\boldsymbol{\beta}|$ :

$$\frac{d^2I}{d\omega d\Omega}=\frac{q^2}{4\pi^2c}\left|\boldsymbol{\varepsilon}\cdot\left(\frac{\Delta\boldsymbol{\beta}+\hat{\mathbf{r}}\times(\boldsymbol{\beta}\times\Delta\boldsymbol{\beta})}{(1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta})^2}\right)\right|^2$$




Also assume $\Delta\boldsymbol{\beta}$ is perpendicular to $\boldsymbol{\beta}$ direction

Expressions (averaging over $\varphi$) for || or $\perp$ polarization:

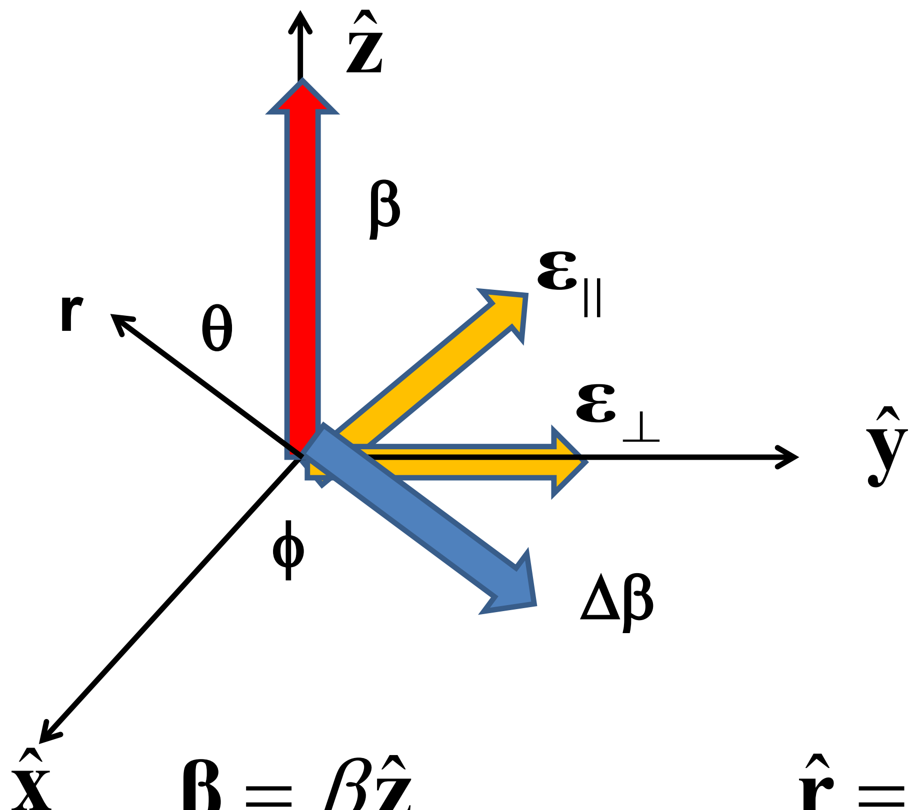
$$\frac{d^2I_{\parallel}}{d\omega d\Omega}=\frac{q^2}{8\pi^2c}|\Delta\boldsymbol{\beta}|^2\frac{(\beta-\cos\theta)^2}{(1-\beta\cos\theta)^4}$$

polarization in ***r*** and ***β*** plane

$$\frac{d^2I_{\perp}}{d\omega d\Omega}=\frac{q^2}{8\pi^2c}|\Delta\boldsymbol{\beta}|^2\frac{1}{(1-\beta\cos\theta)^2}$$

polarization perpendicular to ***r*** and ***β*** plane

Some details:




(using geometry of Fig. 15.2 in Jackson)

$$\boldsymbol{\beta} = \beta \hat{\mathbf{z}} \qquad \hat{\mathbf{r}} = \sin\theta \hat{\mathbf{x}} + \cos\theta \hat{\mathbf{z}}$$

$$\boldsymbol{\varepsilon}_{||} = -\cos\theta \hat{\mathbf{x}} + \sin\theta \hat{\mathbf{z}} \qquad \boldsymbol{\varepsilon}_{\perp} = \hat{\mathbf{y}}$$

$$\Delta\boldsymbol{\beta} = \Delta\beta \left( \cos\phi \hat{\mathbf{x}} + \sin\phi \hat{\mathbf{y}} \right)$$

