PHY 742 Quantum Mechanics II
12-12:50 PM MWF Olin 103

Plan for Lecture 14

Time dependent perturbation theory
Ref: Chapter 15 in E. Carlson's textbook

1. Time harmonic responances & Fermi's Golden Rule
2. Transitions between bound states
3. Transitions between bound and continuum states

Course schedule for Spring 2022

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	Reading	Topic	HW	Due date
1	Mon: 01/10/2022	Chap. 12	Approximate solutions for stationary states -- The variational approach	#1	01/14/2022
2	Wed: 01/12/2022	Chap. 12 C	Approximate solutions for stationary states -- Perturbation theory	#2	01/19/2022
3	Fri: 01/14/2022	Chap. 12 D	Approximate solutions for stationary states -- Degenerate perturbation theory	#3	01/21/2022
	Mon: 01/17/2022		MLK Holiday -- no class		
4	Wed: 01/19/2022	Chap. 12 C & D	Approximate solutions for stationary states -- Additional tricks	#4	01/24/2022
5	Fri: 01/21/2022	Chap. 13	Examples of of the use of perturbation theory	#5	01/26/2022
6	Mon: 01/24/2022	Chap. 13 & 12 B	Hyperfine perturbation and also the WKB approximation	#6	01/28/2022
7	Wed: 01/26/2022	Chap. 14	Scattering theory		
8	Fri: 01/28/2022	Chap. 14	Scattering theory	#7	02/04/2022
9	Mon: 01/31/2022	Chap. 14	Scattering theory	#8	02/07/2022
	Wed: 02/02/2022	No class	Fire caution		
	Fri: 02/04/2022	No class	Fire caution		
10	Mon: 02/07/2022	Chap. 11 (A-C)	Time evolution and Feynman path integrals	#9	02/09/2022
11	Wed: 02/09/2022	Chap. 11 (A-C)	Time evolution and Feynman path integrals	#10	02/11/2022
12	Fri: 02/11/2022	Chap. 15 A	Approximation methods for time evolution of quantum systems	#11	02/14/2022
13	Mon: 02/14/2022	Chap. 15	Approximate time evolution	#12	02/16/2022
14	Wed: 02/16/2022	Chap. 15	Fermi Golden Rule	#13	02/18/2022
15	Fri: 02/18/2022	Chap. 15	Matrix elements and selection rules		
	Mon: 02/21/2022	Chaps. (11-15)	Homework review & presentations		

