PHY 712 Electrodynamics
11-11:50 AM MWF Olin 103

Discussion for Lecture 33:

Special Topics in Electrodynamics:

Cherenkov radiation

References: Jackson Chapter 13.4

Zangwill Chapter 23.7

Smith Chapter 6.4

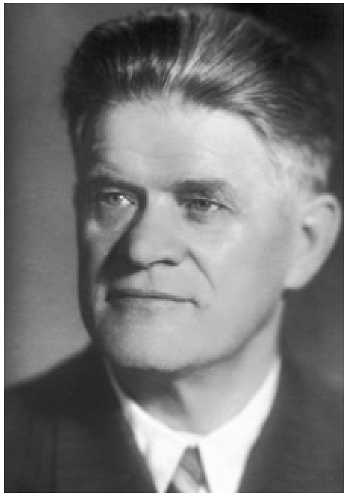
27	Fri: 04/08/2022	Chap. 14	Radiation from moving charges	#22	04/11/2022
28	Mon: 04/11/2022	Chap. 14	Radiation from accelerating charged particles	#23	04/18/2022
29	Wed: 04/13/2022	Chap. 14	Synchrotron radiation		
	Fri: 04/15/2022	No class	*Holiday*		
30	Mon: 04/18/2022	Chap. 14 & 15	Thompson and Compton scattering	#24	04/20/2022
31	Wed: 04/20/2022	Chap. 15	Radiation from collisions of charged particles		
32	Fri: 04/22/2022	Chap. 13	Cherenkov radiation		

Cherenkov radiation

Cherenkov radiation emitted by the core of the Reed Research Reactor located at Reed College in Portland, Oregon, U.S. *Cherenkov radiation*. Photograph. *Encyclopædia Britannica Online*. Web. 12 Apr. 2013. http://www.britannica.com/EBchecked/media/174732

The Nobel Prize in Physics 1958

Pavel A. Cherenkov
Il´ja M. Frank
Igor Y. Tamm



Affiliation at the time of the award: P.N. Lebedev Physical Institute, Moscow, USSR

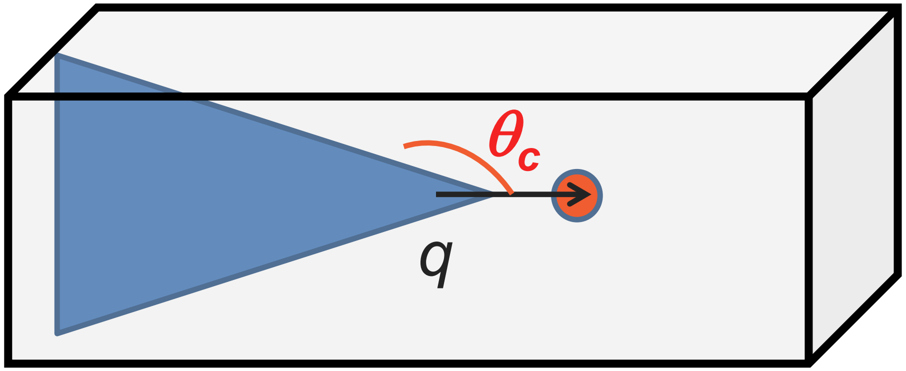
Prize motivation: "for the discovery and the interpretation of the Cherenkov effect."

https://www.nobelprize.org/prizes/physics/1958/ceremony-speech/

References for notes: Glenn S. Smith, *An Introduction to Electromagnetic Radiation* (Cambridge UP, 1997), Andrew Zangwill, Modern Electrodynamics (Cambridge UP, 2013)

