

PHY 712 Electrodynamics

11-11:50 AM MWF in Olin 103

Class notes for Lecture 8:

Finish reading Chap. 2 and start Chap. 3

**Solution of Poisson/Laplace equation
for special geometries –**

A. Cylindrical

B. Spherical

Scheduling ahead -- When to schedule mid-term?

February						
Sun	Mon	Tue	Wed	Thu	Fri	Sat
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28					

March						
Sun	Mon	Tue	Wed	Thu	Fri	Sat
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30	31		

April						
Sun	Mon	Tue	Wed	Thu	Fri	Sat
					1	2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26	27	28	29	30

Last class

← **Spring break – no class**

← **APS meeting – no class**

Possible scheduling options for PHY 712 and 742

- 1. Base mid-term grade entirely on HW and schedule both mid-term exams for the week of March 14**
- 2. Stagger mid-term exams; one for the week of March 14 and the other for the week of Feb 28**
- 3. Make HW grade count more and only have final exam**
- 4. Other?**

Course schedule for Spring 2022

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Mon: 01/10/2022	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/14/2022
2	Wed: 01/12/2022	Chap. 1	Electrostatic energy calculations	#2	01/19/2022
3	Fri: 01/14/2022	Chap. 1	Electrostatic energy calculations	#3	01/21/2022
	Mon: 01/17/2022		MLK Holiday -- no class		
4	Wed: 01/19/2022	Chap. 1 & 2	Electrostatic potentials and fields	#4	01/24/2022
5	Fri: 01/21/2022	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#5	01/26/2022
6	Mon: 01/24/2022	Chap. 1 - 3	Brief introduction to numerical methods	#6	01/28/2022
7	Wed: 01/26/2022	Chap. 2 & 3	Image charge constructions	#7	01/31/2022
8	Fri: 01/28/2022	Chap. 2 & 3	Cylindrical and spherical geometries	#8	02/02/2022
9	Mon: 01/31/2022	Chap. 3 & 4	Spherical geometry and multipole moments		

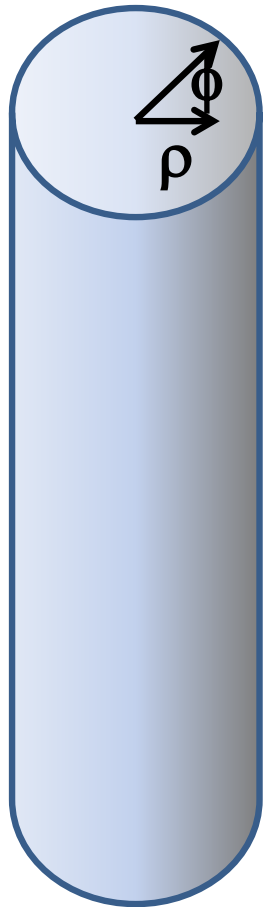
PHY 712 -- Assignment #8

January 28, 2022

Finish reading Chap. 2 and start Chap. 3 in **Jackson**.

1. Consider the long cylindrical shell charge density discussed in Lecture 8, confirming the the electrostatic potential and also calculating the electrostatic field for this system.

Solution of the Poisson/Laplace equation in various geometries
→ cylindrical geometry with no z-dependence (infinitely long wire, for example):



Corresponding orthogonal functions from solution of

Laplace equation: $\nabla^2 \Phi = 0$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

$$\Phi(\rho, \phi) = \Phi(\rho, \phi + m2\pi)$$

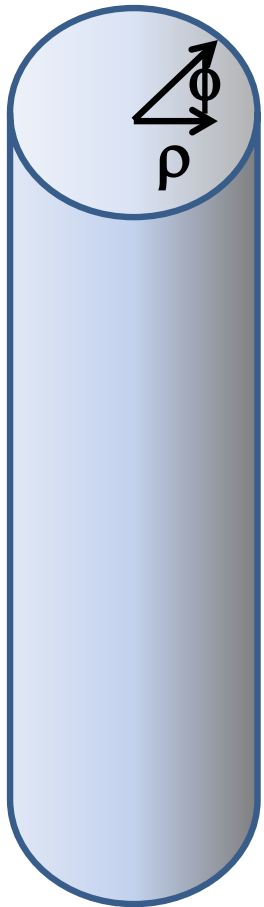
Assume: $\Phi(\rho, \phi) = f(\rho)g(\phi)$

$$\text{Suppose } \frac{d^2 g(\phi)}{d\phi^2} = -m^2 g(\phi)$$

$$g(\phi) = \sin(m\phi + \alpha_m) \quad \Rightarrow m = \text{integer}$$

Solution of the Poisson/Laplace equation in various geometries

→ cylindrical geometry with no z-dependence (infinitely long wire, for example):



Corresponding orthogonal functions from solution of

Laplace equation: $\nabla^2 \Phi = 0$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

Assume: $\Phi(\rho, \phi) = f(\rho)g(\phi)$

$$\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{df_m(\rho)}{d\rho} \right) - \frac{m^2}{\rho^2} f_m(\rho) = 0$$

$$f_0(\rho) = \begin{cases} 1 \\ \ln \rho \end{cases} \quad f_{m>0} = \rho^{\pm m}$$

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no z-dependence (infinitely long wire, for example):

Corresponding orthogonal functions from solution of

Laplace equation : $\nabla^2 \Phi = 0$

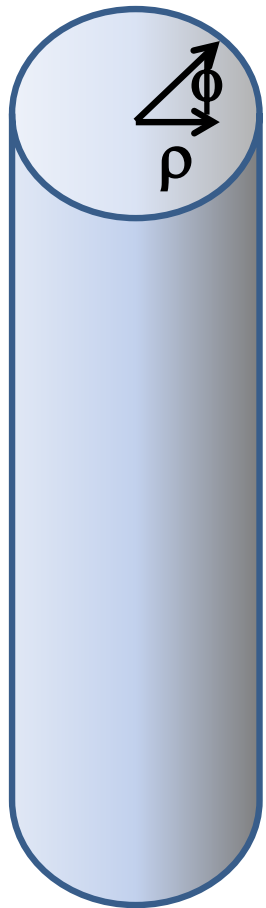
$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

$$\Phi(\rho, \phi) = \Phi(\rho, \phi + m2\pi) \quad \rightarrow m = \text{integer}$$

$\Rightarrow$ General solution of the Laplace equation
in these coordinates :

$$\Phi(\rho, \phi) = A_0 + B_0 \ln(\rho) + \sum_{m=1}^{\infty} (A_m \rho^m + B_m \rho^{-m}) \sin(m\phi + \alpha_m)$$

