

# **PHY 712 Electrodynamics**

**11-11:50 AM MWF Olin 103**

## **Class notes for Lecture 3:**

**Reading: Chapter 1 (especially 1.11) in JDJ;**

- 1. Continued discussion/derivation of Ewald summation methods**
- 2. Example for CsCl**

# PHY 712 Electrodynamics

MWF 11-11:50 AM Olin 103 Webpage: <http://www.wfu.edu/~natalie/s22phy712/>

Instructor: [Natalie Holzwarth](#) Office: 300 OPL e-mail: [natalie@wfu.edu](mailto:natalie@wfu.edu)

## Course schedule for Spring 2022

(Preliminary schedule -- subject to frequent adjustment.)

|   | Lecture date    | JDJ Reading      | Topic                                    | HW                 | Due date   |
|---|-----------------|------------------|------------------------------------------|--------------------|------------|
| 1 | Mon: 01/10/2022 | Chap. 1 & Appen. | Introduction, units and Poisson equation | <a href="#">#1</a> | 01/14/2022 |
| 2 | Wed: 01/12/2022 | Chap. 1          | Electrostatic energy calculations        | <a href="#">#2</a> | 01/19/2022 |
| 3 | Fri: 01/14/2022 | Chap. 1          | Electrostatic energy calculations        | <a href="#">#3</a> | 01/21/2022 |
|   | Mon: 01/17/2022 |                  | MLK Holiday -- no class                  |                    |            |
| 4 | Wed: 01/19/2022 | Chap. 1 & 2      | Electrostatic potentials and fields      |                    |            |

# PHY 712 -- Assignment #3

January 14, 2022

Continue reading Chap. 1 in **Jackson**.

1. Using the Ewald summation methods developed in class, find the electrostatic interaction energy of a NaCl lattice having a cubic lattice constant  $a$ . Check that your result does not depend of the Ewald parameter  $\eta$ . You are welcome to copy (and modify) the maple file used in class. A FORTRAN code is also available upon request.

The details are presented in the pdf file – Extranotes2.pdf

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## Lecture Notes

- Lecture 1 -- Introduction and electrostatics [PP slides](#) [PDF](#) [Extra PP slides](#) [Extra PDF](#)
- Lecture 2 -- Electrostatic energy [PP slides](#) [PDF](#) [Extra PP slides](#) [Extra PDF](#) [Detailed PDF](#)
- Lecture 3 -- Ewald summation methods [PP slides](#) [PDF](#) [CsCl Maple](#) [CsCl PDF](#)

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[Return to main web page for PHY 712](#)

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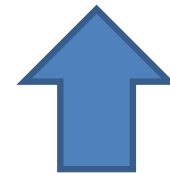
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Last time, we argued that we need to carry out the following summation of all pairs of ions using Ewald's very clever trick with the erf and erfc functions:

$$W = \frac{1}{8\pi\epsilon_0} \sum_{i \neq j} \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} = \frac{1}{8\pi\epsilon_0} \left( \sum_{i \neq j} \frac{q_i q_j \operatorname{erf}(\frac{1}{2}\sqrt{\eta}|\mathbf{r}_i - \mathbf{r}_j|)}{|\mathbf{r}_i - \mathbf{r}_j|} + \sum_{i \neq j} \frac{q_i q_j \operatorname{erfc}(\frac{1}{2}\sqrt{\eta}|\mathbf{r}_i - \mathbf{r}_j|)}{|\mathbf{r}_i - \mathbf{r}_j|} \right)$$



Must be evaluated in  
“reciprocal” space



Can be converged  
as is (in real space)

In order to make progress, we need to systematically enumerate all of the ion positions in the system