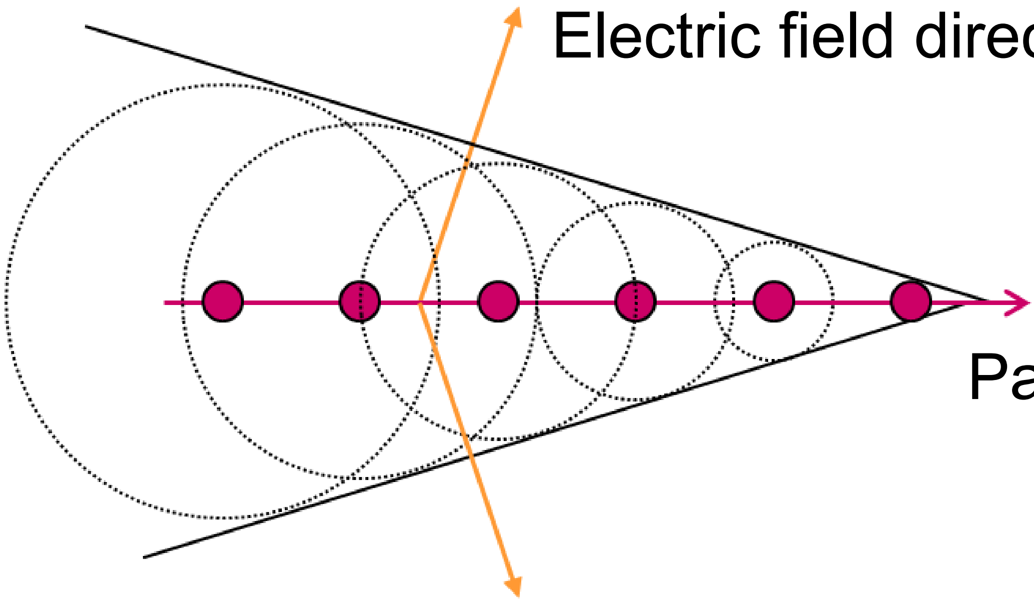
Cherenkov radiation

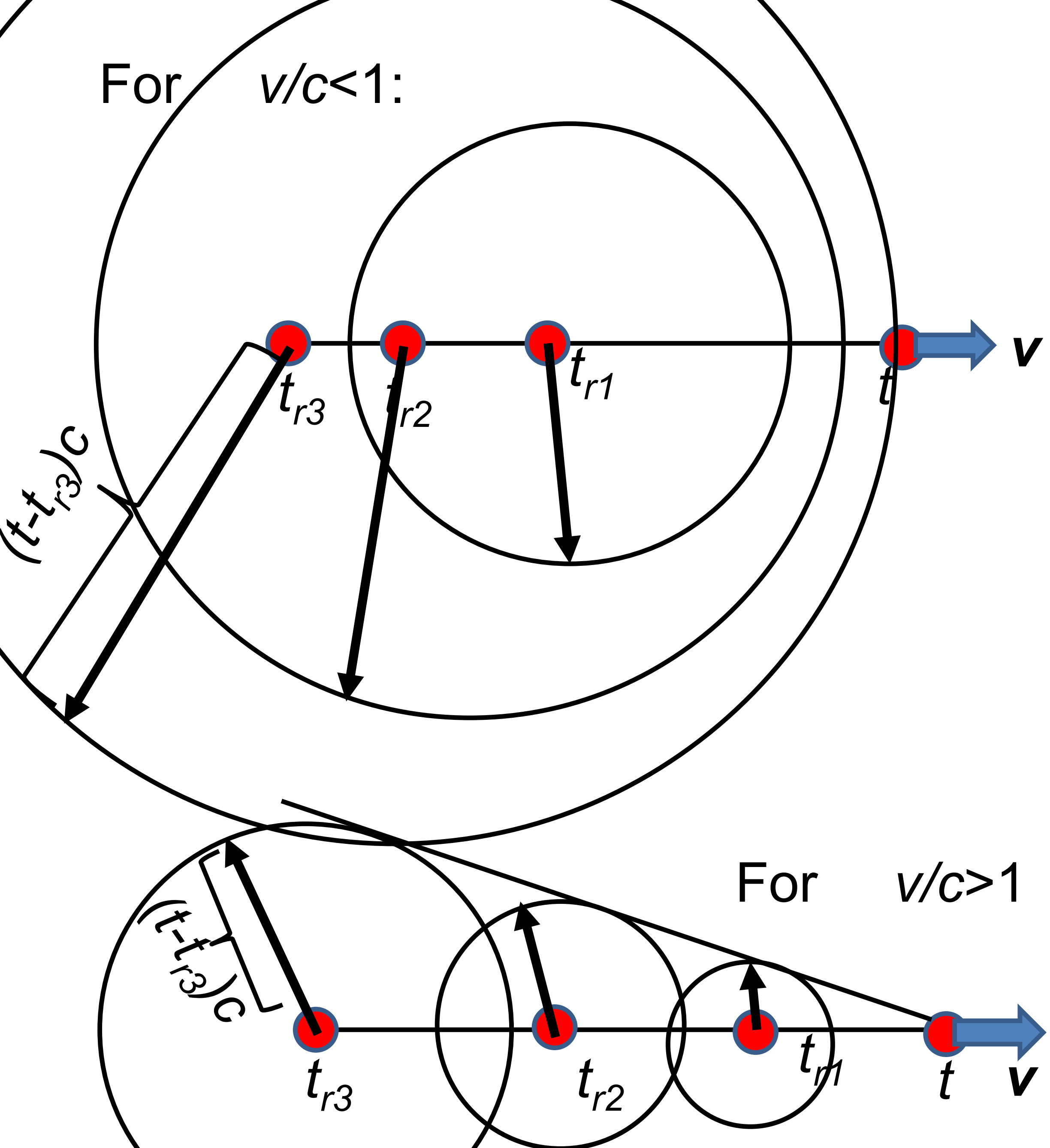
Discovered ~1930; bluish light emitted by energetic charged particles traveling within dielectric materials



Note that some treatments give the critical angle as $\theta_c - \pi/2$.

From: http://large.stanford.edu/courses/2014/ph241/alaeian2/




For v/c<1:
(t-tr3)c
tr3
tr2
tr1
t
v
For v/c>1
(t-tr3)c
tr3
tr2
tr1
t
v

Maxwell's potential equations within a material having permittivity and permeability (Lorentz gauge; cgs Gaussian units)

$$\nabla^2\Phi - \mu\varepsilon\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\frac{4\pi}{\varepsilon}\rho$$

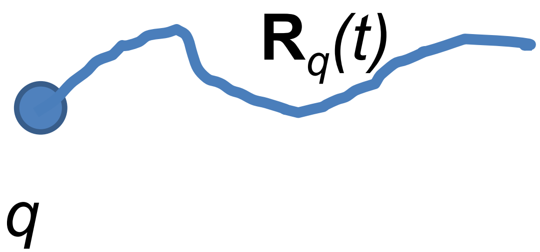
$$\nabla^2\mathbf{A} - \mu\varepsilon\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi\mu}{c}\mathbf{J}$$

Here the values of μ and ε depend on the material and on frequency.