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no z-dependence (infinitely long wire, for example):



Green's function appropriate for this geometry with boundary conditions at $\rho = 0$ and $\rho = \infty$:

$$\left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \right) G(\rho, \rho', \phi, \phi') = -4\pi \frac{\delta(\rho - \rho')}{\rho} \delta(\phi - \phi')$$

It can be shown that the following form can be used:

$$G(\rho, \rho', \phi, \phi') = -\ln(\rho_{>}^2) + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho_{<}}{\rho_{>}} \right)^m \cos(m(\phi - \phi'))$$

Note that the previous example is similar to the construction for the 2-d cartesian case --

For the 2-d cartesian case, for example, we can assume that the Green's function can be written in the form:

$$G(x, x', y, y') = \sum_n u_n(x) u_n(x') g_n(y, y') \quad \text{where} \quad \frac{d^2}{dx^2} u_n(x) = -\alpha_n u_n(x)$$

The y dependence of this equation will have the required

behavior, if we choose:
$$\left[-\alpha_n + \frac{\partial^2}{\partial y^2} \right] g_n(y, y') = -4\pi \delta(y - y'),$$

which in turn can be expressed in terms of the two independent solutions $v_{n_1}(y)$ and $v_{n_2}(y)$ of the homogeneous equation:

$$\left[\frac{d^2}{dy^2} - \alpha_n \right] v_{n_i}(y) = 0,$$

and the Wronskian constant:
$$K_n \equiv \frac{dv_{n_1}}{dy} v_{n_2} - v_{n_1} \frac{dv_{n_2}}{dy}$$

$$\left[-\alpha_n + \frac{\partial^2}{\partial y^2} \right] g_n(y, y') = -4\pi \delta(y - y'),$$

$$g_n(y, y') = \frac{4\pi}{K_n} v_{n_1}(y_<) v_{n_2}(y_>)$$

where: $\left[\frac{d^2}{dy^2} - \alpha_n \right] v_{n_i}(y) = 0,$

and $K_n \equiv \frac{dv_{n_1}}{dy} v_{n_2} - v_{n_1} \frac{dv_{n_2}}{dy}$

For example, choose $v_{n_1}(y) = \sinh(\sqrt{\alpha_n} y)$ and $v_{n_2}(y) = \sinh(\sqrt{\alpha_n} (b - y))$

where $K_n = \sqrt{\alpha_n} \sinh(\sqrt{\alpha_n} b)$

using the identity: $\cosh(r) \sinh(s) + \sinh(r) \cosh(s) = \sinh(r + s)$

$$G(x, x', y, y') = \sum_n u_n(x) u_n(x') \frac{4\pi}{K_n} v_{n_1}(y_<) v_{n_2}(y_>).$$

In the cylindrical geometry case,

$$u_n(x) \rightarrow \{\sin(m\varphi), \cos(m\varphi)\}$$

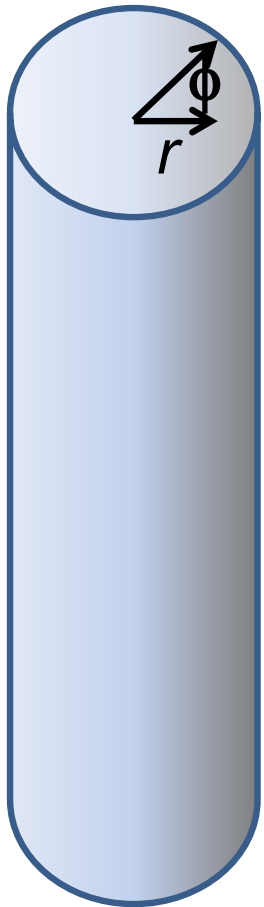
$$v_{n_{1,2}} \rightarrow \{1, \ln(\rho), \rho^m, \rho^{-m}\}$$

$$G(\rho, \rho', \varphi, \varphi') = -\ln(\rho_{>}^2) + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho_{<}}{\rho_{>}} \right)^m \cos(m(\varphi - \varphi'))$$

Comments and details

Change notation

$$\rho \Rightarrow r$$



$$G(r, r', \varphi, \varphi') = -\ln(r_{>}^2) + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{r_{<}}{r_{>}} \right)^m \cos(m(\varphi - \varphi'))$$

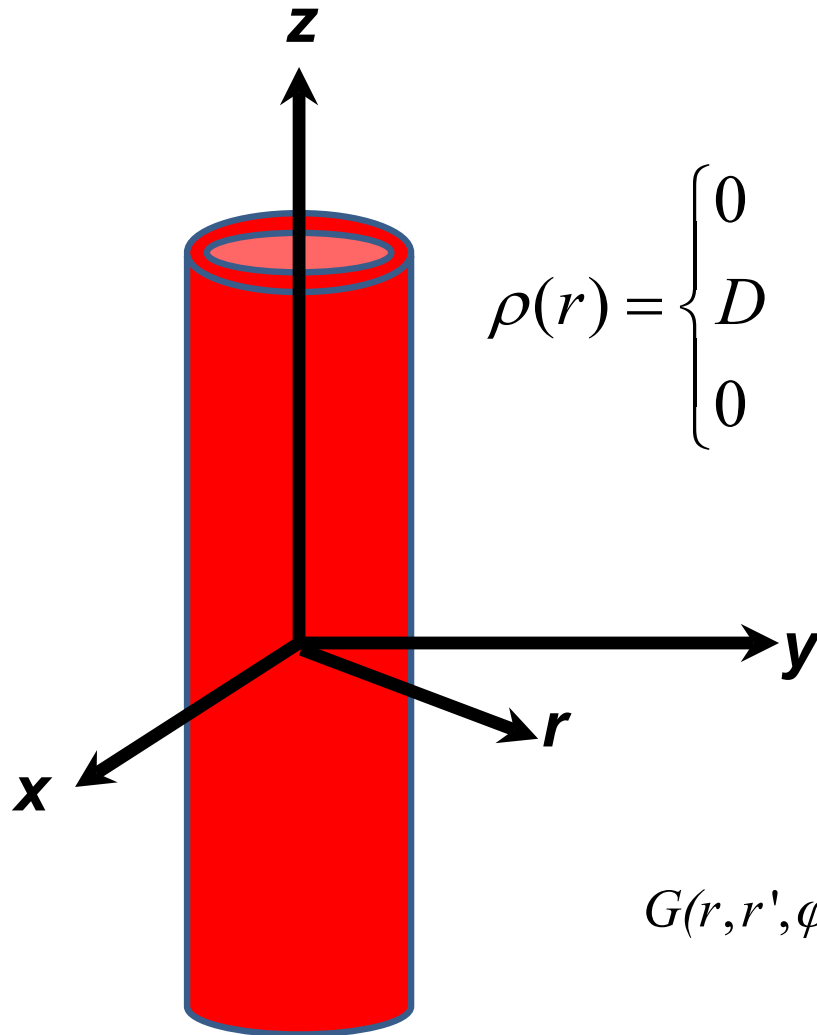
$$\Phi(r, \varphi) = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} d\varphi' \int_0^{\infty} r' dr' G(r, r', \varphi, \varphi') \rho(r', \varphi')$$

