PHY 742 Quantum Mechanics II
12-12:50 AM MWF Olin 103

Plan for Lecture 13

Time dependent perturbation theory
Ref: Chapter 15

1. Review of basic equations
2. Time harmonic perturbations
3. Fermi Golden Rule

Course schedule for Spring 2022

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	Reading	Topic	HW	Due date
1	Mon: 01/10/2022	Chap. 12	Approximate solutions for stationary states -- The variational approach	#1	01/14/2022
2	Wed: 01/12/2022	Chap. 12 C	Approximate solutions for stationary states -- Perturbation theory	#2	01/19/2022
3	Fri: 01/14/2022	Chap. 12 D	Approximate solutions for stationary states -- Degenerate perturbation theory	#3	01/21/2022
	Mon: 01/17/2022		MLK Holiday -- no class		
4	Wed: 01/19/2022	Chap. 12 C & D	Approximate solutions for stationary states -- Additional tricks	#4	01/24/2022
5	Fri: 01/21/2022	Chap. 13	Examples of of the use of perturbation theory	#5	01/26/2022
6	Mon: 01/24/2022	Chap. 13 & 12 B	Hyperfine perturbation and also the WKB approximation	#6	01/28/2022
7	Wed: 01/26/2022	Chap. 14	Scattering theory		
8	Fri: 01/28/2022	Chap. 14	Scattering theory	#7	02/04/2022
9	Mon: 01/31/2022	Chap. 14	Scattering theory	#8	02/07/2022
	Wed: 02/02/2022	No class	Fire caution		
	Fri: 02/04/2022	No class	Fire caution		
10	Mon: 02/07/2022	Chap. 11 (A-C)	Time evolution and Feynman path integrals	#9	02/09/2022
11	Wed: 02/09/2022	Chap. 11 (A-C)	Time evolution and Feynman path integrals	#10	02/11/2022
12	Fri: 02/11/2022	Chap. 15 A	Approximation methods for time evolution of quantum systems	#11	02/14/2022
13	Mon: 02/14/2022	Chap. 15	Approximate time evolution	#12	02/16/2022
14	Wed: 02/16/2022	Chap. 15	Approximate time evolution		

PHY 742 -- Assignment #12

February 14, 2022

Continue reading Chapters 15 in **Professor Carlson's QM textbook.**.

1. For a system with spherically symmetric states in the absence of any perturbing electromagnetic field, in the dipole approximation, there are selection rules for allowed transitions between states. What are these selection rules? Using analysis or demonstration with examples, convince yourself that these selection rules are true.

Review of basic equations for time-dependent perturbations

$$i\hbar\frac{\partial}{\partial t}\left|\psi\right\rangle = H(t)\left|\psi\right\rangle$$

$$H(t) = H^0 + \epsilon H^1(t)$$

We approach the problem using the complete basis set of H^0 :

$$H^0\left|n^0\right\rangle = E_n^0\left|n^0\right\rangle$$

It is reasonable to assume that

$$\left|\psi(t)\right\rangle = \sum_n c_n(t)\left|n^0\right\rangle \equiv \sum_n k_n(t)e^{-iE_n^0 t/\hbar}\left|n^0\right\rangle$$

Treatment of time-dependent perturbations -- continued

$$\left|\psi(t)\right\rangle = \sum_n k_n(t) e^{-iE_n^0 t/\hbar}\left|n^0\right\rangle$$

$$i\hbar\frac{\partial}{\partial t}\left|\psi\right\rangle = \left(H^0 + \epsilon H^1(t)\right)\left|\psi\right\rangle$$

$$\sum_n\left(i\hbar\frac{dk_n(t)}{dt} - \epsilon H^1(t)k_n(t)\right)e^{-iE_n^0 t/\hbar}\left|n^0\right\rangle = 0$$

Projecting this equation with a particular zero-order state $\left\langle f^0\right|$:

$$i\hbar\frac{dk_f(t)}{dt} = \epsilon\sum_n\left\langle f^0\left|H^1(t)\right|n^0\right\rangle k_n(t) e^{i\left(E_f^0 - E_n^0\right)t/\hbar}$$