Source: charged particle moving on trajectory $\mathbf{R}_q(t)$:

$$\rho(\mathbf{r},t) = q\delta(\mathbf{r} - \mathbf{R}_q(t))$$

$$\mathbf{J}(\mathbf{r},t) = q\dot{\mathbf{R}}_q(t)\delta(\mathbf{r} - \mathbf{R}_q(t))$$



Liénard-Wiechert potential solutions:

$$\Phi(\mathbf{r},t) = \frac{q}{\varepsilon} \frac{1}{\left| R(t_r) - \boldsymbol{\beta}_n \cdot \mathbf{R}(t_r) \right|}$$

$$\mathbf{A}(\mathbf{r},t) = q\mu \frac{\boldsymbol{\beta}_n}{\left| R(t_r) - \boldsymbol{\beta}_n \cdot \mathbf{R}(t_r) \right|}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r)$$

$$\boldsymbol{\beta}_n(t_r) \equiv \frac{\dot{\mathbf{R}}_q(t_r)}{c_n} \qquad c_n \equiv \frac{c}{\sqrt{\mu\varepsilon}} \equiv \frac{c}{n}$$

$$t_r = t - \frac{R(t_r)}{c_n}$$

Example --

$$\beta_n \equiv \frac{v}{c_n} \qquad c_n \equiv \frac{c}{\sqrt{\mu\varepsilon}} \equiv \frac{c}{n}$$

Consider water with $n \approx 1.3$

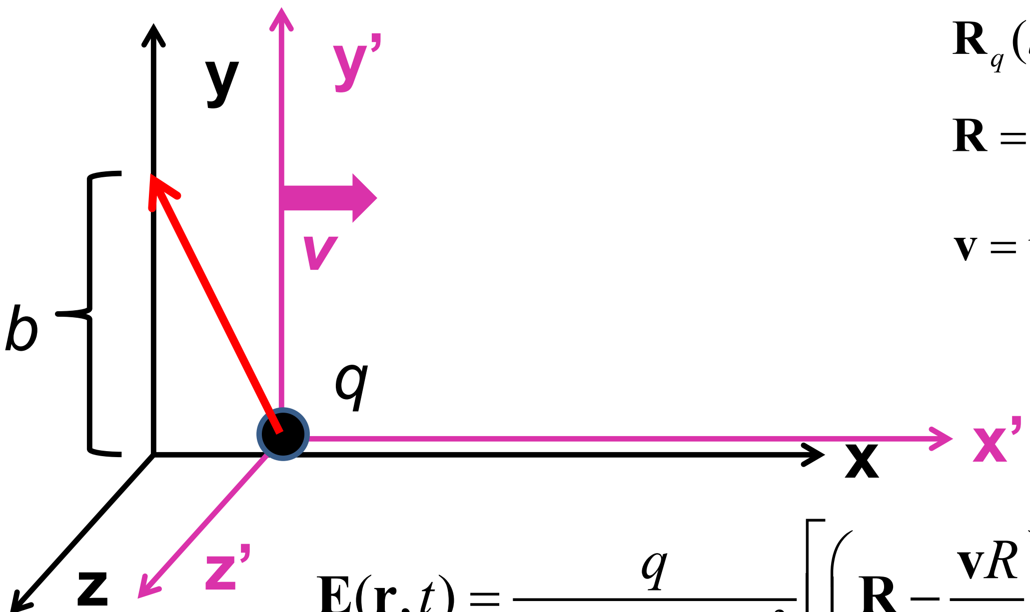
Which of these particles could produce Cherenkov radiation?

1. A neutron with speed c?
2. An electron with speed 0.6c?
3. A proton with speed 0.6c?
4. An electron with speed 0.8c?
5. An alpha particle with speed 0.8c?
6. None of these?

Further comment –

As discussed particularly in Chap. 13 of Jackson, a particle moving within a medium is likely to be slowed down so that the Cherenkov effect will only happen while $\beta_n > 1$.

Recall – in Lecture 26, we considered a particle moving at constant velocity v in vacuum:



$$\mathbf{R}_q(t_r) = vt_r\hat{\mathbf{x}} \qquad \mathbf{r} = b\hat{\mathbf{y}}$$

$$\mathbf{R} = b\hat{\mathbf{y}} - vt_r\hat{\mathbf{x}} \qquad R = \sqrt{v^2 t_r^2 + b^2}$$

$$\mathbf{v} = v\hat{\mathbf{x}} \qquad t_r = t - \frac{R}{c}$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right)\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2}\right)\right]$$

Some details

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R}-\frac{\mathbf{v}R}{c}\right)\left(1-\frac{v^2}{c^2}\right)\right]$$

$$\mathbf{B}(\mathbf{r},t)=\frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1-\frac{v^2}{c^2}\right)\right]$$

For our example:

$$\mathbf{R}_q(t_r)=vt_r\hat{\mathbf{x}} \qquad \mathbf{r}=b\hat{\mathbf{y}}$$

$$\mathbf{R}=b\hat{\mathbf{y}}-vt_r\hat{\mathbf{x}} \qquad R=\sqrt{v^2t_r^2+b^2}$$

$$\mathbf{v}=v\hat{\mathbf{x}} \qquad t_r=t-\frac{R}{c}$$

t_r must be a solution to a quadratic equation: where $\frac{v}{c}\leq 1;\ \ \gamma=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