Note that: For this extended charge distribution, Coulomb's law in its original form diverges:

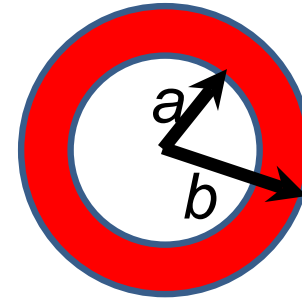
$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

Note that in this case, we have assumed that the surface integral contributions are trivial.

Example – uniform cylindrical shell:



Top view:



$$\rho(r) = \begin{cases} 0 & r < a \\ D & a \leq r \leq b \\ 0 & r > b \end{cases}$$

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2 \Phi(r, \phi) = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \Phi(r, \phi)}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi(r, \phi)}{\partial^2 \phi}$$

$$G(r, r', \phi, \phi') = -\ln(r_{>}^2) + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{r_{<}}{r_{>}} \right)^m \cos(m(\phi - \phi'))$$

$$\Phi(r, \phi) = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} d\phi' \int_0^{\infty} r' dr' G(r, r', \phi, \phi') \rho(r', \phi')$$

Question – Why only $m=0$ for this case?

$$G(r, r', \varphi, \varphi') = -\ln(r_{>}^2) + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{r_{<}}{r_{>}} \right)^m \cos(m(\varphi - \varphi'))$$

$$\Phi(r, \varphi) = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} d\varphi' \int_0^{\infty} r' dr' G(r, r', \varphi, \varphi') \rho(r', \varphi')$$

Note that $\int_0^{2\pi} d\varphi' \cos(m(\varphi - \varphi')) = 0$ for $m > 0$

Some details

$$G(r, r', \varphi, \varphi') = -\ln(r_{>}^2) + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{r_{<}}{r_{>}} \right)^m \cos(m(\varphi - \varphi'))$$

$$\Phi(r, \varphi) = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} d\varphi' \int_0^{\infty} r' dr' G(r, r', \varphi, \varphi') \rho(r', \varphi')$$

$$\text{In our case: } \Phi(r, \varphi) = \frac{2\pi D}{4\pi\epsilon_0} \int_a^b r' dr' (-\ln(r_{>}^2)) = \frac{D}{\epsilon_0} \int_a^b r' dr' (-\ln(r_{>}))$$

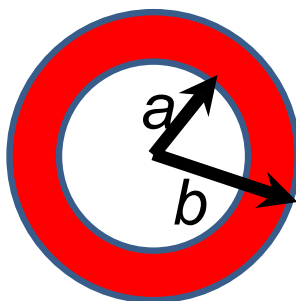
$$\text{For } 0 \leq r < a: \quad \Phi(r, \varphi) = \frac{D}{\epsilon_0} \int_a^b r' dr' (-\ln(r'))$$

$$\text{For } a \leq r < b: \quad \Phi(r, \varphi) = \frac{D}{\epsilon_0} \left(\int_a^r r' dr' (-\ln(r)) + \int_r^b r' dr' (-\ln(r')) \right)$$

$$\text{For } r > b: \quad \Phi(r, \varphi) = \frac{D}{\epsilon_0} \int_a^b r' dr' (-\ln(r))$$

Example continued -- $m=0$ only --

Top view:



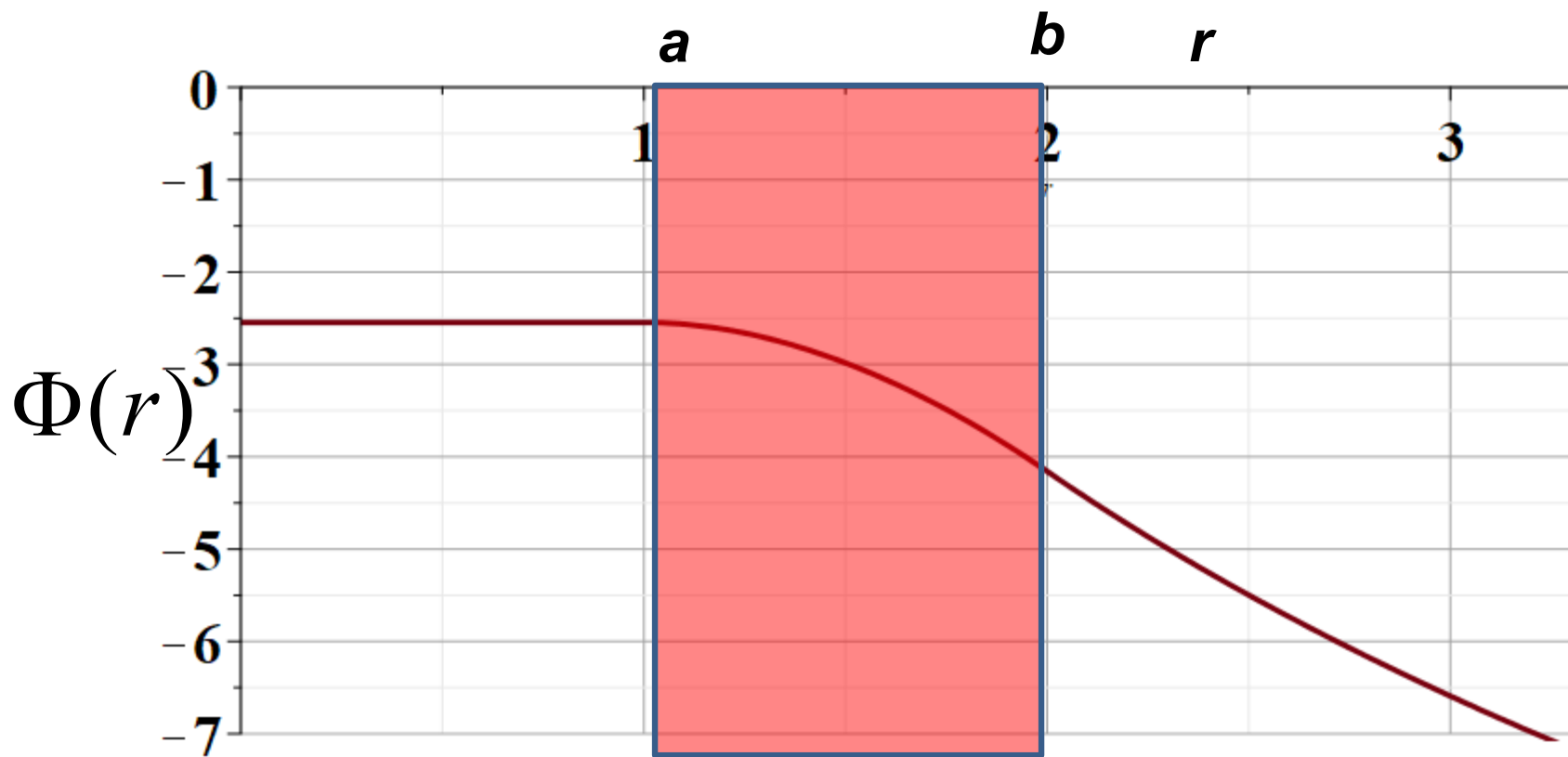
$$\rho(r) = \begin{cases} 0 & 0 < r < a \\ D & a \leq r \leq b \\ 0 & r > b \end{cases}$$