Treatment of time-dependent perturbations -- continued

$$i\hbar \frac{dk_f(t)}{dt} = \epsilon \sum_n \left\langle f^0 \middle| H^1(\mathbf{r},t) \middle| n^0 \right\rangle k_n(t) e^{i\left(E_f^0 - E_n^0\right)t/\hbar}$$

Perturbation expansion for time-dependent coefficients:

$$k_n(t) = k_n^0 + \epsilon k_n^1(t) + \epsilon^2 k_n^2(t) + \ldots$$

Zero order equation: $\dfrac{dk_n^0}{dt} = 0$

s-order equation for $s > 0$:

$$\frac{dk_m^s}{dt} = \frac{1}{i\hbar} \sum_n \left\langle m^0 \middle| H^1(\mathbf{r},t) \middle| n^0 \right\rangle k_n^{s-1}(t) e^{i\left(E_m^0 - E_n^0\right)t/\hbar}$$

Recap --

$$\left|\psi(t)\right\rangle = \sum_n k_n(t) e^{-iE_n^0 t/\hbar} \left|n^0\right\rangle$$

Perturbation expansion for time-dependent coefficients:

$$k_n(t) = k_n^0 + \epsilon k_n^1(t) + \epsilon^2 k_n^2(t) + \ldots$$

Zero order equation: $\dfrac{dk_n^0}{dt} = 0$

s-order equation for $s > 0$:

$$\frac{dk_m^s}{dt} = \frac{1}{i\hbar} \sum_n \left\langle m^0 \middle| H^1(\mathbf{r},t) \middle| n^0 \right\rangle k_n^{s-1}(t) e^{i\left(E_m^0 - E_n^0\right)t/\hbar}$$

If $k_1^0 = \delta_{n1}$, $\quad \dfrac{dk_m^1}{dt} = \dfrac{1}{i\hbar} \left\langle m^0 \middle| H^1(\mathbf{r},t) \middle| 1^0 \right\rangle e^{i\left(E_m^0 - E_1^0\right)t/\hbar}$

Example – particle of mass *m* in a one-dimensional infinite well of length *a* --

$$\left|n^0\right\rangle = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) \qquad E_n^0 = \frac{\hbar^2\pi^2}{2ma^2}n^2 \qquad \text{for } n = 1,2,3....$$

$$\text{For } k_1^0 = \delta_{n1}, \qquad \frac{dk_m^1}{dt} = \frac{1}{i\hbar}\left\langle m^0\middle|H^1(\mathbf{r},t)\middle|1^0\right\rangle e^{i\left(E_m^0 - E_1^0\right)t/\hbar}$$

$$\text{Suppose } H^1(\mathbf{r},t) = W(\mathbf{r})\Theta(t)$$

$$\text{Then } k_m^1(t) = \frac{1}{i\hbar}\left\langle m^0\middle|W(\mathbf{r})\middle|1^0\right\rangle \int_{-\infty}^{t} dt'\Theta(t')e^{i\left(E_m^0 - E_1^0\right)t'/\hbar}$$

➔Perturbation depends both on spatial matrix element and on time matrix element.

Continued -- Example – particle of mass *m* in a one-dimensional infinite well of length *a* --

$$\left|n^0\right\rangle = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) \qquad E_n^0 = \frac{\hbar^2\pi^2}{2ma^2}n^2 \qquad \text{for } n = 1, 2, 3....$$

For $k_1^0 = \delta_{n1}$, and $H^1(\mathbf{r},t) = W(\mathbf{r})\Theta(t)$

$$k_m^1(t) = \frac{1}{i\hbar}\left\langle m^0\middle|W(\mathbf{r})\middle|1^0\right\rangle \int_{-\infty}^{t} dt'\Theta(t')e^{i\left(E_m^0 - E_1^0\right)t'/\hbar}$$

Suppose $\Theta(t) = \dfrac{1}{\sqrt{2\pi T^2}}e^{-t^2/\left(2T^2\right)}$ Let $\omega_{m1} \equiv \dfrac{\left(E_m^0 - E_1^0\right)}{\hbar}$