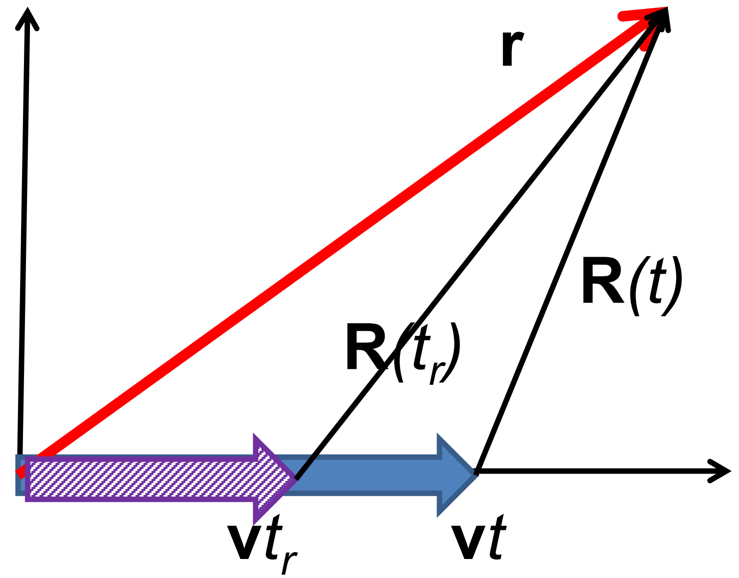
$$t_r-t=-\frac{R}{c} \qquad \Rightarrow \qquad t_r^2-2\gamma^2 tt_r+\gamma^2t^2-\gamma^2b^2/c^2=0$$

with the physical solution:

$$t_r=\gamma\left(\gamma t-\sqrt{(\gamma^2-1)t^2+b^2/c^2}\right)=\gamma\left(\gamma t-\frac{\sqrt{(v\gamma t)^2+b^2}}{c}\right)$$

For Cherenkov case --

Consider a particle moving at constant velocity **v**; $v > c_n$



Some algebra

$$\mathbf{R}(t) = \mathbf{r} - \mathbf{v}t$$

$$\mathbf{R}(t_r) = \mathbf{r} - \mathbf{v}t_r = \mathbf{R}(t) + \mathbf{v}(t - t_r)$$

$$(t - t_r)c_n = R(t_r) = \left|\mathbf{R}(t) + \mathbf{v}(t - t_r)\right|$$

Quadratic equation for $(t - t_r)c_n$:

$$\left((t - t_r)c_n\right)^2 = R^2(t) + 2\mathbf{R}(t)\cdot\boldsymbol{\beta}_n (t - t_r)c_n + \beta_n^{\,2}\left((t - t_r)c_n\right)^2$$

$$\left(\beta_n^{\,2} - 1\right)\left((t - t_r)c_n\right)^2 + 2\mathbf{R}(t)\cdot\boldsymbol{\beta}_n (t - t_r)c_n + R^2(t) = 0$$

Quadratic equation for $(t-t_r)c_n$:

$$\left(\beta_n^{\ 2}-1\right)\left((t-t_r)c_n\right)^2+2\mathbf{R}(t)\cdot\boldsymbol{\beta}_n(t-t_r)c_n+R^2(t)=0$$

For $\beta_n>1$, how can the equality be satisfied?

1. No problem
2. It cannot be satisfied.
3. It can only be satisfied for special conditions

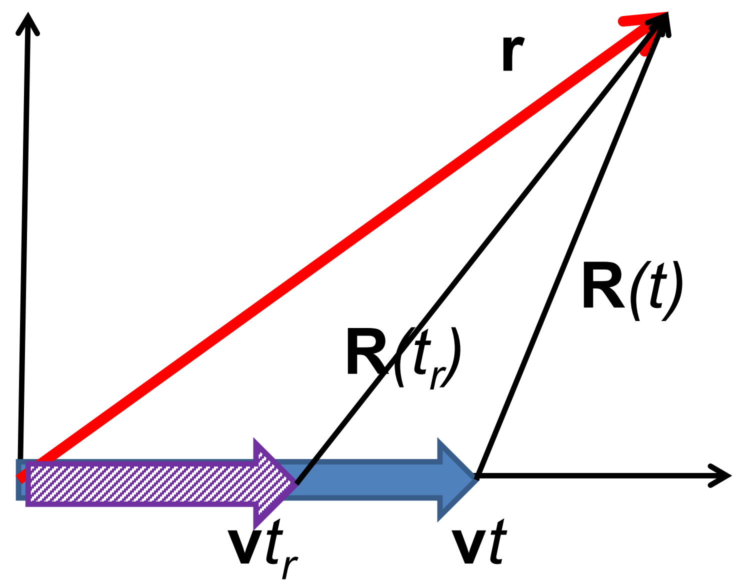
From solution of quadratic equation:

$$(t-t_r)c_n=\frac{-\mathbf{R}(t)\cdot\boldsymbol{\beta}_n\pm\sqrt{\left(\mathbf{R}(t)\cdot\boldsymbol{\beta}_n\right)^2-\left(\beta_n^{\ 2}-1\right)R^2(t)}}{\beta_n^{\ 2}-1}$$

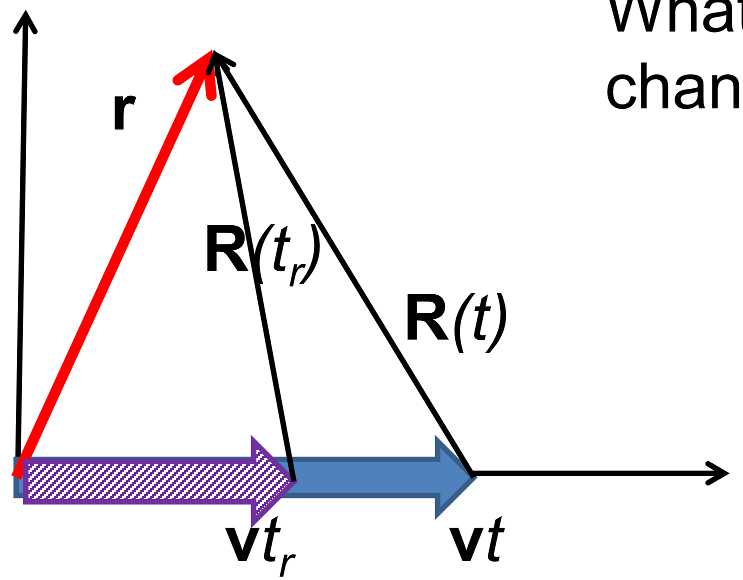
$\Rightarrow \mathbf{R}(t)\cdot\boldsymbol{\beta}_n<0$ (initial diagram is incorrect!)