$$G(r, r', \varphi, \varphi') = -\ln(r_{>}^2) + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{r_{<}}{r_{>}} \right)^m \cos(m(\varphi - \varphi'))$$

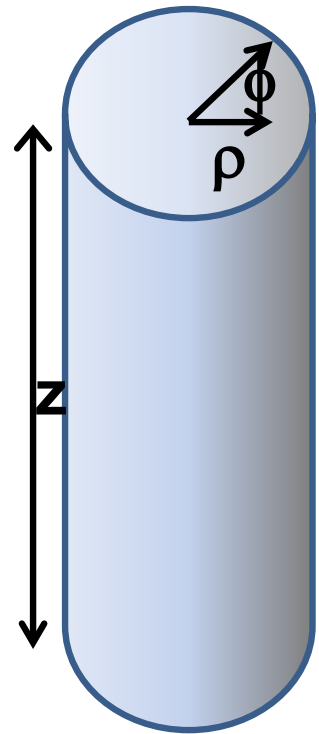
$$\Phi(r, \varphi) = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} d\varphi' \int_0^{\infty} r' dr' G(r, r', \varphi, \varphi') \rho(r', \varphi')$$

$$\Phi(r) = \begin{cases} \frac{D}{4\epsilon_0} (b^2 - a^2 - b^2 \ln(b^2) + a^2 \ln(a^2)) & 0 < r < a \\ \frac{D}{4\epsilon_0} (b^2 - r^2 - b^2 \ln(b^2) + a^2 \ln(r^2)) & a \leq r \leq b \\ \frac{D}{4\epsilon_0} (a^2 - b^2) \ln(r^2) & r > b \end{cases}$$

Example continued -- $m=0$ only --



Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with z-dependence

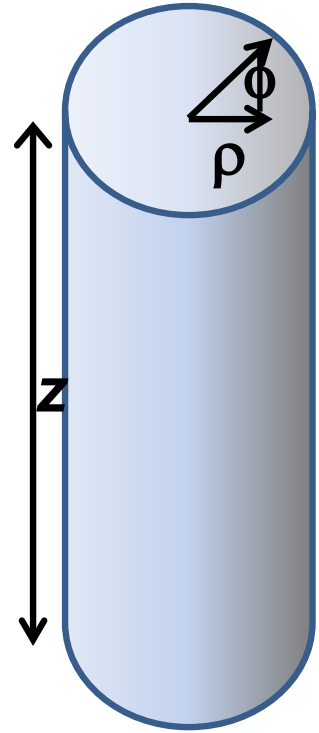


Laplace equation : $\nabla^2 \Phi = 0$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\Phi(\rho, \phi, z) = R(\rho)Q(\phi)Z(z)$$

Cylindrical geometry continued:



Laplace equation : $\nabla^2 \Phi = 0$

$$\Phi(\rho, \phi, z) = R(\rho)Q(\phi)Z(z)$$

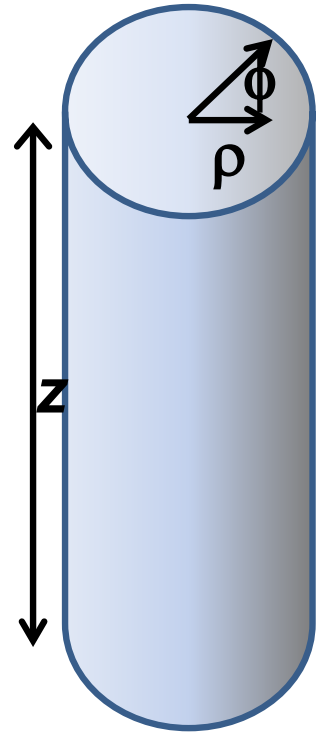
One possibility :

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0 \quad \Rightarrow Z(z) = \sinh(kz), \cosh(kz), e^{\pm kz}$$

$$\frac{d^2 Q}{d\phi^2} + m^2 Q = 0 \quad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2} \right) R = 0 \quad \Rightarrow J_m(k\rho), N_m(k\rho)$$

Cylindrical geometry continued:



Laplace equation : $\nabla^2 \Phi = 0$

$$\Phi(\rho, \phi, z) = R(\rho)Q(\phi)Z(z)$$

Another possibility :

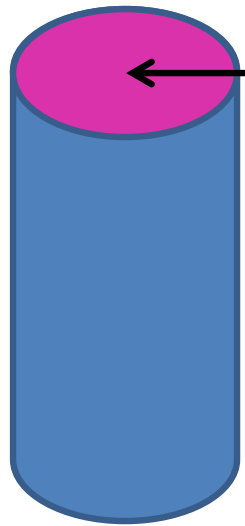
$$\frac{d^2 Z}{dz^2} + k^2 Z = 0 \quad \Rightarrow Z(z) = \sin(kz), \cos(kz), e^{\pm ikz}$$

$$\frac{d^2 Q}{d\phi^2} + m^2 Q = 0 \quad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + \left(-k^2 - \frac{m^2}{\rho^2} \right) R = 0 \quad \Rightarrow I_m(k\rho), K_m(k\rho)$$