Then $k_m^1(t \to \infty) = \dfrac{1}{i\hbar}\left\langle m^0\middle|W(\mathbf{r})\middle|1^0\right\rangle e^{-\omega_{m1}^2 T^2/2}$

Treatment of time-dependent perturbations -- continued

1st-order equation, assuming that $k_n^0 = \delta_{nI}$

$$\frac{dk_m^1}{dt} = \frac{1}{i\hbar}\left\langle m^0 \middle| H^1(t) \middle| I^0 \right\rangle e^{i\left(E_m^0 - E_I^0\right)t/\hbar}$$

Example:

Suppose that $H^1(t) = \tilde{H}^1 h(t)$

where $$h(t) \equiv \begin{cases} 0 & \text{for } t < 0 \text{ and } t > T \\ 2\sin\omega t & \text{for } 0 < t < T \end{cases}$$

Treatment of time-dependent perturbations -- continued

For this example

$$\frac{dk_m^1}{dt} = 0 \quad \text{for } t < 0 \text{ or } t > T$$

$$\frac{dk_m^1}{dt} = \frac{2}{i\hbar}\frac{\left\langle m^0 \middle| \tilde{H}^1 \middle| I^0 \right\rangle}{2i}\left(e^{i(\omega+\omega_{mI})t} - e^{i(-\omega+\omega_{mI})t}\right)$$

$$\text{for } 0 < t < T$$

$$\Rightarrow k_m^1(t) = \frac{2}{i\hbar}\frac{\left\langle m^0 \middle| \tilde{H}^1 \middle| I^0 \right\rangle}{2}\left(\frac{e^{i(\omega+\omega_{mI})T} - 1}{\omega + \omega_{mI}} - \frac{e^{i(-\omega+\omega_{mI})T} - 1}{-\omega + \omega_{mI}}\right)$$

where $\omega_{mI} \equiv \left(E_m^0 - E_I^0\right)/\hbar \qquad \text{for } t > T$

For $t > T$

$$k_m^1(t) = \frac{2}{i\hbar}\frac{\left\langle m^0 \middle| \tilde{H}^1 \middle| I^0 \right\rangle}{2}\left(\frac{e^{i(\omega+\omega_{mI})T}-1}{\omega+\omega_{mI}} - \frac{e^{i(-\omega+\omega_{mI})T}-1}{-\omega+\omega_{mI}}\right)$$

where $\omega_{mI} \equiv \left(E_m^0 - E_I^0\right)/\hbar$

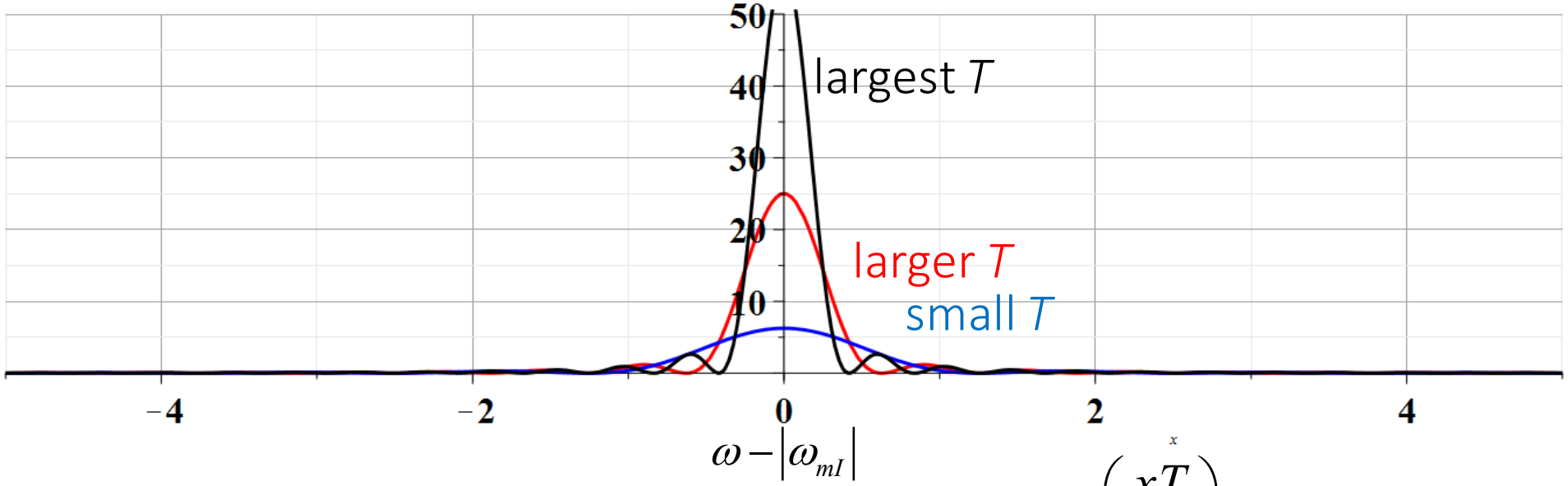
$$\left|k_m^1(t)\right|^2 \approx \frac{4}{\hbar^2}\left|\left\langle m^0 \middle| \tilde{H}^1 \middle| I^0 \right\rangle\right|^2 \times$$