Using CsCl as our example system

● Cs

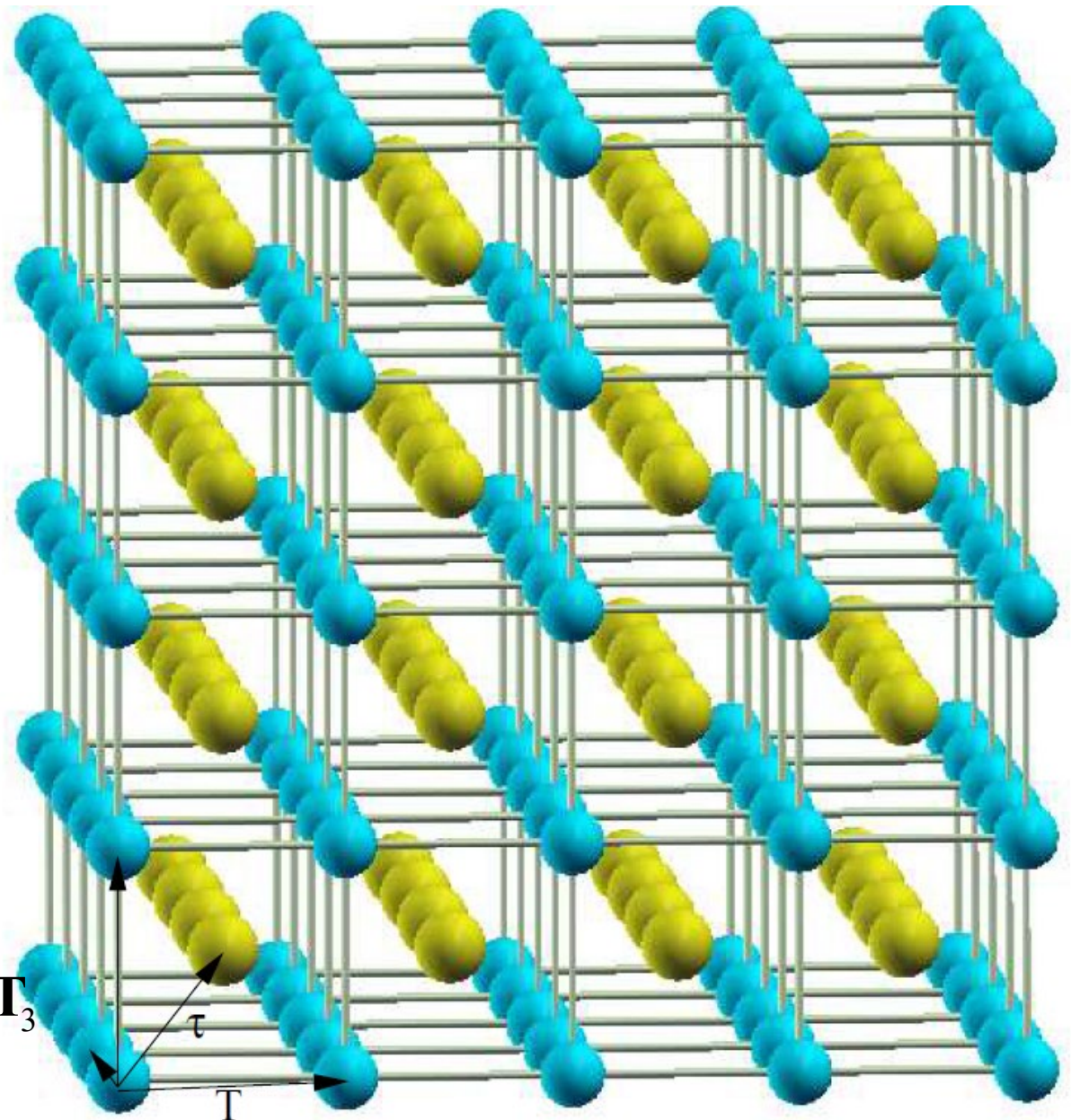
● Cl

Notice that a unit cell contains one Cl and one Cs

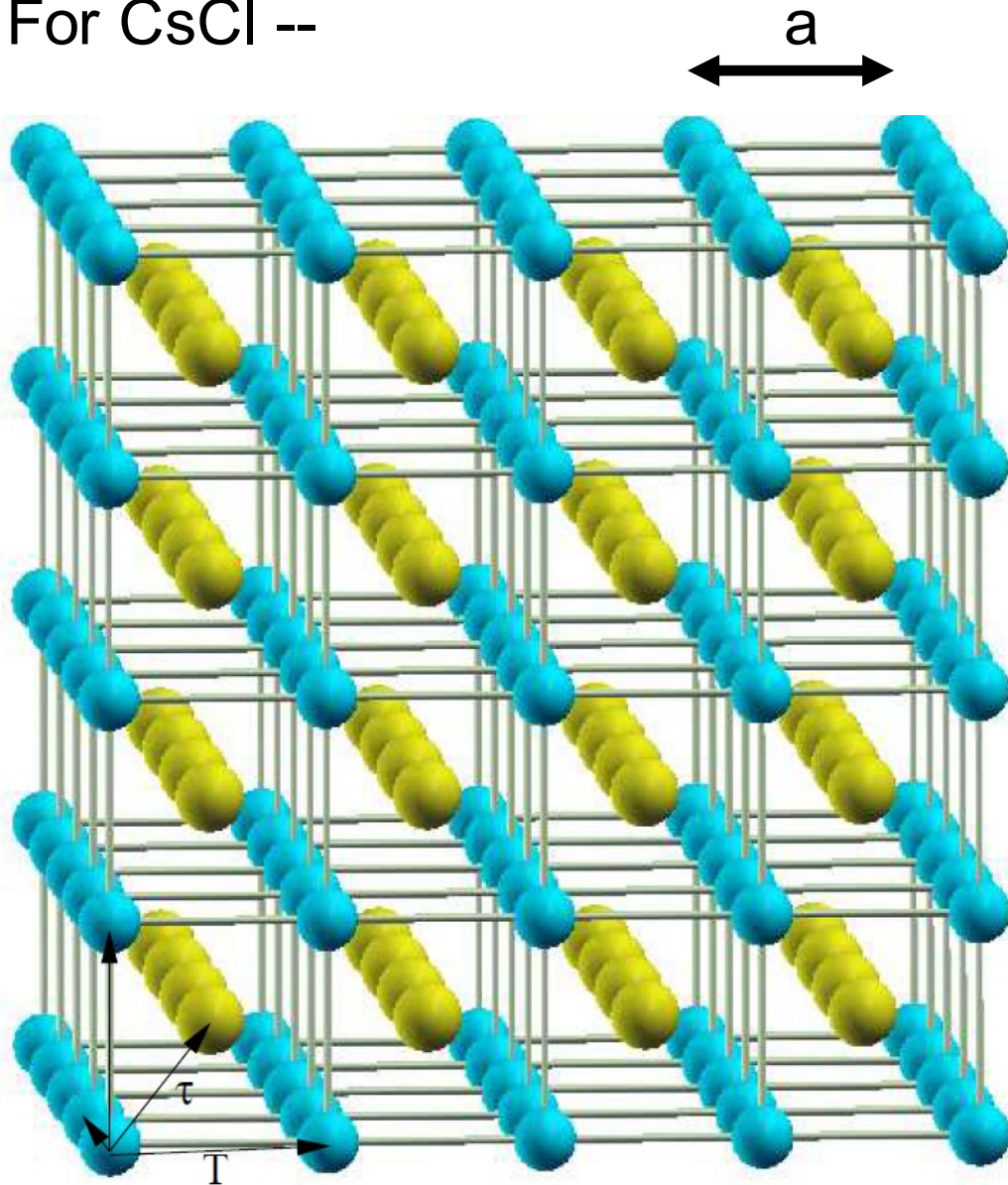
$$\mathbf{r}_i = \boldsymbol{\tau}_\alpha + \mathbf{T}$$

for  $\alpha = \text{Cl or Cs}$

$$\text{and } \mathbf{T} = n_1 \mathbf{T}_1 + n_2 \mathbf{T}_2 + n_3 \mathbf{T}_3$$



For CsCl --



$$\tau_{\text{Cs}} = 0 \text{ and } \tau_{\text{Cl}} = \frac{a}{2}(\hat{x} + \hat{y} + \hat{z}).$$

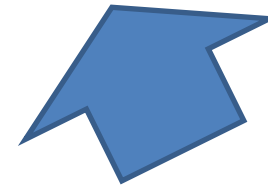
$$\mathbf{T}_1 = a\hat{x} \quad \mathbf{T}_2 = a\hat{y} \quad \mathbf{T}_3 = a\hat{z}.$$

Note that in general --

$$\sum_{ij} = N \sum_{\alpha\beta\mathbf{T}}$$