Moreover, there are two retarded time solutions!

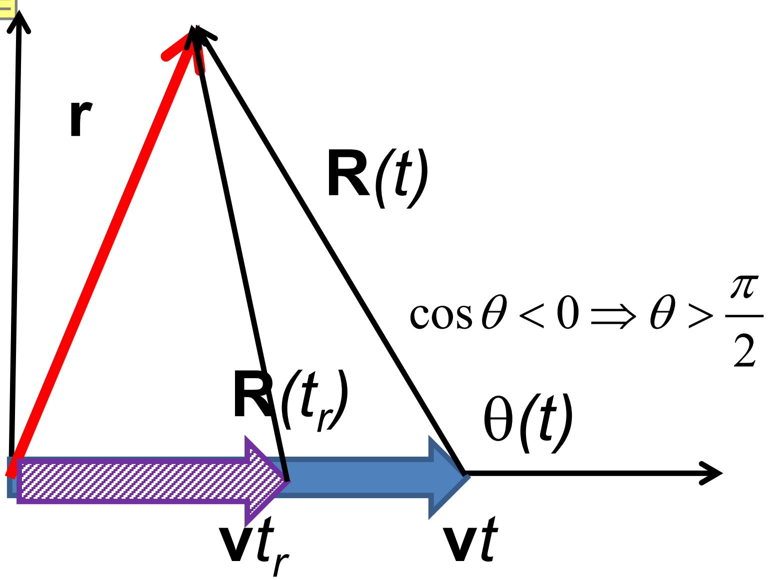
Original diagram:



New diagram:



What is the significance of changing the diagram?



$$\mathbf{R}(t_r) = \mathbf{r} - \mathbf{v}t_r = \mathbf{R}(t) + \mathbf{v}(t - t_r)$$

$$(t - t_r)c_n = R(t_r)$$

$$R(t_r) - \mathbf{R}(t_r)\cdot\boldsymbol{\beta}_n =$$

$$(t - t_r)c_n(1 - \beta_n^2) - \mathbf{R}(t)\cdot\boldsymbol{\beta}_n$$

$$= R(t_r)(1 - \beta_n^2) - \mathbf{R}(t)\cdot\boldsymbol{\beta}_n$$

$$R(t_r) = \frac{-\mathbf{R}(t)\cdot\boldsymbol{\beta}_n \pm \sqrt{(\mathbf{R}(t)\cdot\boldsymbol{\beta}_n)^2 - (\beta_n{}^2 - 1)R^2(t)}}{\beta_n{}^2 - 1}$$

$$R(t_r) = \frac{R(t)}{\beta_n{}^2 - 1}\left(-\beta_n \cos\theta \pm \sqrt{1 - \beta_n{}^2 \sin^2\theta}\right) = (t - t_r)c_n$$

$$R(t_r) - \mathbf{R}(t_r)\cdot\boldsymbol{\beta}_n = \mp R(t)\sqrt{1 - \beta_n{}^2 \sin^2\theta}$$

Recall the Liénard-Wiechert potential solutions:

$$\Phi(\mathbf{r},t) = \frac{q}{\varepsilon} \frac{1}{\left| R(t_r) - \boldsymbol{\beta}_n \cdot \mathbf{R}(t_r) \right|}$$

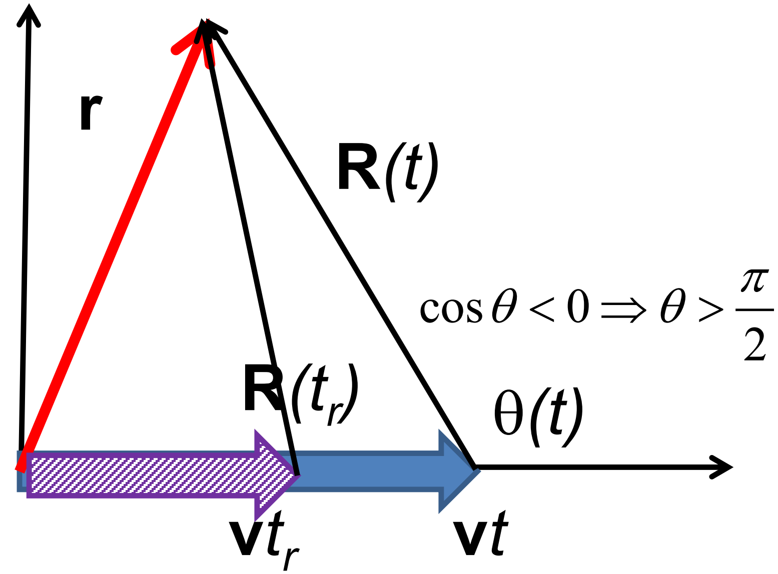
$$\mathbf{A}(\mathbf{r},t) = q\mu \frac{\boldsymbol{\beta}_n}{\left| R(t_r) - \boldsymbol{\beta}_n \cdot \mathbf{R}(t_r) \right|}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r)$$