$$\left(\frac{\sin^2\left(\left(\omega+\omega_{mI}\right)T/2\right)}{\left(\omega+\omega_{mI}\right)^2} + \frac{\sin^2\left(\left(-\omega+\omega_{mI}\right)T/2\right)}{\left(-\omega+\omega_{mI}\right)^2}\right)$$

$$\approx \frac{4\pi T}{2\hbar^2}\left|\left\langle m^0 \middle| \tilde{H}^1 \middle| I^0 \right\rangle\right|^2 \left(\delta\left(\omega+\omega_{mI}\right) + \delta\left(-\omega+\omega_{mI}\right)\right)$$

Treatment of time-dependent perturbations -- continued

Behavior of $k_m^1(T)$ in the neighborhood of $\omega = |\omega_{mI}|$

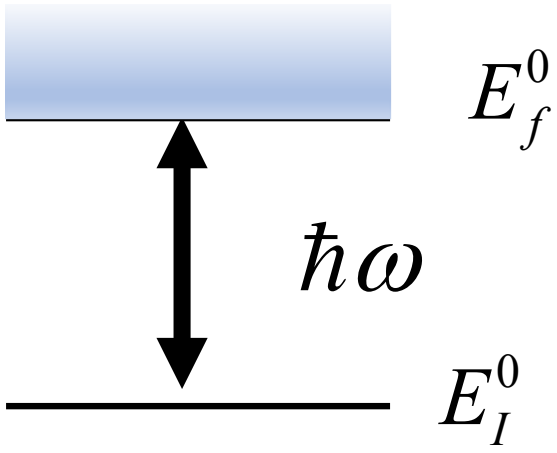


Note that $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2\left(\frac{xT}{2}\right)}{x^2} dx = \frac{\pi T}{2}$

Estimating the rate of transitions *I* → *f*

$$\mathcal{R}_{I\to f} = \frac{\left|k^1_{I\to f}(t)\right|^2}{T} \approx \frac{2\pi}{\hbar}\left|\left\langle f^0\middle|\tilde{H}^1\middle|I^0\right\rangle\right|^2 \left(\delta\left(\hbar\omega + E_f^0 - E_I^0\right) + \delta\left(-\hbar\omega + E_f^0 - E_I^0\right)\right)$$

Fermi "Golden" rule



Example – Zero order system in the presence of an electromagnetic field

Full Hamiltonian: $H(\mathbf{r},t)=\frac{1}{2m}\left(\mathbf{p}-q\mathbf{A}(\mathbf{r},t)\right)^2+V(\mathbf{r})+qU(\mathbf{r},t)$

Zero order Hamiltonian: $H^0(\mathbf{r})=-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{r})$