Note that, using  $\mathbf{r}_i = \boldsymbol{\tau}_\alpha + \mathbf{T}$

$$W = \frac{1}{8\pi\epsilon_0} \sum_{i \neq j} \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} = \frac{N}{8\pi\epsilon_0} \sum_{\alpha, \beta, \mathbf{T}} \frac{q_\alpha q_\beta}{|\boldsymbol{\tau}_{\alpha\beta} + \mathbf{T}|}$$

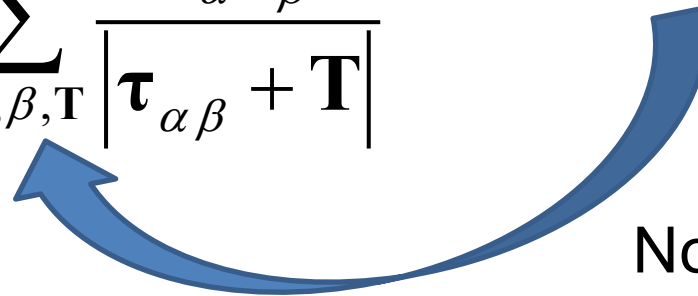


omit term  $\alpha = \beta$

when  $\mathbf{T}=0$

$$\boldsymbol{\tau}_{\alpha\beta} \equiv \boldsymbol{\tau}_\alpha - \boldsymbol{\tau}_\beta$$

$$\frac{W}{N} = \frac{1}{8\pi\epsilon_0} \sum_{\alpha, \beta, \mathbf{T}} \frac{q_\alpha q_\beta}{|\boldsymbol{\tau}_{\alpha\beta} + \mathbf{T}|}$$