$$\boldsymbol{\beta}_n(t_r) \equiv \frac{\dot{\mathbf{R}}_q(t_r)}{c_n} \qquad c_n \equiv \frac{c}{\sqrt{\mu\varepsilon}} \equiv \frac{c}{n}$$

$$t_r = t - \frac{R(t_r)}{c_n}$$

Liénard-Wiechert potentials for two solutions:



$$\Phi(\mathbf{r},t) = \frac{q}{\varepsilon}\frac{1}{\left|\mp R(t)\sqrt{1-\beta_n{}^2\sin^2\theta}\right|}$$

$$\mathbf{A}(\mathbf{r},t) = q\mu\frac{\boldsymbol{\beta}_n}{\left|\mp R(t)\sqrt{1-\beta_n{}^2\sin^2\theta}\right|}$$

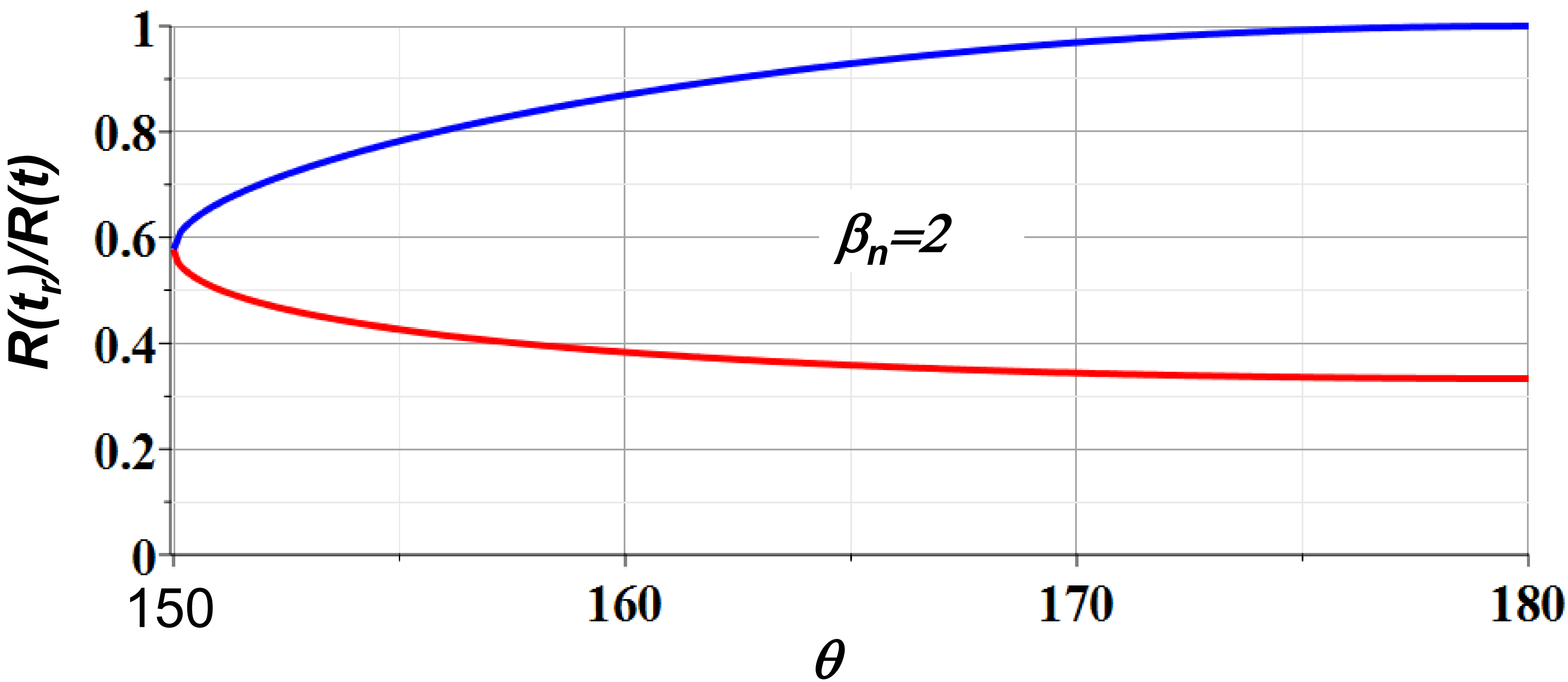
For $\beta_n > 1$, the range of θ is limited further:

$$R(t_r) = \frac{R(t)}{\beta_n{}^2 - 1}\left(-\beta_n\cos\theta \pm \sqrt{1-\beta_n{}^2\sin^2\theta}\right) \geq 0$$

$$\Rightarrow \left|\sin\theta\right| \leq \frac{1}{\beta_n} \equiv \left|\sin\theta_c\right| \quad \text{and} \quad \pi \geq \theta_c \geq \pi/2 \qquad \cos\,\theta_c = -\sqrt{1-\frac{1}{\beta_n^2}}$$