Solutions of Laplace equation inside cylindrical shape

Example with non-trivial boundary value at $z=L$

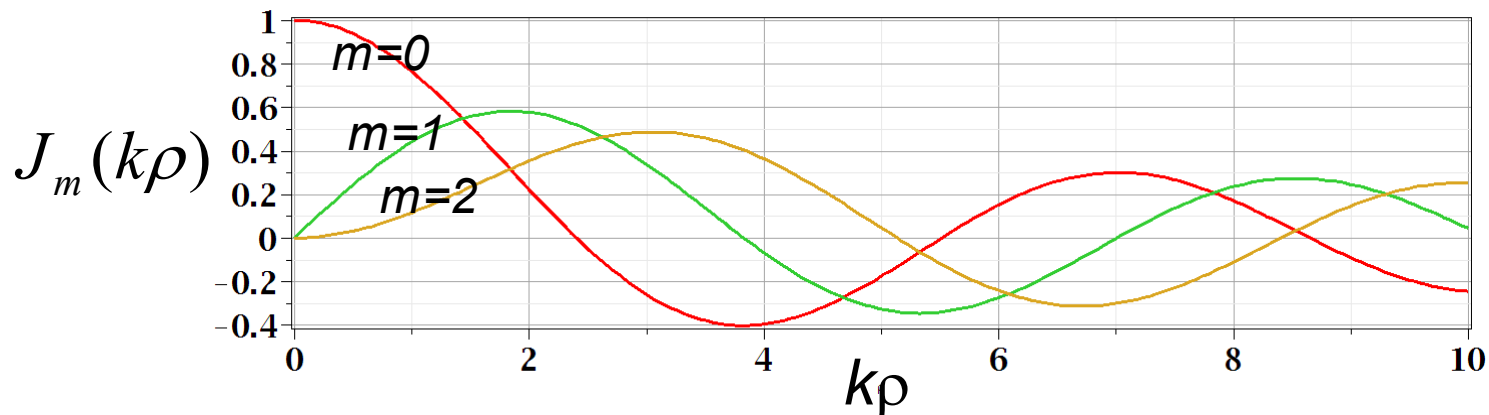


$$\Phi(\rho, \phi, z = L) = V(\rho, \phi)$$

$$\Phi(\rho, \phi, z) = 0 \quad \text{on all other boundaries}$$

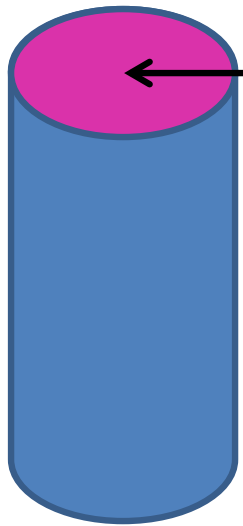
$$\Phi(\rho, \phi, z) = \sum_{n,m} A_{mn} J_m(k_{mn} \rho) \sinh(k_{mn} z) \sin(m\phi + \alpha_{mn})$$

$$\text{where } J_m(k_{mn} a) = 0$$



Solutions of Laplace equation inside cylindrical shape

Example with non-trivial boundary value at $z=L$



$$\Phi(\rho, \varphi, z = L) = V(\rho, \varphi)$$

$$\Phi(\rho, \varphi, z) = 0 \quad \text{on all other boundaries}$$

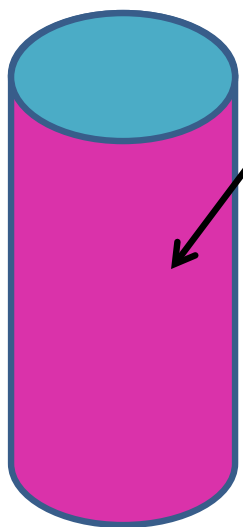
$$\Phi(\rho, \varphi, z) = \sum_{n,m} A_{mn} J_m(k_{mn}\rho) \sinh(k_{mn}z) \sin(m\varphi + \alpha_{mn})$$

If $V(\rho, \varphi)$ is an even function of φ so that $\alpha_{mn} = \pi / 2$:

$$A_{mn} = \frac{\int_0^{2\pi} d\varphi \cos(m\varphi) \int_0^a \rho d\rho J_m(k_{mn}\rho) V(\rho, \varphi)}{\sinh(k_{mn}L) \int_0^{2\pi} d\varphi \cos^2(m\varphi) \int_0^a \rho d\rho J_m^2(k_{mn}\rho)}$$

Solutions of Laplace equation inside cylindrical shape

Example with non-trivial boundary value at $\rho=a$

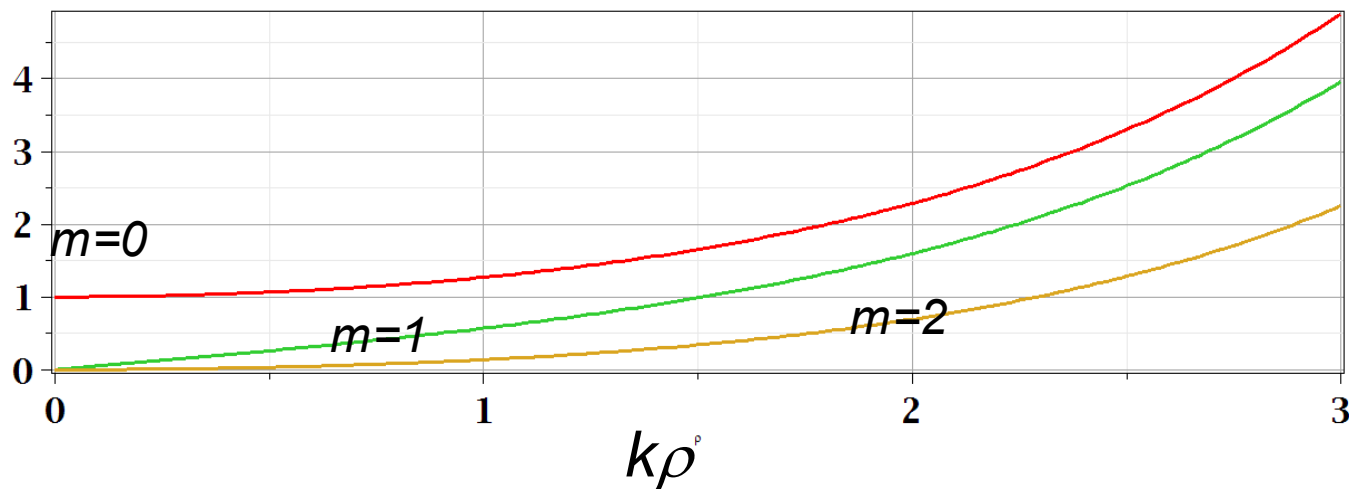


$$\Phi(\rho = a, \phi, z) = V(\phi, z)$$

$$\Phi(\rho, \phi, z) = 0 \quad \text{on all other boundaries}$$

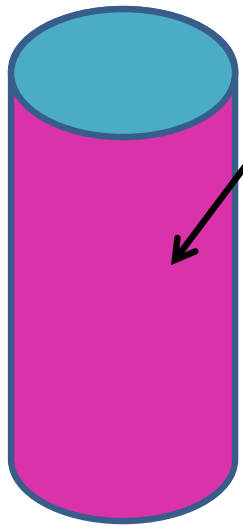
$$\Phi(\rho, \phi, z) = \sum_{n,m} A_{mn} I_m \left(\frac{n\pi\rho}{L} \right) \sin \left(\frac{n\pi z}{L} \right) \sin(m\phi + \alpha_{mn})$$