Note – this works for CsCl  
and more generally

$$\frac{W}{N} = \frac{1}{8\pi\epsilon_0} \sum_{\alpha, \beta, \mathbf{T}} \frac{q_\alpha q_\beta}{|\boldsymbol{\tau}_{\alpha\beta} + \mathbf{T}|} = w_1 + w_2$$

$$w_2 \equiv \frac{1}{8\pi\epsilon_0} \sum_{\alpha, \beta, \mathbf{T}} \frac{q_\alpha q_\beta \operatorname{erfc}\left(\frac{1}{2}\sqrt{\eta}|\boldsymbol{\tau}_{\alpha\beta} + \mathbf{T}|\right)}{|\boldsymbol{\tau}_{\alpha\beta} + \mathbf{T}|} \equiv \frac{e^2}{8\pi\epsilon_0} \mathcal{J}_2$$

For CsCl:

$$\tau_{\text{Cs}} = 0 \text{ and } \tau_{\text{Cl}} = \frac{a}{2}(\hat{x} + \hat{y} + \hat{z}).$$

$$q_{\text{Cs}} = e$$

$$q_{\text{Cl}} = -e$$

$$\mathbf{T}_1 = a\hat{x} \quad \mathbf{T}_2 = a\hat{y} \quad \mathbf{T}_3 = a\hat{z}.$$

$$\mathcal{J}_2 \equiv \sum_{\mathbf{T} \neq \mathbf{0}} 2 \frac{\operatorname{erfc}(\frac{1}{2}\sqrt{\eta}|\mathbf{T}|)}{|\mathbf{T}|} - \sum_{\mathbf{T}} 2 \frac{\operatorname{erfc}(\frac{1}{2}\sqrt{\eta}|\tau_{\text{Cl}} + \mathbf{T}|)}{|\tau_{\text{Cl}} + \mathbf{T}|}.$$

For evaluating the summation in reciprocal space, additional considerations are needed --

It can be shown that --

$$\sum_{\mathbf{T}} \delta^3(\mathbf{r} - \mathbf{T}) = \frac{1}{\Omega} \sum_{\mathbf{G}} e^{i\mathbf{G} \cdot \mathbf{r}},$$



Volume of unit cell  
in real space

In this discussion, will assume we have a 3-dimensional periodic system. It can be easily generalized to 1- or 2- dimensional systems. In general, a translation vector can be described a linear combination of the three primitive translation vectors  $\mathbf{T}_1$ ,  $\mathbf{T}_2$ , and  $\mathbf{T}_3$ :

$$\mathbf{T} = n_1 \mathbf{T}_1 + n_2 \mathbf{T}_2 + n_3 \mathbf{T}_3, \quad (12)$$

where  $\{n_1, n_2, n_3\}$  are integers. Note that the unit cell volume  $\Omega$  can be expressed in terms of the primitive translation vectors according to:

$$\Omega = |\mathbf{T}_1 \cdot (\mathbf{T}_2 \times \mathbf{T}_3)|. \quad (13)$$

The reciprocal lattice vectors  $\mathbf{G}$  can generally be written as a linear combination of the three primitive reciprocal lattice vectors  $\mathbf{G}_1$ ,  $\mathbf{G}_2$ , and  $\mathbf{G}_3$ :

$$\mathbf{G} = m_1 \mathbf{G}_1 + m_2 \mathbf{G}_2 + m_3 \mathbf{G}_3, \quad (14)$$

where  $\{m_1, m_2, m_3\}$  are integers. The primitive reciprocal lattice vectors are determined from the primitive translation vectors according to the identities:

$$\mathbf{G}_i \cdot \mathbf{T}_j = 2\pi \delta_{ij}. \quad (15)$$

Note that the “volume” of the primitive reciprocal lattice is given by

$$|\mathbf{G}_1 \cdot (\mathbf{G}_2 \times \mathbf{G}_3)| = \frac{(2\pi)^3}{\Omega}. \quad (16)$$

### CsCl structure

There are two kinds of sites –  $\tau_{\text{Cs}} = 0$  and  $\tau_{\text{Cl}} = \frac{a}{2}(\hat{x} + \hat{y} + \hat{z})$ .