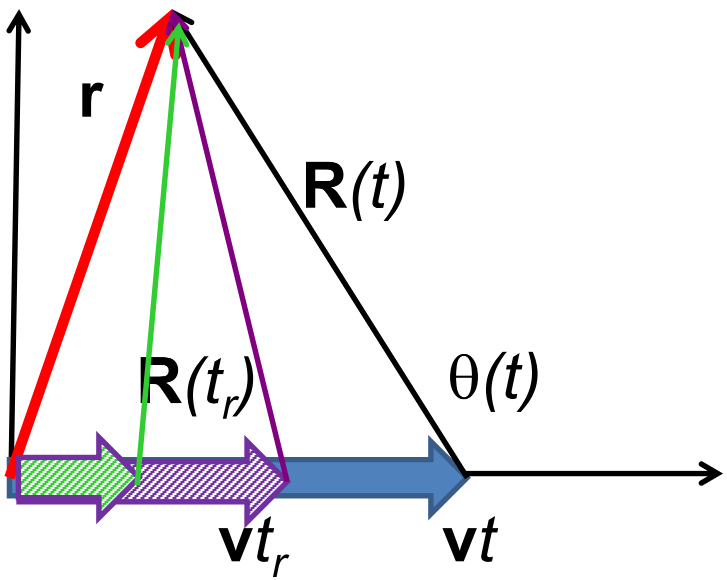
In this range, $\theta \geq \theta_c$

$$R(t_r) = \frac{R(t)}{\beta_n^{\,2} - 1}\left(-\beta_n \cos\theta \pm \sqrt{1 - \beta_n^{\,2}\sin^2\theta}\right)$$



$\theta_c = 150^o$ for this case

Physical fields for $\beta_n > 1$ -- two retarded solutions contribute



$$\theta \leq \sin^{-1}\left(\frac{1}{\beta_n}\right)$$

$$\text{Define} \quad \cos\theta_C \equiv -\sqrt{1-\frac{1}{\beta_n{}^2}}$$

$$\Rightarrow \cos\theta \leq \cos\theta_C$$

Adding two solutions; in terms of Heaviside $\Theta(x)$:

$$\Phi(\mathbf{r},t) = \frac{2q}{\varepsilon}\frac{1}{R(t)\sqrt{1-\beta_n{}^2\sin^2\theta}}\Theta(\cos\theta_C - \cos\theta(t))$$

$$\mathbf{A}(\mathbf{r},t) = 2q\mu\frac{\boldsymbol{\beta}_n}{R(t)\sqrt{1-\beta_n{}^2\sin^2\theta}}\Theta(\cos\theta_C - \cos\theta(t))$$

Physical fields for $\beta_n > 1$

$$\Phi(\mathbf{r},t) = \frac{2q}{\varepsilon} \frac{1}{R(t)\sqrt{1-\beta_n{}^2 \sin^2\theta}} \Theta(\cos\theta_C - \cos\theta(t))$$

$$\mathbf{A}(\mathbf{r},t) = 2q\mu \frac{\boldsymbol{\beta}_n}{R(t)\sqrt{1-\beta_n{}^2 \sin^2\theta}} \Theta(\cos\theta_C - \cos\theta(t))$$

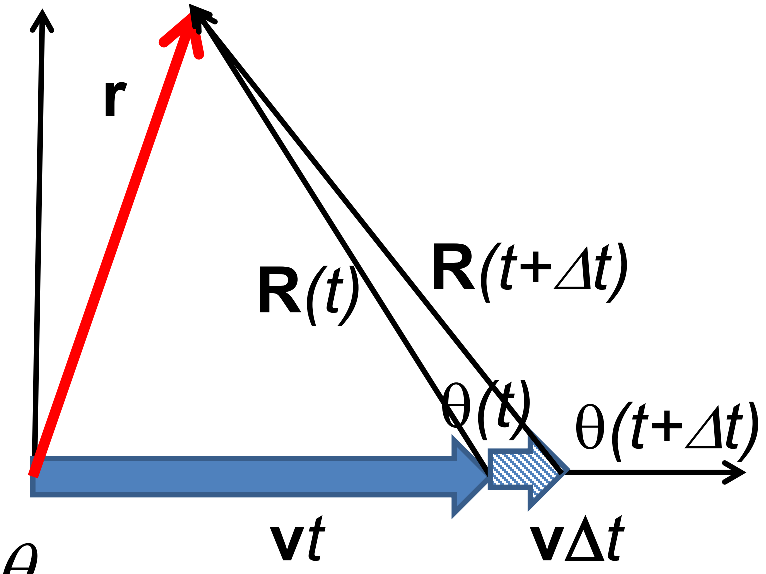
$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi - \frac{1}{c_n}\frac{\partial \mathbf{A}}{\partial t} \qquad \mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}$$

$$\mathbf{E}(\mathbf{r},t) = \frac{2q}{\varepsilon} \frac{\hat{\mathbf{R}}}{(R(t))^2 \sqrt{1-\beta_n{}^2 \sin^2\theta}} \quad \times$$

$$\left(-\frac{\beta_n{}^2 - 1}{1-\beta_n{}^2 \sin^2\theta} \Theta(\cos\theta_C - \cos\theta(t)) + \sqrt{\beta_n{}^2 - 1}\; \delta(\cos\theta_C - \cos\theta(t)) \right)$$

$$\mathbf{B}(\mathbf{r},t) = -\beta_n \sin\theta \left(\hat{\theta} \times \mathbf{E}(\mathbf{r},t) \right)$$

Intermediate steps:



$$\frac{d\theta}{dt} = \frac{v\sin\theta}{R} \qquad \frac{dR}{dt} = -v\cos\theta$$

Using instantaneous polar coordinates: $\nabla \equiv \hat{\mathbf{R}}\frac{\partial}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial}{\partial \theta}$

$$\nabla\Theta\left(\cos\theta_C - \cos\theta(t)\right) = \delta\left(\cos\theta_C - \cos\theta(t)\right)\frac{\sin\theta(t)}{R(t)}\hat{\boldsymbol{\theta}}$$

$$\frac{\partial\Theta\left(\cos\theta_C - \cos\theta(t)\right)}{\partial t} = \delta\left(\cos\theta_C - \cos\theta(t)\right)\frac{v\sin^2\theta(t)}{R(t)}$$