**Note: This is a wild assumption!**

# Some details -- continued:

$$\hat{\mathbf{r}} = \sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{z}}$$

$$\boldsymbol{\varepsilon}_{\perp} = \hat{\mathbf{y}} \qquad \boldsymbol{\varepsilon}_{||} = -\cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{z}}$$

**Consistent with radiation from charged particles.**

$$\boldsymbol{\beta} = \beta\hat{\mathbf{z}}$$

**Convenient geometry**

$$\Delta\boldsymbol{\beta} = \Delta\beta\left(\cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}\right)$$

**Wild guess**

$$\Delta\boldsymbol{\beta} + \hat{\mathbf{r}} \times \left(\boldsymbol{\beta} \times \Delta\boldsymbol{\beta}\right) = \Delta\boldsymbol{\beta}\left(1 - \hat{\mathbf{r}} \cdot \boldsymbol{\beta}\right) + \boldsymbol{\beta}\left(\hat{\mathbf{r}} \cdot \Delta\boldsymbol{\beta}\right)$$

$$\boldsymbol{\varepsilon}_{\perp} \cdot \left(\Delta\boldsymbol{\beta} + \hat{\mathbf{r}} \times \left(\boldsymbol{\beta} \times \Delta\boldsymbol{\beta}\right)\right) = \Delta\beta\sin\phi\left(1 - \beta\cos\theta\right)$$

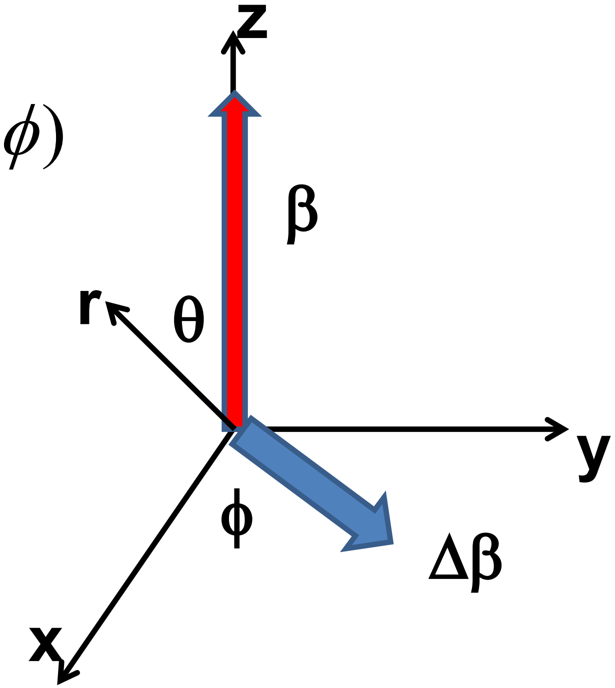
$$\boldsymbol{\varepsilon}_{||} \cdot \left(\Delta\boldsymbol{\beta} + \hat{\mathbf{r}} \times \left(\boldsymbol{\beta} \times \Delta\boldsymbol{\beta}\right)\right) = \Delta\beta\cos\phi\left(\beta - \cos\theta\right)$$

# Radiation during collisions -- continued

Intensity expressions: (averaging over $\phi$)

$$\frac{d^2 I_{\parallel}}{d\omega d\Omega} = \frac{q^2}{8\pi^2 c} |\Delta\boldsymbol{\beta}|^2 \frac{(\beta - \cos\theta)^2}{(1-\beta\cos\theta)^4}$$

$$\frac{d^2 I_{\perp}}{d\omega d\Omega} = \frac{q^2}{8\pi^2 c} |\Delta\boldsymbol{\beta}|^2 \frac{1}{(1-\beta\cos\theta)^2}$$