### Example --

$$\mathbf{T}_1 = a\hat{x} \quad \mathbf{T}_2 = a\hat{y} \quad \mathbf{T}_3 = a\hat{z}.$$

$$\mathbf{G}_1 = \frac{2\pi}{a}\hat{x} \quad \mathbf{G}_2 = \frac{2\pi}{a}\hat{y} \quad \mathbf{G}_3 = \frac{2\pi}{a}\hat{z}.$$

## “Proof” of identity --

Consider the geometric series

$$\sum_{k=-M}^{+M} e^{ik(\mathbf{G}_1 \cdot \mathbf{r})} = \frac{\sin\left((M + \frac{1}{2})\mathbf{G}_1 \cdot \mathbf{r}\right)}{\sin(\mathbf{G}_1 \cdot \mathbf{r}/2)}. \quad (17)$$

The behavior of the right hand side of Eq. (17) is that it is small in magnitude very except when the denominator vanishes. This occurs whenever  $\mathbf{G}_1 \cdot \mathbf{r}/2 = n_1\pi$ , where  $n_1$  represents any integer. If we take the limit  $M \rightarrow \infty$ , we find that the function represents the behavior of a sum of delta functions:

$$\lim_{M \rightarrow \infty} \frac{\sin\left((M + \frac{1}{2})\mathbf{G}_1 \cdot \mathbf{r}\right)}{\sin(\mathbf{G}_1 \cdot \mathbf{r}/2)} = 2\pi \sum_{n_1} \delta(\mathbf{G}_1 \cdot (\mathbf{r} - n_1 \mathbf{T}_1)). \quad (18)$$

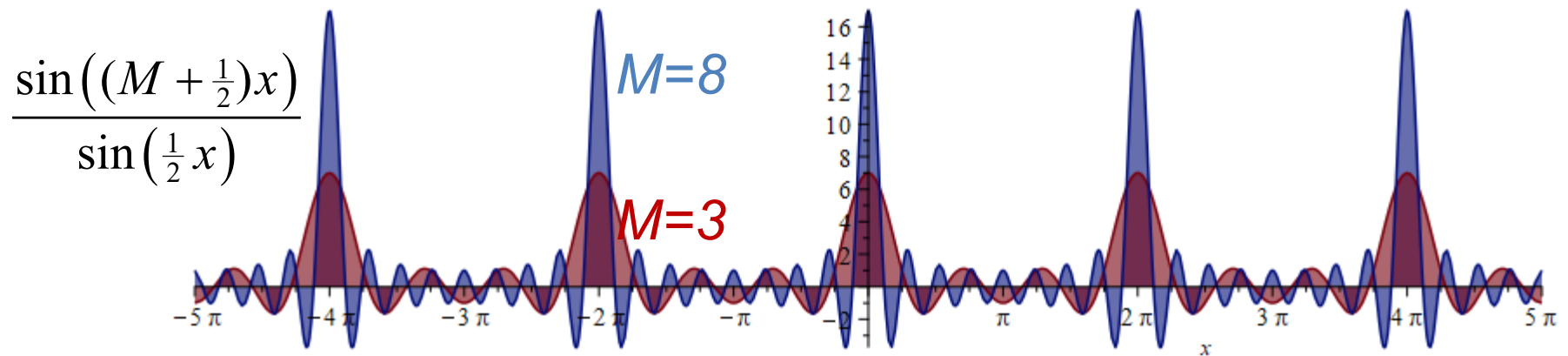
The summation over all lattice translations  $n_1 \mathbf{T}_1$  is due to the fact that  $\sin(\mathbf{G}_1 \cdot \mathbf{r}/2) = 0$  whenever  $\mathbf{r} = n_1 \mathbf{T}_1$ . Carrying out the geometric summations in the right hand side of Eq. 7 for all three reciprocal lattice vectors and taking the limit as in Eq. 18,

$$\sum_{\mathbf{G}} e^{i\mathbf{G} \cdot \mathbf{r}} = (2\pi)^3 \sum_{n_1, n_2, n_3} \delta(\mathbf{G}_1 \cdot (\mathbf{r} - n_1 \mathbf{T}_1)) \delta(\mathbf{G}_2 \cdot (\mathbf{r} - n_2 \mathbf{T}_2)) \delta(\mathbf{G}_3 \cdot (\mathbf{r} - n_3 \mathbf{T}_3)). \quad (19)$$

Some details --

$$\sum_{k=-M}^{+M} e^{ik(\mathbf{G}_1 \cdot \mathbf{r})} = \frac{\sin\left((M + \frac{1}{2})\mathbf{G}_1 \cdot \mathbf{r}\right)}{\sin(\mathbf{G}_1 \cdot \mathbf{r}/2)}.$$

Behavior of function for various values of  $M$ :