PHY 742 -- Assignment #13

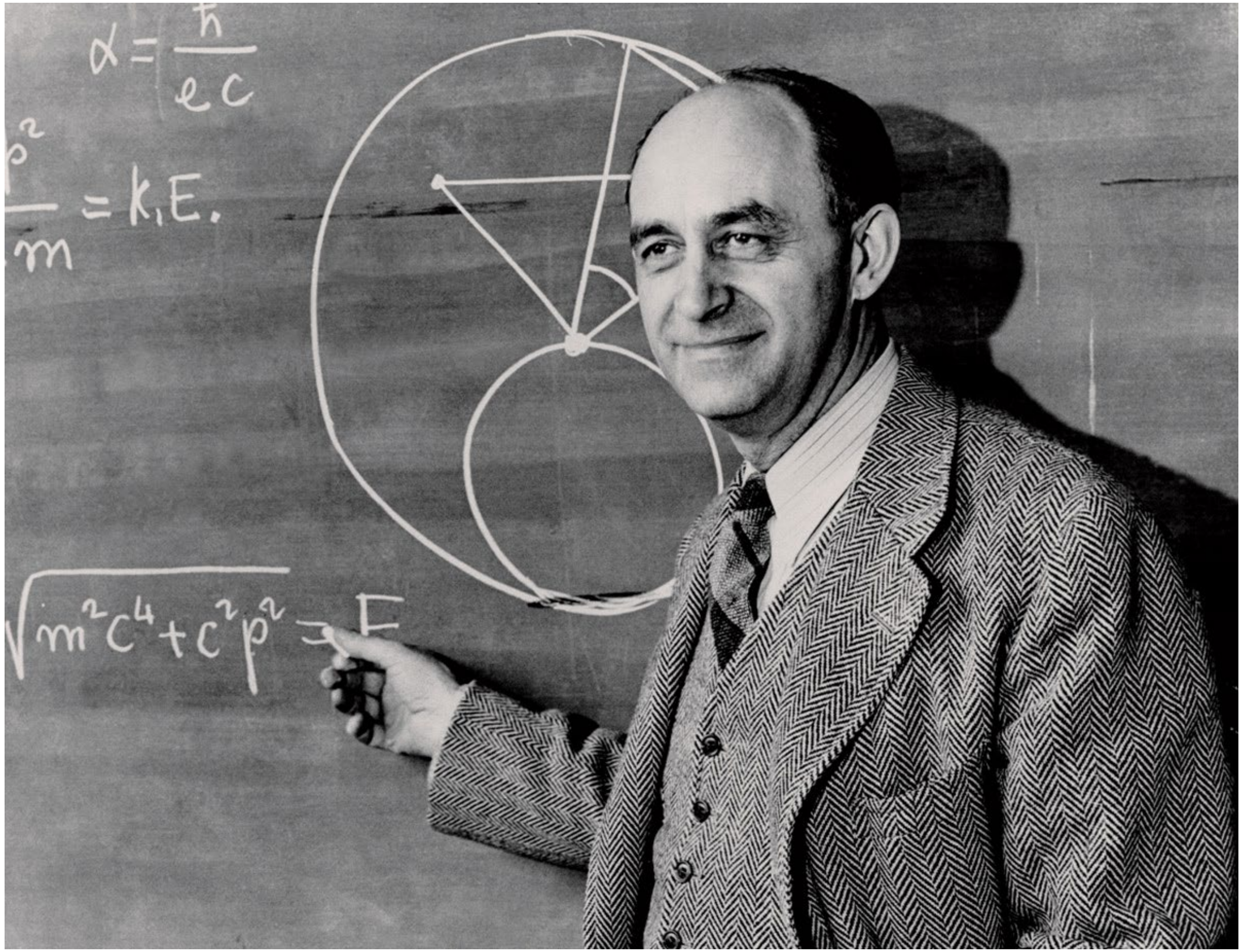
February 16, 2022

Finish reading Chapter 15 in **Professor Carlson's QM textbook.**.

1. In class we considered the transition matrix element between two eigenstates of a hydrogen like ion -- $|i^0=100\rangle$ and $|f^0=210\rangle$ for $\langle f^0|z|i^0\rangle$. Evaluate the corresponding matrix element $\langle f^0|p_z|i^0\rangle$.

Famous Italian & American physicist

Enrico Fermi 1901-1954



Fermi Golden rule for time harmonic resonant transitions

Considering a time harmonic perturbation acting for a time T of the form:

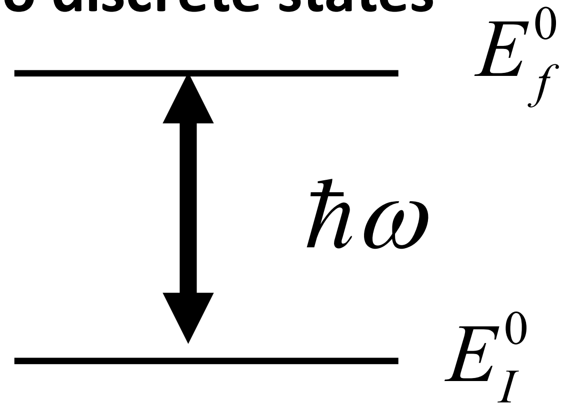
$$H^1(\mathbf{r},t) = \tilde{H}^1(\mathbf{r})\ 2\sin(\omega t) \qquad \text{or} \qquad H^1(\mathbf{r},t) = \tilde{H}'^1(\mathbf{r})\ 2\cos(\omega t)$$

Estimating the rate of transitions *I* → *f*

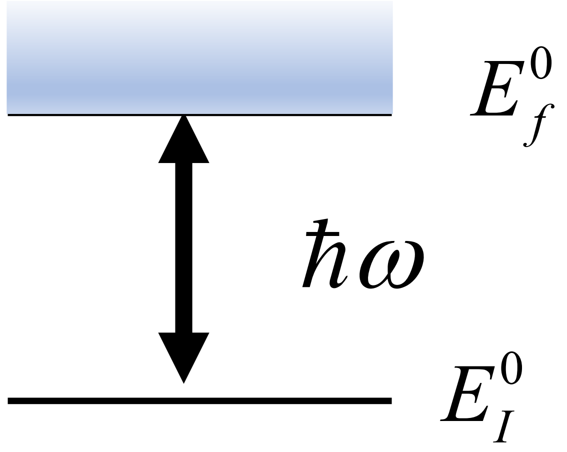
$$\mathcal{R}_{I\to f} = \frac{\left|k^1_{I\to f}(t)\right|^2}{T} \approx \frac{2\pi}{\hbar}\left|\left\langle f^0\middle|\tilde{H}^1\middle|I^0\right\rangle\right|^2 \left(\delta\left(\hbar\omega + E^0_f - E^0_I\right) + \delta\left(-\hbar\omega + E^0_f - E^0_I\right)\right)$$

Fermi "Golden" rule

Transition between two discrete states



Transition between discrete and continuum states



Example – Zero order system in the presence of an electromagnetic field

Full Hamiltonian: $H(\mathbf{r},t)=\frac{1}{2m}\left(\mathbf{p}-q\mathbf{A}(\mathbf{r},t)\right)^2+V(\mathbf{r})+qU(\mathbf{r},t)$

Zero order Hamiltonian: $H^0(\mathbf{r})=-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{r})$

First order Hamiltonian: $H^1(\mathbf{r},t)=\frac{-q}{m}\mathbf{A}(\mathbf{r},t)\cdot\mathbf{p}+\frac{i\hbar q}{2m}\left(\nabla\cdot\mathbf{A}(\mathbf{r},t)\right)+qU(\mathbf{r},t)$

Time dependent electric field: $\mathbf{F}(\mathbf{r},t)=-\nabla U(\mathbf{r},t)-\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}$

In most cases, either term will be active, but generally not both.