First order Hamiltonian: $H^1(\mathbf{r},t)=\frac{-q}{m}\mathbf{A}(\mathbf{r},t)\cdot\mathbf{p}+\frac{i\hbar q}{2m}\left(\nabla\cdot\mathbf{A}(\mathbf{r},t)\right)+qU(\mathbf{r},t)$

Time dependent electric field: $\mathbf{F}(\mathbf{r},t)=-\nabla U(\mathbf{r},t)-\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}$

Example – Zero order system in the presence of an electromagnetic field -- continued

Time dependent electric field: $\mathbf{F}(\mathbf{r},t) = -\nabla U(\mathbf{r},t) - \dfrac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t}$

Suppose $\mathbf{F}(\mathbf{r},t) = \begin{cases} 0 & \text{for } t < 0 \text{ or } t > T \\ F_0 \hat{\mathbf{z}}\left(2\sin(\omega t)\right) & \text{for } 0 < t < T \end{cases}$

Possibility #1 -- scalar representation: $U(\mathbf{r},t) = -F_0 z\left(2\sin(\omega t)\right)$ for $0 < t < T$

$$\mathbf{A}(\mathbf{r},t) = 0$$

Possibility #2 -- vector representation: $U(\mathbf{r},t) = 0$

$$\mathbf{A}(\mathbf{r},t) = -\frac{F_0}{\omega}\hat{\mathbf{z}}\left(2\cos(\omega t)\right) \quad \text{for } 0 < t < T$$

Example – Zero order system in the presence of an electromagnetic field -- continued

First order Hamiltonian: $H^1(\mathbf{r},t) = \frac{-q}{m}\mathbf{A}(\mathbf{r},t)\cdot\mathbf{p} + \frac{i\hbar q}{2m}\left(\nabla\cdot\mathbf{A}(\mathbf{r},t)\right) + qU(\mathbf{r},t)$

Convenient notation: $H^1(\mathbf{r},t) \equiv \tilde{H}^1(\mathbf{r})\left(2\sin(\omega t)\right)$ for $0 < t < T$

Possibility #1 -- scalar representation: $\tilde{H}^1(\mathbf{r}) = -qF_0 z$

Possibility #2 -- vector representation: $\tilde{H}^1(\mathbf{r}) = \frac{qF_0}{m\omega}p_z$

Note that for this example, $\left(\nabla\cdot\mathbf{A}(\mathbf{r},t)\right) = 0$

Example – Zero order system in the presence of an electromagnetic field -- continued

Suppose the charged particle is an electron: $q = -e$

$\tilde{H}^1 = eF_0 z$ representing field as scalar potential

$\tilde{H}'^1 = -\dfrac{eF_0 p_z}{\omega m}$ representing field as vector potential

Note that these two are equivalent in the exact basis of H^0 :

$$\frac{p_z}{m} = \frac{1}{i\hbar}\left[z, \frac{\mathbf{p}^2}{2m}\right] = \frac{1}{i\hbar}\left[z, H^0\right]$$

$$\left\langle f^0 \middle| \frac{p_z}{m} \middle| I^0 \right\rangle = \frac{1}{i\hbar}\left\langle f^0 \middle| \left[z, H^0\right] \middle| I^0 \right\rangle = -\frac{E_f^0 - E_I^0}{i\hbar}\left\langle f^0 \middle| z \middle| I^0 \right\rangle$$

$$= i\omega_{fI}\left\langle f^0 \middle| z \middle| I^0 \right\rangle \qquad \text{for } \hbar\omega_{fi} \equiv E_f^0 - E_I^0$$

Example – Zero order system in the presence of an electromagnetic field -- continued

$$\left\langle f^0 \middle| \tilde{H}^1 \middle| I^0 \right\rangle = eF \left\langle f^0 \middle| z \middle| I^0 \right\rangle \quad \text{representing field as scalar potential}$$

$$\left\langle f^0 \middle| \tilde{H}'^1 \middle| I^0 \right\rangle = -\frac{eF}{\omega m} \left\langle f^0 \middle| p_z \middle| I^0 \right\rangle \quad \text{representing field as vector potential}$$

$$= -\frac{eF}{\omega} i\omega_{fi} \left\langle f^0 \middle| z \middle| I^0 \right\rangle$$

$$\Rightarrow \text{When } \omega = \omega_{fi}, \qquad \left| \left\langle f^0 \middle| \tilde{H}'^1 \middle| I^0 \right\rangle \right|^2 = \left| \left\langle f^0 \middle| \tilde{H}^1 \middle| I^0 \right\rangle \right|^2$$