$I_m(k\rho)$



Solutions of Laplace equation inside cylindrical shape

Example with non-trivial boundary value at $\rho=a$



$$\Phi(\rho = a, \varphi, z) = V(\varphi, z)$$

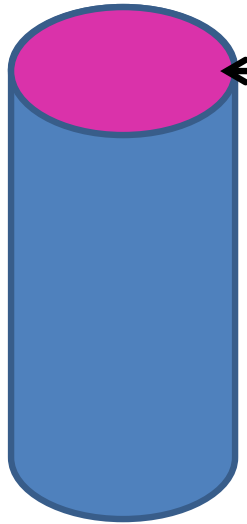
$$\Phi(\rho, \varphi, z) = 0 \quad \text{on all other boundaries}$$

$$\Phi(\rho, \varphi, z) = \sum_{n,m} A_{mn} I_m \left(\frac{n\pi\rho}{L} \right) \sin \left(\frac{n\pi z}{L} \right) \sin(m\varphi + \alpha_{mn})$$

If $V(z, \varphi)$ is an even function of φ so that $\alpha_{mn} = \pi/2$:

$$A_{mn} = \frac{\int_0^{2\pi} d\varphi \cos(m\varphi) \int_0^L dz \sin \left(\frac{n\pi z}{L} \right) V(z, \varphi)}{I_m \left(\frac{n\pi a}{L} \right) \int_0^{2\pi} d\varphi \cos^2(m\varphi) \int_0^L dz \sin^2 \left(\frac{n\pi z}{L} \right)}$$

Green's function for Dirchelet boundary value inside cylindar:



$$\Phi(\rho, \phi, z = L) = V(\rho, \phi)$$

$$\Phi(\rho = a, \phi, z) = 0, \quad \Phi(\rho, \phi, z = 0) = 0$$

Expansion in terms of Bessel function zeros : $J_m(k_{mn}a) = 0$

$$G(\rho, \rho', \phi, \phi', z, z') =$$

$$\frac{8\pi}{\pi a^2} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{e^{im(\phi-\phi')} J_m(k_{mn}\rho) J_m(k_{mn}\rho') \sinh(k_{mn}z_{<}) \sinh(k_{mn}(L-z_{>}))}{k_{mn} (J_{m+1}(k_{mn}a))^2 \sinh(k_{mn}L)}$$

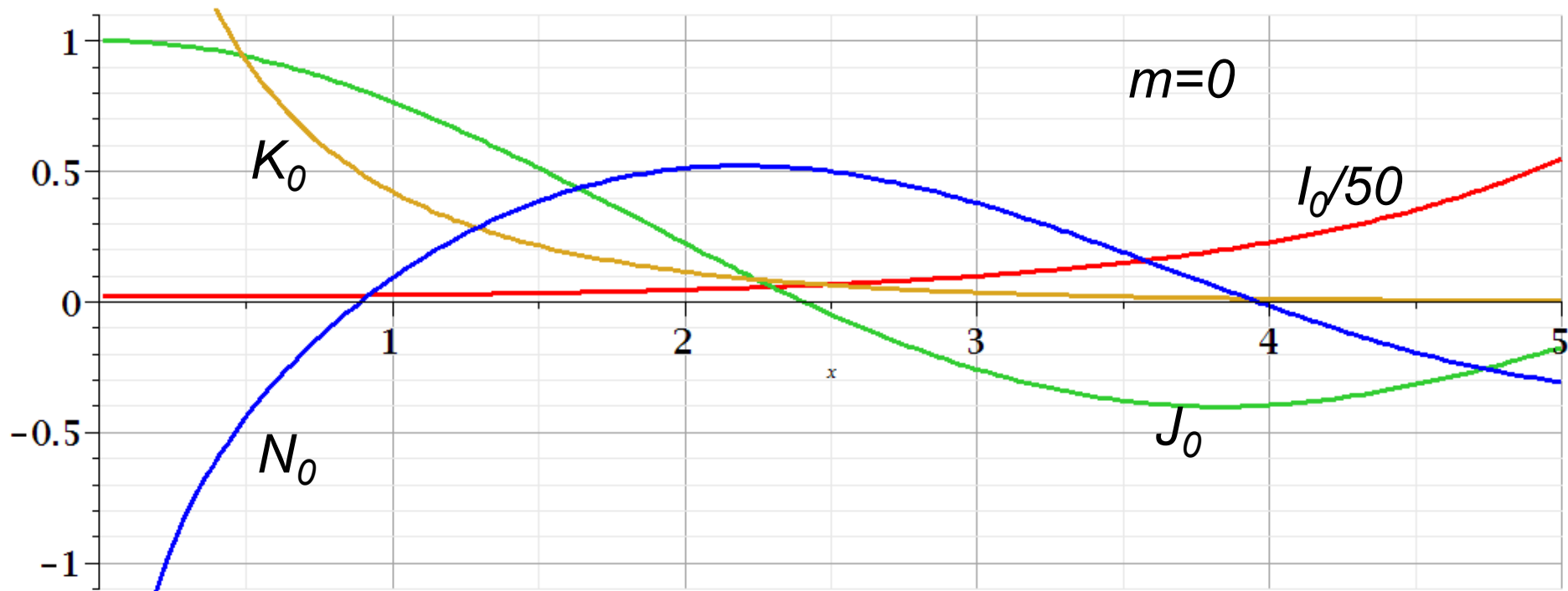
$$\begin{aligned} \Phi(\rho, \phi, z) = & \frac{1}{4\pi\epsilon_0} \int_V d\phi' \rho' d\rho' dz' G(\rho, \rho', \phi, \phi', z, z') \rho(\rho', \phi', z') \\ & + \frac{1}{4\pi} \int_{S; z'=L} d\phi' \rho' d\rho' \left. \frac{\partial G(\rho, \rho', \phi, \phi', z, z')}{\partial z'} \right|_{z'=L} V(\rho', \phi') \end{aligned}$$