Final result --

$$\sum_{\mathbf{G}} e^{i\mathbf{G}\cdot\mathbf{r}} = \frac{(2\pi)^3}{|\mathbf{G}_1 \cdot (\mathbf{G}_2 \times \mathbf{G}_3)|} \sum_{\mathbf{T}} \delta^3(\mathbf{r} - \mathbf{T}) = \Omega \sum_{\mathbf{T}} \delta^3(\mathbf{r} - \mathbf{T})$$

$$\sum_{\mathbf{T}} \delta^3(\mathbf{r} - \mathbf{T}) = \frac{1}{\Omega} \sum_{\mathbf{G}} e^{i\mathbf{G}\cdot\mathbf{r}},$$

$$\frac{W}{N} = \frac{1}{8\pi\epsilon_0} \sum_{\alpha,\beta,\mathbf{T}} \frac{q_\alpha q_\beta}{|\boldsymbol{\tau}_{\alpha\beta} + \mathbf{T}|} = w_1 + w_2$$

$$w_1 \equiv \frac{1}{8\pi\epsilon_0} \sum_{\alpha,\beta,\mathbf{T}} \frac{q_\alpha q_\beta \operatorname{erf}\left(\frac{1}{2}\sqrt{\eta}|\boldsymbol{\tau}_{\alpha\beta} + \mathbf{T}|\right)}{|\boldsymbol{\tau}_{\alpha\beta} + \mathbf{T}|} \equiv \frac{e^2}{8\pi\epsilon_0} \mathcal{J}_1$$

$$\begin{aligned} & \sum_{\alpha,\beta,\mathbf{T}} \frac{q_\alpha q_\beta \operatorname{erf}\left(\frac{1}{2}\sqrt{\eta}|\boldsymbol{\tau}_{\alpha\beta} + \mathbf{T}|\right)}{|\boldsymbol{\tau}_{\alpha\beta} + \mathbf{T}|} \\ &= \sum_{\alpha,\beta} \left( \sum_{\mathbf{T}} \frac{q_\alpha q_\beta \operatorname{erf}\left(\frac{1}{2}\sqrt{\eta}|\boldsymbol{\tau}_{\alpha\beta} + \mathbf{T}|\right)}{|\boldsymbol{\tau}_{\alpha\beta} + \mathbf{T}|} - \delta_{\alpha\beta} q_\alpha^2 \lim_{x \rightarrow 0} \left( \frac{\operatorname{erf}\left(\frac{1}{2}\sqrt{\eta}x\right)}{x} \right) \right) \end{aligned}$$

$$\lim_{x \rightarrow 0} \left( \frac{\operatorname{erf}\left(\frac{1}{2}\sqrt{\eta}x\right)}{x} \right) = \sqrt{\frac{\eta}{\pi}}$$

$$\sum_{\mathbf{T}} \frac{\text{erf}(\frac{1}{2}\sqrt{\eta}|\tau_{\alpha\beta} + \mathbf{T}|)}{|\tau_{\alpha\beta} + \mathbf{T}|} = \int d^3r \sum_{\mathbf{T}} \delta^3(\mathbf{r} - \mathbf{T}) \frac{\text{erf}(\frac{1}{2}\sqrt{\eta}|\tau_{\alpha\beta} + \mathbf{r}|)}{|\tau_{\alpha\beta} + \mathbf{r}|}$$

$$\frac{1}{\Omega} \sum_{\mathbf{G}} \int d^3r e^{i\mathbf{G}\cdot\mathbf{r}} \frac{\text{erf}(\frac{1}{2}\sqrt{\eta}|\tau_{\alpha\beta} + \mathbf{r}|)}{|\tau_{\alpha\beta} + \mathbf{r}|} =$$

$$\frac{4\pi}{\Omega} \left( \sum_{\mathbf{G} \neq \mathbf{0}} \frac{e^{-i\mathbf{G}\cdot\tau_{\alpha\beta}} e^{-G^2/\eta^2}}{G^2} + \frac{1}{2} \int_0^{\frac{1}{2}\sqrt{\eta}} \frac{du}{u^3} \right)$$



divergent term

Derivation includes :

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad + \text{ extreme cleverness}$$

Summary --

$$\begin{aligned}
 & \sum_{\alpha, \beta, \mathbf{T}} \frac{q_{\alpha} q_{\beta} \operatorname{erf}\left(\frac{1}{2} \sqrt{\eta} \left| \boldsymbol{\tau}_{\alpha \beta} + \mathbf{T} \right| \right)}{\left| \boldsymbol{\tau}_{\alpha \beta} + \mathbf{T} \right|} \\
 &= \sum_{\alpha, \beta} \left( \sum_{\mathbf{T}} \frac{q_{\alpha} q_{\beta} \operatorname{erf}\left(\frac{1}{2} \sqrt{\eta} \left| \boldsymbol{\tau}_{\alpha \beta} + \mathbf{T} \right| \right)}{\left| \boldsymbol{\tau}_{\alpha \beta} + \mathbf{T} \right|} - \delta_{\alpha \beta} q_{\alpha}^2 \sqrt{\frac{\eta}{\pi}} \right) \\
 &= \sum_{\alpha, \beta} q_{\alpha} q_{\beta} \left( \frac{4\pi}{\Omega} \sum_{\mathbf{G} \neq 0} \frac{e^{-i\mathbf{G} \cdot \boldsymbol{\tau}_{\alpha \beta}} e^{-G^2/\eta}}{G^2} + \frac{4\pi}{\Omega} \int_0^{\frac{1}{2}\sqrt{\eta}} \frac{du}{u^3} - \delta_{\alpha \beta} q_{\alpha}^2 \sqrt{\frac{\eta}{\pi}} \right)
 \end{aligned}$$