Fermi golden rule: $$\mathcal{R}_{I\to f} = \frac{\left| k^1_{I\to f}(t) \right|^2}{T} \approx \frac{2\pi}{\hbar} \left| \left\langle f^0 \middle| \tilde{H}^1 \middle| I^0 \right\rangle \right|^2 \delta\left(\hbar\left(\omega - \omega_{fi}\right)\right)$$

Example -- H-like atom in presence of electric field

$$\tilde{H}^1 = -eFz \quad \text{representing field as scalar potential}$$
$$= -eFr\cos\theta$$

Some H^0 eigenstates for H-like ion:

$$\left| I^0 = 1s \right\rangle = \left(\frac{Z^3}{a_0^3 \pi} \right)^{1/2} e^{-Zr/a_0} \qquad E_I^0 = -\frac{Z^2 e^2}{8\pi\epsilon_0 a_0}$$

$$\left| f^0 = 2p_0 \right\rangle = \left(\frac{Z^3}{32 a_0^3 \pi} \right)^{1/2} \frac{Zr}{a_0} e^{-Zr/2a_0} \cos\theta \qquad E_f^0 = -\frac{Z^2 e^2}{8\pi\epsilon_0 a_0} \frac{1}{4}$$

$$\left\langle f^0 \middle| \tilde{H}^1 \middle| I^0 \right\rangle = \left\langle f^0 \middle| -eFr\cos\theta \middle| I^0 \right\rangle$$

H^0 eigenstates for H-like ion: $\left| I^0 = 1s \right\rangle = \left(\frac{Z^3}{a_0^3 \pi} \right)^{1/2} e^{-Zr/a_0}$

$$\left| f^0 = 2p_0 \right\rangle = \left(\frac{Z^3}{32 a_0^3 \pi} \right)^{1/2} \frac{Zr}{a_0} e^{-Zr/2a_0} \cos\theta$$

$$\left\langle f^0 \middle| \tilde{H}^1 \middle| I^0 \right\rangle = -eF \frac{Z^3}{a_0^3 \pi} \left(\frac{1}{32} \right)^{1/2} 2\pi \frac{2}{3} \int_0^\infty r^3 dr \frac{Zr}{a_0} e^{-3Zr/2a_0}$$

$$= -eF \frac{Z^3}{a_0^3 \pi} \left(\frac{1}{32} \right)^{1/2} 2\pi \frac{2}{3} \left(\frac{a_0}{Z} \right)^4 \int_0^\infty x^4 dx \; e^{-\frac{3}{2}x}$$

$$= -\frac{eFa_0}{\sqrt{2}Z} \frac{256}{243}$$

Summary of results for resonant transitions for H-like ion *1s→2p$_0$*

$$\mathcal{R}_{I\to f} \approx \frac{2\pi}{\hbar}\left|\left\langle f^0\left|\tilde{H}^1\right|I^0\right\rangle\right|^2 \delta\left(-\hbar\omega + E_f^0 - E_I^0\right)$$

$$\left\langle f^0\left|\tilde{H}^1\right|I^0\right\rangle = -\frac{eFa_0}{\sqrt{2}Z}\frac{256}{243}$$

$$\hbar\omega = E_f^0 - E_I^0 = \frac{3}{4}\frac{Z^2e^2}{8\pi\epsilon_0 a_0} = 10.204\, Z^2 \text{ eV}$$