Comments on cylindrical Bessel functions

$$\left(\frac{d^2}{du^2} + \frac{1}{u} \frac{d}{du} + \left(\pm 1 - \frac{m^2}{u^2} \right) \right) F_m^\pm(u) = 0$$

$$F_m^+(u) = J_m(u), N_m(u), H_m(u) \equiv J_m(u) \pm iN_m(u)$$

$$F_m^-(u) = I_m(u), K_m(u)$$

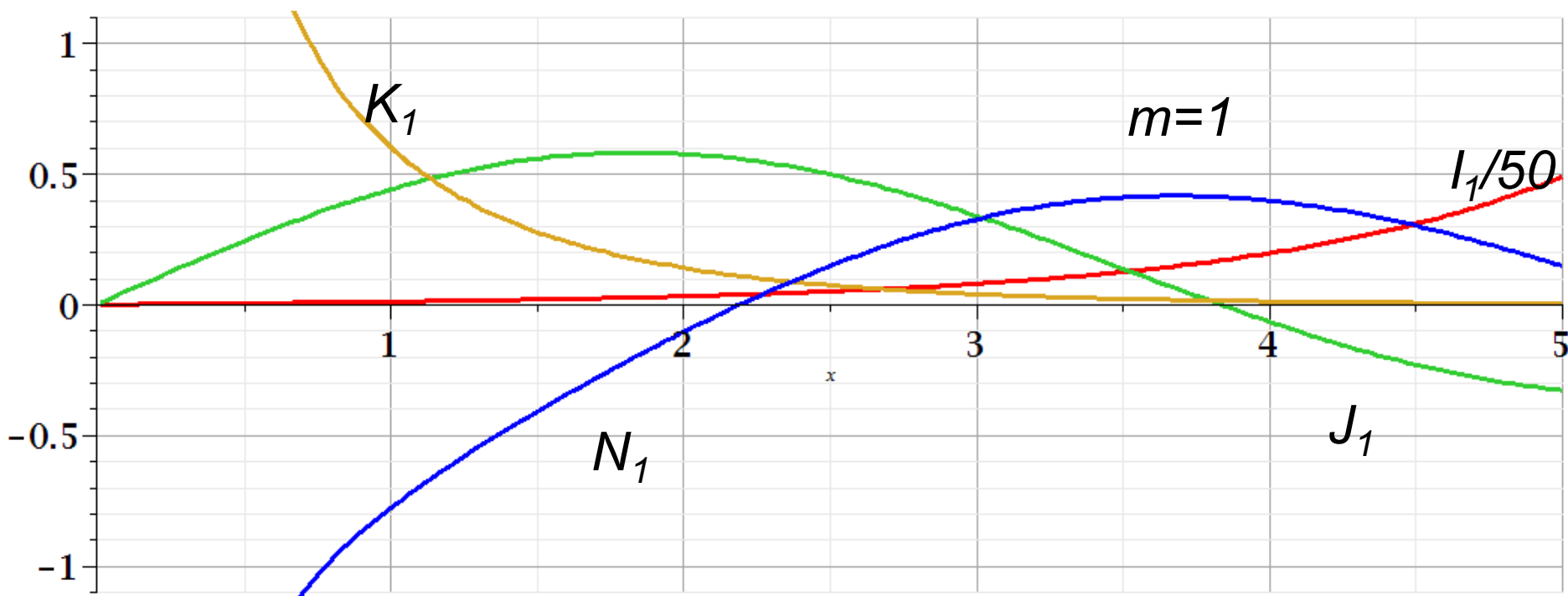


Comments on cylindrical Bessel functions

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$$F_m^+(u) = J_m(u), N_m(u), H_m(u) \equiv J_m(u) \pm iN_m(u)$$

$$F_m^-(u) = I_m(u), K_m(u)$$



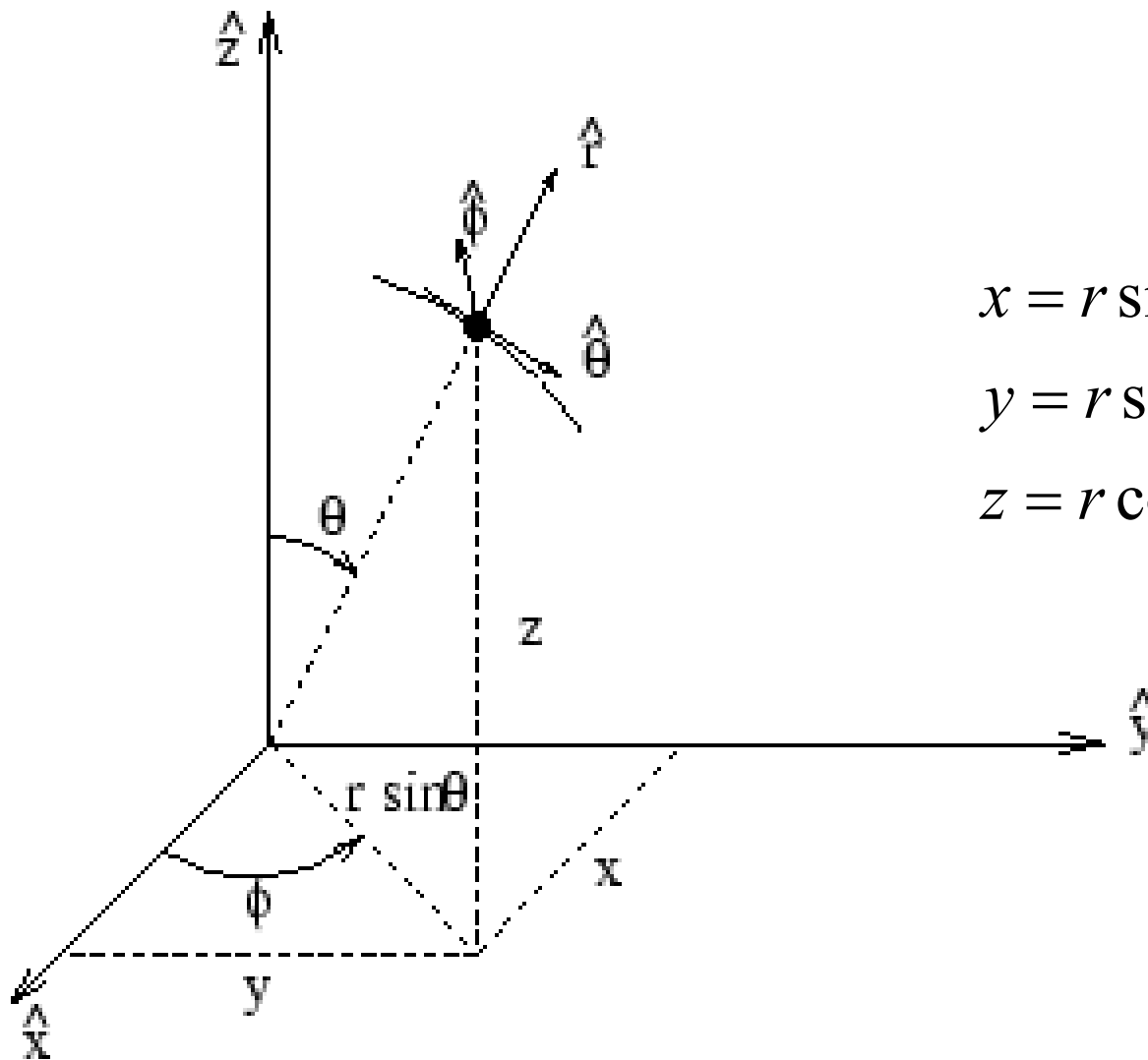
Some useful identities involving cylindrical Bessel functions