Relativistic collision at low $\omega$ and with small $|\Delta\boldsymbol{\beta}|$ and $\Delta\boldsymbol{\beta}$ perpendicular to plane of $\hat{\mathbf{r}}$ and $\boldsymbol{\beta}$, as a function of $\theta$ where $\hat{\mathbf{r}} \cdot \boldsymbol{\beta} = \beta\cos\theta$;

Integrating over solid angle:

$$\frac{dI}{d\omega} = \int d\Omega \left( \frac{d^2 I_{\parallel}}{d\omega d\Omega} + \frac{d^2 I_{\perp}}{d\omega d\Omega} \right) = \frac{2}{3\pi} \frac{q^2}{c} \gamma^2 \; |\Delta\boldsymbol{\beta}|^2$$

Some more details:

$$\int d\Omega \frac{d^2 I_{\parallel}}{d\omega d\Omega} = \frac{q^2}{8\pi^2 c} |\Delta\boldsymbol{\beta}|^2 2\pi \int_{-1}^{1} d\cos\theta \frac{(\beta - \cos\theta)^2}{(1-\beta\cos\theta)^4}$$

$$= \frac{q^2}{4\pi c} |\Delta\boldsymbol{\beta}|^2 \frac{2}{3} \frac{1}{(1-\beta^2)}$$

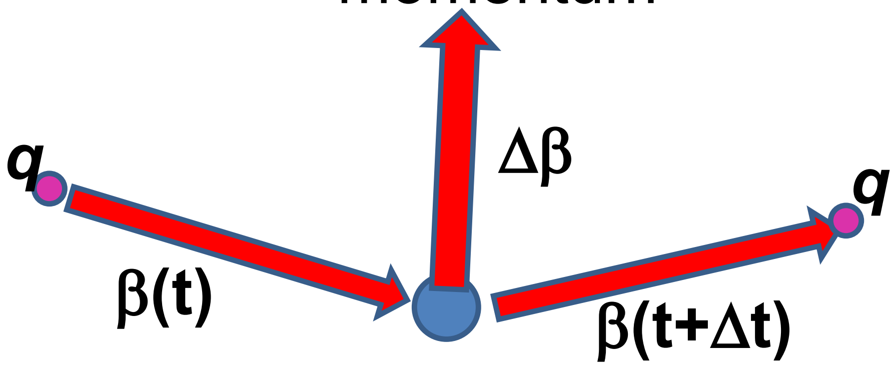
$$\int d\Omega \frac{d^2 I_{\perp}}{d\omega d\Omega} = \frac{q^2}{8\pi^2 c} |\Delta\boldsymbol{\beta}|^2 \int_{-1}^{1} d\cos\theta \frac{1}{(1-\beta\cos\theta)^2}$$

$$= \frac{q^2}{4\pi c} |\Delta\boldsymbol{\beta}|^2 \frac{2}{(1-\beta^2)}$$

$$\frac{dI}{d\omega} = \int d\Omega \left( \frac{d^2 I_{\parallel}}{d\omega d\Omega} + \frac{d^2 I_{\perp}}{d\omega d\Omega} \right) = \frac{2}{3\pi} \frac{q^2}{c} \gamma^2 \; |\Delta\boldsymbol{\beta}|^2$$

Estimation of Δβ

Need to consider the mechanics of collision; it is convenient to parameterize in terms of momentum --




Momentum transfer:

$$Qc \equiv \left|\mathbf{p}(t+\tau)-\mathbf{p}(t)\right|c \approx \gamma M c^2 \left|\boldsymbol{\Delta\beta}\right|$$