Divergent term

Note that for a neutral system

$\sum_{\alpha, \beta} q_{\alpha} q_{\beta} = 0$  and the divergent term vanishes while the others remain

Problem – Electrostatic energy of a periodic non-neutral system is not defined!

Solution -- Add uniform compensating charge and find the electrostatic energy of periodic neutral system, carefully analyzing diverging terms to determine a convergent expression.

$$\frac{W}{N} = \sum_{\alpha\beta} \frac{q_{\alpha}q_{\beta}}{8\pi\epsilon_0} \left( \frac{4\pi}{\Omega} \sum_{\mathbf{G} \neq 0} \frac{e^{-i\mathbf{G} \cdot \boldsymbol{\tau}_{\alpha\beta}} e^{-G^2/\eta}}{G^2} - \sqrt{\frac{\eta}{\pi}} \delta_{\alpha\beta} + \sum'_{\mathbf{T}} \frac{\text{erfc}(\frac{1}{2}\sqrt{\eta}|\boldsymbol{\tau}_{\alpha\beta} + \mathbf{T}|)}{|\boldsymbol{\tau}_{\alpha\beta} + \mathbf{T}|} \right) - \frac{4\pi Q^2}{8\pi\epsilon_0\Omega\eta}$$

Where  $Q \equiv \sum_{\alpha} q_{\alpha}$

## Some details --

Neutral charge density  $\rho(\mathbf{r}) = \sum_{i=1} q_i \delta(\mathbf{r} - \mathbf{r}_i) - \frac{Q}{\Omega}$

where  $Q = \frac{1}{N} \sum_{i=1} q_i$

$$W = \frac{1}{2} \int d^3r \rho(\mathbf{r}) \Phi(\mathbf{r})$$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{4\pi\epsilon_0} \left( \sum_{i=1} \frac{q_i}{|\mathbf{r} - \mathbf{r}_i|} - \frac{Q}{\Omega} \int d^3r' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right)$$

Trick --

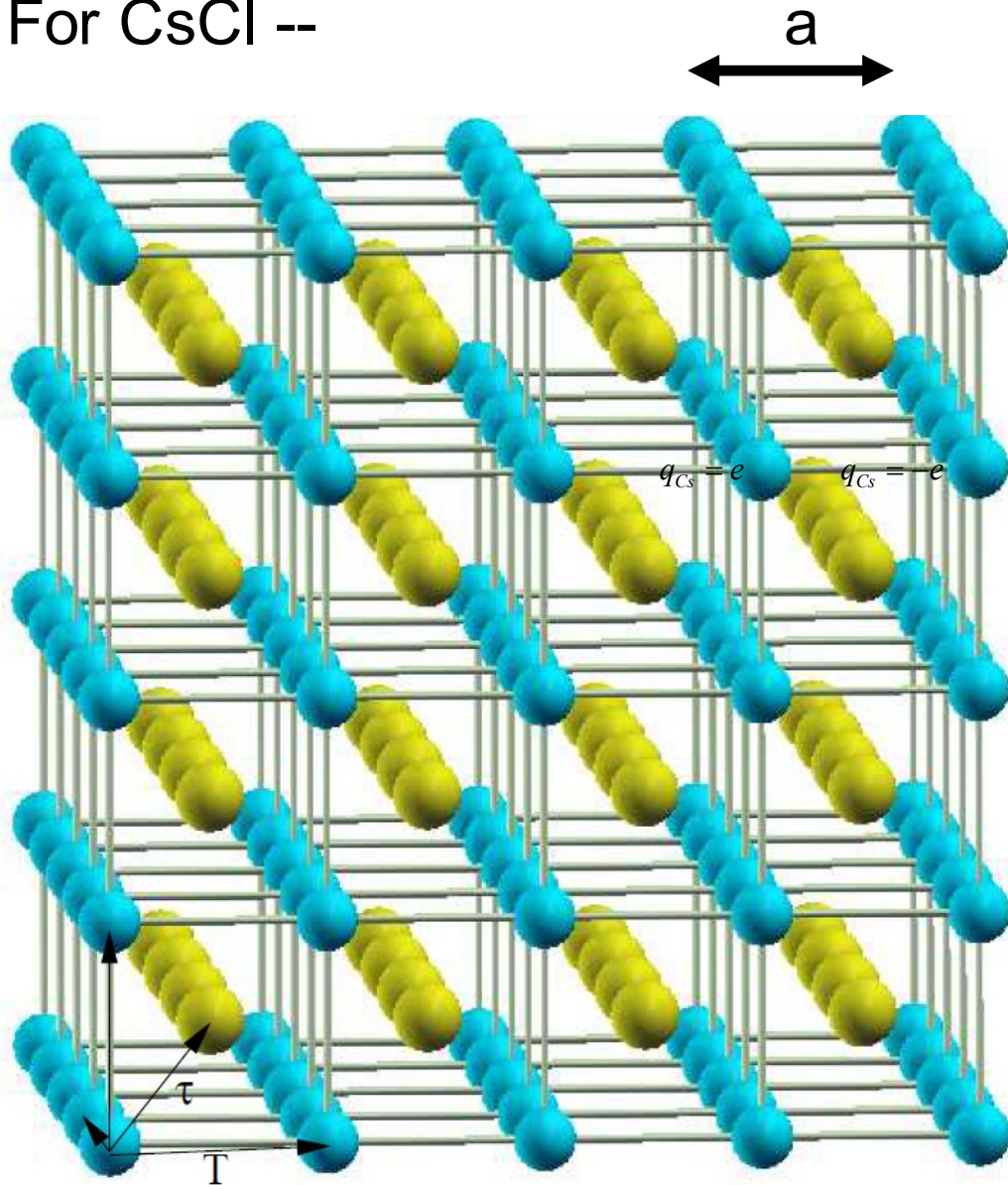
Let  $r'' \equiv |\mathbf{r} - \mathbf{r}'|$