Digression: Notion of oscillator strength for transition between states *l*→*n*:

$$f_{nl} = \frac{2m}{\hbar^2}\left(E_n^0 - E_l^0\right)\left|\left\langle n^0 \middle| z \middle| l^0 \right\rangle\right|^2$$

$$\left[z,\left[z,H^0\right]\right] = z^2 H^0 + H^0 z^2 - 2zH^0 z = \frac{i\hbar}{m}\left[z,p_z\right] = -\frac{\hbar^2}{m}$$

$$\left\langle l^0 \middle| \left[z,\left[z,H^0\right]\right] \middle| l^0 \right\rangle = 2E_l^0 \left\langle l^0 \middle| z^2 \middle| l^0 \right\rangle - 2\left\langle l^0 \middle| zH^0 z \middle| l^0 \right\rangle = -\frac{\hbar^2}{m}\left\langle l^0 \middle| l^0 \right\rangle$$

Inserting resolution of the identity: $1 = \sum_n \left| n^0 \right\rangle \left\langle n^0 \right|$

$$\sum_n \left(2E_l^0 \left\langle l^0 \middle| z \middle| n^0 \right\rangle \left\langle n^0 \middle| z \middle| l^0 \right\rangle - 2\left\langle l^0 \middle| z \middle| n^0 \right\rangle E_n^0 \left\langle n^0 \middle| z \middle| l^0 \right\rangle\right) = -\frac{h^2}{m}\left\langle l^0 \middle| l^0 \right\rangle$$

$$\frac{2m}{\hbar^2}\sum_n \left(E_n^0 - E_l^0\right)\left\langle l^0 \middle| z \middle| n^0 \right\rangle \left\langle n^0 \middle| z \middle| l^0 \right\rangle = \sum_n f_{nl} = \left\langle l^0 \middle| l^0 \right\rangle = 1$$

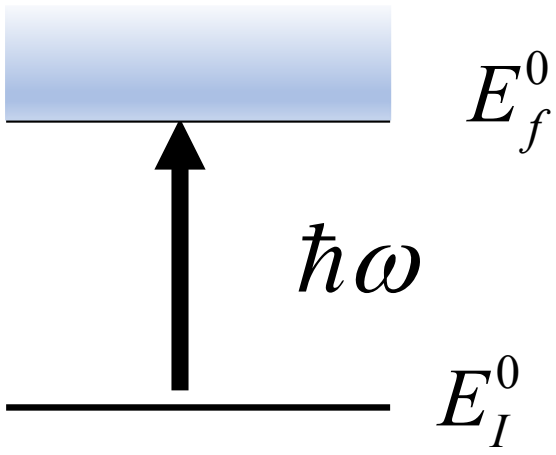
sum rule for oscillator strength

Digression – "selection rules" for transitions between spherically symmetric states in due to interaction with an electromagnetic field in the dipole approximation --

$$\left\langle f^0 \middle| \tilde{H}^1 \middle| I^0 \right\rangle = \left\langle f^0 \middle| -eFr\cos\theta \middle| I^0 \right\rangle$$

Symmetry analysis of the matrix element finds non-trivial matrix elements for $\ell_f - \ell_I = \pm 1$ and $m_f - m_I = 0, \pm 1$.

Example when the final state is in the continuum spectrum

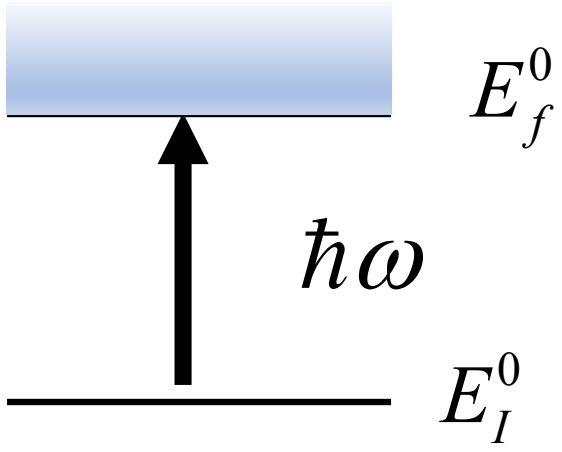


Exact solutions for continuum states of H-like ions are related to confluent hypergeometric functions