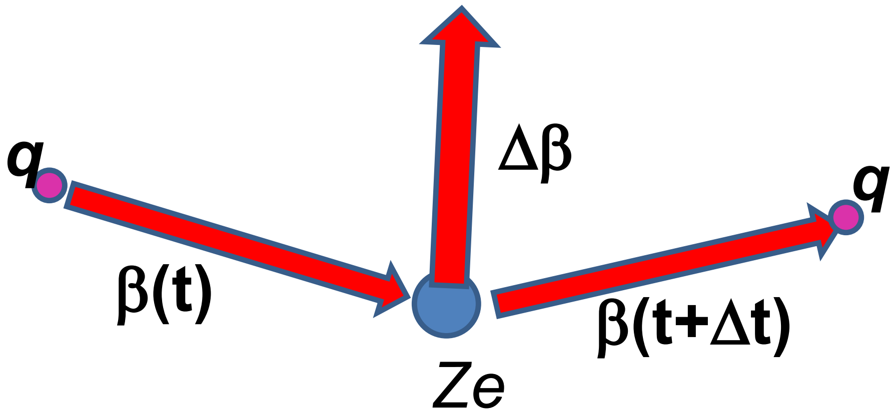
mass of particle having charge *q*

$$\frac{dI}{d\omega} = \frac{2}{3\pi}\ \frac{q^2}{c}\gamma^2 \left|\boldsymbol{\Delta\beta}\right|^2 \approx \frac{2}{3\pi}\ \frac{q^2}{M^2c^3}Q^2$$

What are the conditions for the validity of this result?

# What are possible sources for the momentum transfer Q?

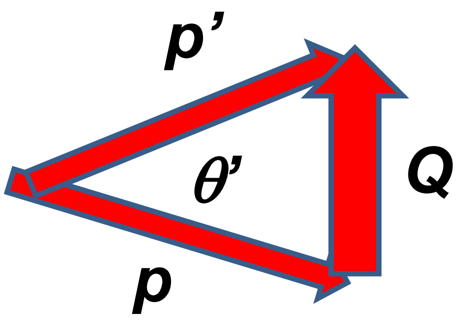
# Estimation of Δβ or *Q* -- for the case of Rutherford scattering




Assume that target nucleus (charge *Ze*) has mass >>M;

Rutherford scattering cross-section in center of mass analysis:

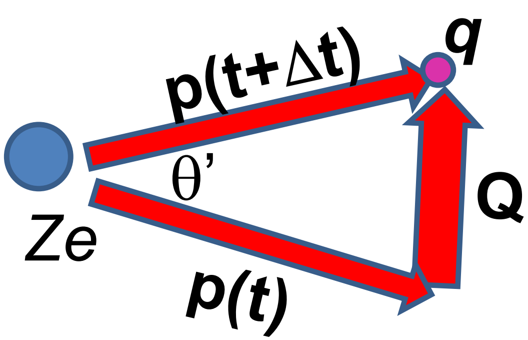
$$\frac{d\sigma}{d\Omega} = \left(\frac{2Zeq}{pv}\right)^2 \frac{1}{\left(2\sin\left(\theta'/2\right)\right)^4}$$




Assuming elastic scattering:

$$Q^2 = \left(2p\sin\left(\theta'/2\right)\right)^2 = 2p^2\left(1-\cos\theta'\right)$$

# Case of Rutherford scattering -- continued




Rutherford scattering cross-section:

$$\frac{d\sigma}{d\Omega} = \left(\frac{2Zeq}{pv}\right)^2 \frac{1}{\left(2\sin\left(\theta'/2\right)\right)^4}$$

$$\frac{d\sigma}{dQ} = \int_{\varphi'} \frac{d\sigma}{d\Omega} \left|\frac{d\Omega}{dQ}\right| d\varphi'$$

$$d\Omega = d\varphi' d\cos\theta'$$

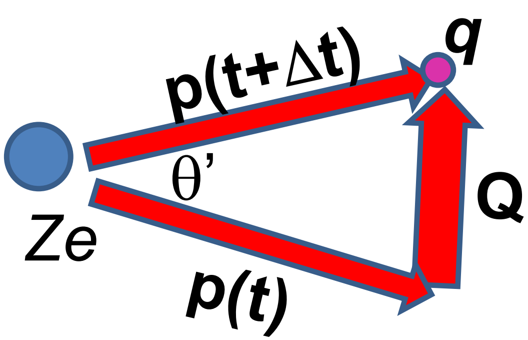
$$Q^2 = \left(2p\sin\left(\theta'/2\right)\right)^2 = 2p^2\left(1-\cos\theta'\right)$$

$$dQ = -\frac{p^2}{Q} d\cos\theta'$$

$$\Rightarrow \quad \frac{d\sigma}{dQ} = 8\pi\left(\frac{Zeq}{\beta c}\right)^2 \frac{1}{Q^3}$$