$$\left(\frac{d^2}{du^2} + \frac{1}{u} \frac{d}{du} + \left(1 - \frac{m^2}{u^2} \right) \right) J_m(u) = 0 \quad \text{for integer } m$$

Properties of Bessel functions in terms of zeros: x_{mn} ; $J_m(x_{mn}) = 0$

$$\int_0^a \rho d\rho J_m(x_{mn}\rho / a) J_m(x_{nn}\rho / a) = \frac{a^2}{2} (J_{m+1}(x_{mn}))^2 \delta_{nn},$$

Poisson and Laplace equation in spherical polar coordinates



$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

<http://www.uic.edu/classes/eecs/eecs520/textbook/node32.html>

Poisson and Laplace equation in spherical polar coordinates -- continued

Laplace equation for electrostatic potential $\Phi(r, \theta, \varphi)$:

$$\frac{1}{r} \frac{\partial^2}{\partial r^2} (r\Phi) + \frac{1}{r^2} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right) \Phi = 0$$

$$\Phi(r, \theta, \varphi) = \sum_{lm} R_{lm}(r) Y_{lm}(\theta, \varphi)$$

Spherical harmonic functions:

$$\left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right) Y_{lm}(\theta, \varphi) = -l(l+1) Y_{lm}(\theta, \varphi)$$

Properties of spherical harmonic functions

$$Y_{lm}(\theta, \phi) = (-1)^m Y_{l(-m)}^*(\theta, \phi) \quad (\text{standard Condon - Shortley convention})$$

$$\int d\Omega Y_{lm}(\theta, \phi) Y_{l'm'}^*(\theta, \phi) \equiv \int \sin \theta d\theta d\phi Y_{lm}(\theta, \phi) Y_{l'm'}^*(\theta, \phi) = \delta_{ll'} \delta_{mm'}$$

Completeness :

$$\sum_{lm} Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi') = \delta(\hat{\mathbf{r}} - \hat{\mathbf{r}}') \equiv \delta(\cos \theta - \cos \theta') \delta(\phi - \phi')$$

Relationship to Legendre polynomials :

$$Y_{l0}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos \theta)$$

Useful identity:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi')$$

Example for isolated charge density $\rho(\mathbf{r})$ with electrostatic potential vanishing for $r \rightarrow \infty$:

$$\begin{aligned} \Phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \\ &= \frac{1}{4\pi\epsilon_0} \int d^3r' \rho(\mathbf{r}') \left(\sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi') \right) \end{aligned}$$

Example -- continued

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \rho(\mathbf{r}') \left(\sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi') \right)$$

$$\text{Suppose: } \rho(\mathbf{r}') = \frac{Q}{a^3 \pi^{3/2}} e^{-r'^2/a^2}$$

$$\int d\Omega' Y_{lm}^*(\theta', \varphi') = \sqrt{4\pi} \delta_{l0} \delta_{m0}$$

$$\begin{aligned} \Phi(\mathbf{r}) &= \frac{4\pi}{4\pi\epsilon_0} \int_0^\infty r'^2 dr' \frac{r_{<}^0}{r_{>}^1} \frac{Q}{a^3 \pi^{3/2}} e^{-r'^2/a^2} \\ &= \frac{Q}{4\pi\epsilon_0} \frac{\text{erf}(r/a)}{r} \end{aligned}$$

Useful identity:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi')$$

Elements of "proof":

$$\begin{aligned} \frac{1}{|\mathbf{r} - \mathbf{r}'|} &= \frac{1}{r_{>} \left(1 + \left(\frac{r_{<}}{r_{>}} \right)^2 - 2 \left(\frac{r_{<}}{r_{>}} \right) \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' \right)^{1/2}} = \\ &= \frac{1}{r_{>}} \left(1 + \left(\frac{r_{<}}{r_{>}} \right) \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' + \left(\frac{r_{<}}{r_{>}} \right)^2 \left(\frac{3}{2} (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}')^2 - \frac{1}{2} \right) + \dots \right) \\ &= \frac{1}{r_{>}} \left(\sum_{l=0}^{\infty} \left(\frac{r_{<}}{r_{>}} \right)^l P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') \right) \end{aligned}$$

Useful identity:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi')$$

Elements of "proof" -- continued :

Sum rule for spherical harmonics :

$$P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^*(\hat{\mathbf{r}}')$$

Note that for $\hat{\mathbf{r}} = \hat{\mathbf{r}}'$, $P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') = 1$

$$\Rightarrow \frac{4\pi}{2l+1} \sum_{m=-l}^l |Y_{lm}(\hat{\mathbf{r}})|^2 = 1$$

Some spherical harmonic functions:

$$Y_{00}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1(\pm 1)}(\hat{\mathbf{r}}) = \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi}$$

$$Y_{10}(\hat{\mathbf{r}}) = \sqrt{\frac{3}{4\pi}} \cos \theta$$

$$Y_{2(\pm 2)}(\hat{\mathbf{r}}) = \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{\pm 2i\phi}$$

$$Y_{2(\pm 1)}(\hat{\mathbf{r}}) = \mp \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{\pm i\phi}$$

$$Y_{20}(\hat{\mathbf{r}}) = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$$