Absorption of radiation in the case of photo emission

$$\left|f^0\right\rangle = R_{El}(r)Y_{lm}(\hat{\mathbf{r}})$$

$$\left(-\frac{\hbar^2}{2m}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}\right)-\frac{Ze^2}{4\pi\epsilon_0 r}-E\right)R_{El}(r)=0$$



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§33.2(iii) Irregular Solutions $G_\ell(\eta,\rho), H_\ell(\eta,\rho)$

§33.2(iv) Wronskians and Cross-Product

§33.2(i) Coulomb Wave Equation

33.2.1 $$\frac{d^2w}{d\rho^2}+\left(1-\frac{2\eta}{\rho}-\frac{\ell(\ell+1)}{\rho^2}\right)w=0,$$

$\ell = 0, 1, 2, \ldots$

This differential equation has a regular singularity at $\rho = 0$ with indices $\ell + 1$ and $-\ell$ and an irregular singularity of rank 1 at $\rho = \infty$ (§§2.7(i), 2.7(ii)). There are two turning points, that is, points at which $d^2w/d\rho^2 = 0$ (§2.8(i)). The outer one is given by

33.2.2 $$\rho_{\mathrm{tp}}(\eta,\ell) = \eta + (\eta^2 + \ell(\ell+1))^{1/2}.$$

§33.2(ii) Regular Solution $F_\ell(\eta,\rho)$

The function $F_\ell(\eta,\rho)$ is recessive (§2.7(iii)) at $\rho = 0$, and is defined by

33.2.3 $$F_\ell(\eta,\rho) = C_\ell(\eta)2^{-\ell-1}(\mp\mathrm{i})^{\ell+1}M_{\pm\mathrm{i}\eta,\ell+\frac{1}{2}}(\pm 2\mathrm{i}\rho),$$

or equivalently

33.2.4 $$F_\ell(\eta,\rho) = C_\ell(\eta)\rho^{\ell+1}e^{\mp\mathrm{i}\rho}M(\ell+1\mp\mathrm{i}\eta, 2\ell+2, \pm 2\mathrm{i}\rho),$$

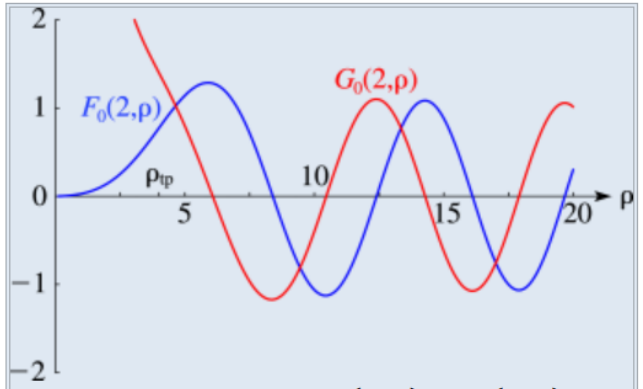

Figure 33.3.3: $F_\ell(\eta,\rho)$, $G_\ell(\eta,\rho)$ with $\ell = 0$, $\eta = 2$. The turning point is at $\rho_{\mathrm{tp}}(2,0) = 4$.

Absorption of radiation in the case of photo emission approximating final state as a plane wave (Born approximation)

$$\left|f^0\right\rangle \approx \mathcal{N} e^{i\mathbf{k}\cdot\mathbf{r}} \qquad \text{where } k = \sqrt{(2mE_f^0/\hbar^2)}$$

For initial state: $$\left|I^0 = 1s\right\rangle = \left(\frac{Z^3}{a_0^3\pi}\right)^{1/2} e^{-Zr/a_0}$$

$$\mathcal{R}_{I\to f} \approx \frac{2\pi}{\hbar}\left|\left\langle f^0\left|\tilde{H}^1\right|I^0\right\rangle\right|^2 \delta\left(-\hbar\omega + E_f^0 - E_I^0\right)$$

$$\left\langle f^0\left|\tilde{H}^1\right|I^0\right\rangle = \left\langle f^0\left|-eFr\cos\theta\right|I^0\right\rangle$$

To be continued …