Power radiated:

$$\frac{dP(t)}{d\Omega} = \left(R(t)\right)^2 \hat{\mathbf{R}} \cdot \mathbf{S}(t) = \left(R(t)\right)^2 \frac{c_n}{4\pi} \left|\mathbf{E}\left(\mathbf{R}(t), t\right)\right|^2 = \left|\boldsymbol{a}(t)\right|^2$$

where

$$\mathbf{E}(\mathbf{r},t) = \frac{2q}{\varepsilon} \frac{\hat{\mathbf{R}}}{\left(R(t)\right)^2 \sqrt{1-\beta_n{}^2 \sin^2\theta}} \quad \times$$

$$\left(\frac{\beta_n{}^2 - 1}{1-\beta_n{}^2 \sin^2\theta} \Theta(\cos\theta_C - \cos\theta(t)) + \sqrt{\beta_n{}^2 - 1}\ \delta(\cos\theta_C - \cos\theta(t)) \right)$$

Spectral analysis using Parseval's theorem:

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} \left|\boldsymbol{a}(t)\right|^2 dt = \int_{-\infty}^{\infty} \left|\tilde{\boldsymbol{a}}(\omega)\right|^2 d\omega = \int_0^{\infty} \left(\left|\tilde{\boldsymbol{a}}(\omega)\right|^2 + \left|\tilde{\boldsymbol{a}}(-\omega)\right|^2 \right) d\omega = \int_0^{\infty} \frac{\partial^2 I}{\partial\Omega\partial\omega} d\omega$$

$$\tilde{\boldsymbol{a}}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\ \boldsymbol{a}(t)\ e^{i\omega t}$$

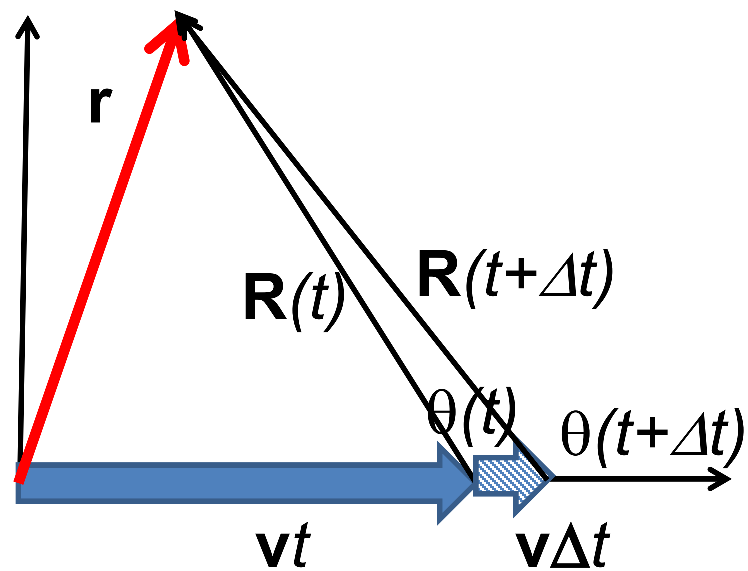
$$\boldsymbol{a}(t) = \frac{K}{R(t)\sqrt{1-\beta_n^2\sin^2\theta}}\left(-\frac{\beta_n^2-1}{1-\beta_n^2\sin^2\theta}\Theta(\cos\theta_C - \cos\theta(t)) + \sqrt{\beta_n^2-1}\;\delta(\cos\theta_C - \cos\theta(t))\right)$$

Denote $t = 0$ corresponding the angle θ_c

$$\theta(t) = \theta_c + \Delta\theta(t) \text{ where } \Delta\theta(t) \approx vt\frac{\sin\theta_c}{R(0)}$$

$$\cos\theta_c - \cos\theta(t) \approx \frac{c_n t}{\beta_n R(0)}$$

$$1-\beta_n^2\sin^2\theta(t) \approx -\frac{2c_n t\sqrt{\beta_n^2-1}}{R(0)}$$



Approximate amplitude near $t \approx 0$:

$$\boldsymbol{a}(t) \approx K\frac{\left(\beta_n^2-1\right)^{1/4}}{(2c_n)^{3/2}\sqrt{R(0)}}\left(\frac{\delta(t)}{\sqrt{t}} - \frac{\Theta(t)}{2\sqrt{t^3}}\right)$$

Approximate Fourier amplitude:

$$\tilde{\boldsymbol{a}}(\omega) \approx K\sqrt{\frac{\pi}{2}}\frac{\left(\beta_n^2-1\right)^{1/4}(1-i)}{(c_n)^{3/2}\sqrt{R(0)}}\sqrt{\omega}$$

Noting that $\beta_n = \frac{c}{n(\omega)} = \frac{c}{\sqrt{\epsilon(\omega)}}$

$$\frac{\partial^2 I}{\partial\Omega\partial\omega} \propto \omega\left(1 - \frac{c^2}{v^2\epsilon(\omega)}\right)$$

When the dust clears --

Frequency dependence of intensity:

$$\frac{dI}{d\omega} \approx \frac{q^2}{c^2}\omega\left(1 - \frac{1}{\beta^2\epsilon(\omega)}\right)$$

From this expression, how would you explain that Cherenkov radiation is typically observed as a blue glow?

1. It is still a mystery.
2. It is obvious from the result.