**Does the algebra work out?**

# Case of Rutherford scattering -- continued




Differential radiation cross section :

$$\frac{d^2\chi}{d\omega dQ} = \frac{dI}{d\omega}\frac{d\sigma}{dQ} = \left(\frac{2}{3\pi}\frac{q^2}{M^2c^3}Q^2\right)\left(8\pi\left(\frac{Zeq}{\beta c}\right)^2\frac{1}{Q^3}\right)$$

$$= \frac{16}{3}\frac{(Ze)^2}{c}\left(\frac{q^2}{Mc^2}\right)^2\frac{1}{\beta^2}\frac{1}{Q}$$

# Differential radiation cross section -- continued

Integrating over momentum transfer

$$\frac{d\chi}{d\omega} = \int_{Q_{\min}}^{Q_{\max}} dQ \frac{d^2\chi}{d\omega dQ} = \frac{16}{3}\frac{(Ze)^2}{c}\left(\frac{q^2}{Mc^2}\right)^2 \frac{1}{\beta^2}\ln\left(\frac{Q_{\max}}{Q_{\min}}\right)$$

How do the limits of $Q$ occur?

Jackson suggests that these come from the limits of validity of the analysis.

1. Seems like cheating?
2. Perhaps fair?

Comment on frequency dependence --

Original expression for radiation intensity:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c} \left| \int dt \; e^{i\omega\left(t - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t)/c\right)} \frac{d}{dt}\left[ \frac{\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \boldsymbol{\beta}\right)}{1 - \hat{\mathbf{r}}\cdot\boldsymbol{\beta}} \right] \right|^2$$

In the previous derivations, we have assumed that

$$\omega\left(t - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t)/c\right) << 1.$$

$$\omega\left(t - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t)/c\right) = \omega\left( t - \hat{\mathbf{r}}\cdot\int_0^t dt'\boldsymbol{\beta}(t') \right) \approx \omega\tau\left(1 - \hat{\mathbf{r}}\cdot\langle\boldsymbol{\beta}\rangle\right)$$

In the non-relativistic case, this means $\omega\tau \ll 1.$

Here $\tau$ is the effective collision time.

How to estimate the collision time?

Jackson uses the following analysis in terms of the impact parameter *b:*

Using classical mechanics and assuming $v \ll c$:

$$\tau \approx \frac{b}{v} \ll \frac{1}{\omega} \quad \text{and} \quad Q \approx \frac{2Zeq}{bv}$$

$$\text{Assume that } Q_{\min} = \frac{2Zeq}{b_{\max} v} = \frac{2Zeq\omega}{v^2}$$

# Differential radiation cross section -- continued

Radiation cross section in terms of momentum transfer

$$\frac{d\chi}{d\omega} = \int_{Q_{\min}}^{Q_{\max}} dQ \frac{d^2\chi}{d\omega dQ} = \frac{16}{3}\frac{(Ze)^2}{c}\left(\frac{q^2}{Mc^2}\right)^2 \frac{1}{\beta^2}\ln\left(\frac{Q_{\max}}{Q_{\min}}\right)$$

Note that: $Q^2 = 2p^2(1-\cos\theta') \quad \Rightarrow Q_{\max} = 2p$

In general, $Q_{\min}$ is determined by the collision time

condition $\omega\tau < 1 \quad \Rightarrow Q_{\min} \approx \frac{2Zeq\omega}{v^2}$

Radiation cross section for classical non - relativistic process

$$\frac{d\chi}{d\omega} = \frac{16}{3}\frac{(Ze)^2}{c}\left(\frac{q^2}{Mc^2}\right)^2 \frac{1}{\beta^2}\ln\left(\frac{\lambda M v^3}{Zeq\omega}\right)$$

λ= “fudge factor” of order unity

What could be the origin of the fudge factor?

What do you take away from this analysis

1. Disgust?
2. Admiration?
3. Motivation to avoid charged particles?