$$\frac{Q}{\Omega} \int d^3r' \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{Q}{\Omega} 4\pi \int_0^\infty r'' dr'' = \frac{Q}{\Omega} 4\pi \int_0^\infty \frac{1}{u^3} du$$

$$W = \frac{1}{8\pi\epsilon_0} \left( \sum_{i \neq j} \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} - \frac{4\pi N Q^2}{\Omega} \int_0^\infty \frac{1}{u^3} du \right)$$

➔ divergent term  
cancels when  
combined with  
Ewald's construction

For CsCl --



$$\tau_{Cs} = 0 \text{ and } \tau_{Cl} = \frac{a}{2}(\hat{x} + \hat{y} + \hat{z}).$$

$$T_1 = a\hat{x} \quad T_2 = a\hat{y} \quad T_3 = a\hat{z}.$$

$$q_{Cs} = e$$

$$q_{Cl} = -e$$

$$\frac{W}{N} = \sum_{\alpha\beta} \frac{q_\alpha q_\beta}{8\pi\epsilon_0} \left( \frac{4\pi}{\Omega} \sum_{\mathbf{G} \neq 0} \frac{e^{-i\mathbf{G} \cdot \boldsymbol{\tau}_{\alpha\beta}} e^{-G^2/\eta}}{G^2} - \sqrt{\frac{\eta}{\pi}} \delta_{\alpha\beta} + \sum'_{\mathbf{T}} \frac{\text{erfc}(\frac{1}{2}\sqrt{\eta}|\boldsymbol{\tau}_{\alpha\beta} + \mathbf{T}|)}{|\boldsymbol{\tau}_{\alpha\beta} + \mathbf{T}|} \right) - \frac{4\pi Q^2}{8\pi\epsilon_0\Omega\eta}$$

$$\frac{W}{N} = \frac{e^2}{8\pi\epsilon_0} (\mathcal{I}_1 + \mathcal{I}_2)$$

$$\mathcal{I}_1 \equiv \frac{4\pi}{\Omega} \sum_{\mathbf{G} \neq 0} 2 \frac{(1 - e^{i\mathbf{G} \cdot \boldsymbol{\tau}_{Cl}}) e^{-|\mathbf{G}|^2/\eta}}{|\mathbf{G}|^2} - 2\sqrt{\frac{\eta}{\pi}}$$

$$\mathcal{I}_2 \equiv \sum_{\mathbf{T} \neq 0} 2 \frac{\text{erfc}(\frac{1}{2}\sqrt{\eta}|\mathbf{T}|)}{|\mathbf{T}|} - \sum_{\mathbf{T}} 2 \frac{\text{erfc}(\frac{1}{2}\sqrt{\eta}|\boldsymbol{\tau}_{Cl} + \mathbf{T}|)}{|\boldsymbol{\tau}_{Cl} + \mathbf{T}|}$$

➔ Can be evaluated using Maple or Mathematica or other programming language