Example – Zero order system in the presence of an electromagnetic field -- continued

Time dependent electric field: $\mathbf{F}(\mathbf{r},t) = -\nabla U(\mathbf{r},t) - \frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t}$

Suppose $\mathbf{F}(\mathbf{r},t) = \begin{cases} 0 & \text{for } t<0 \text{ or } t>T \\ F_0\hat{\mathbf{z}}(2\sin(\omega t)) & \text{for } 0<t<T \end{cases}$

Possibility #1 -- scalar representation: $U(\mathbf{r},t) = -F_0 z(2\sin(\omega t)) \quad \text{for } 0<t<T$

$$\mathbf{A}(\mathbf{r},t) = 0$$

Possibility #2 -- vector representation: $U(\mathbf{r},t) = 0$

$$\mathbf{A}(\mathbf{r},t) = -\frac{F_0}{\omega}\hat{\mathbf{z}}(2\cos(\omega t)) \quad \text{for } 0<t<T$$

Example – Zero order system in the presence of an electromagnetic field -- continued

First order Hamiltonian: $H^1(\mathbf{r},t) = \frac{-q}{m}\mathbf{A}(\mathbf{r},t)\cdot\mathbf{p} + \frac{i\hbar q}{2m}\left(\nabla\cdot\mathbf{A}(\mathbf{r},t)\right) + qU(\mathbf{r},t)$

Convenient notation: $H^1(\mathbf{r},t) \equiv \tilde{H}^1(\mathbf{r})\left(2\sin(\omega t)\right)$ for $0 < t < T$

Possibility #1 -- scalar representation: $\tilde{H}^1(\mathbf{r}) = -qF_0 z$

Possibility #2 -- vector representation: $\tilde{H}^1(\mathbf{r}) = \frac{qF_0}{m\omega}p_z$

Note that for this example, $\left(\nabla\cdot\mathbf{A}(\mathbf{r},t)\right) = 0$

Example – Zero order system in the presence of an electromagnetic field -- continued

Suppose the charged particle is an electron: $q = -e$

$\tilde{H}^1 = eF_0 z$ representing field as scalar potential

$\tilde{H}'^1 = -\frac{eF_0 p_z}{\omega m}$ representing field as vector potential

Note that these two are equivalent in the exact basis of H^0 :

$$\frac{p_z}{m} = \frac{1}{i\hbar}\left[z, \frac{\mathbf{p}^2}{2m}\right] = \frac{1}{i\hbar}\left[z, H^0\right]$$

$$\left\langle f^0 \middle| \frac{p_z}{m} \middle| I^0 \right\rangle = \frac{1}{i\hbar}\left\langle f^0 \middle| \left[z, H^0\right] \middle| I^0 \right\rangle = -\frac{E_f^0 - E_I^0}{i\hbar}\left\langle f^0 \middle| z \middle| I^0 \right\rangle$$

$$= i\omega_{fI}\left\langle f^0 \middle| z \middle| I^0 \right\rangle \quad \text{for } \hbar\omega_{fi} \equiv E_f^0 - E_I^0$$

Similar to your HW problem….

Example -- H-like atom in presence of electric field

$$\tilde{H}^1 = -eF_0 z \quad \text{representing field as scalar potential}$$
$$= -eF_0 r\cos\theta$$

Some H^0 eigenstates for H-like ion:

$$\left|I^0 = 1s\right\rangle = \left(\frac{Z^3}{a_0^3\pi}\right)^{1/2} e^{-Zr/a_0} \qquad E_I^0 = -\frac{Z^2e^2}{8\pi\epsilon_0 a_0}$$

$$\left|f^0 = 2p_0\right\rangle = \left(\frac{Z^3}{32a_0^3\pi}\right)^{1/2} \frac{Zr}{a_0} e^{-Zr/2a_0}\cos\theta \qquad E_f^0 = -\frac{Z^2e^2}{8\pi\epsilon_0 a_0}\frac{1}{4}$$

Digression on matrix elements --

For transition matrix elements between states of spherically symmetric systems, we typically must evaluate "Gaunt" coefficents:

$$\left\langle f^0 \middle| \tilde{H}^1 \middle| I^0 \right\rangle \propto \int d\Omega \; Y^*_{l_1 m_1}(\theta,\phi) Y_{l_2 m_2}(\theta,\phi) Y_{l_3 m_3}(\theta,\phi)$$

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$$\int_0^{2\pi}\int_0^{\pi} Y_{l_1,m_1}(\theta,\phi) Y_{l_2,m_2}(\theta,\phi) Y_{l_3,m_3}(\theta,\phi) \sin\theta \, d\theta \, d\phi = \left(\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}\right)^{\frac{1}{2}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}.$$

3j symbols

34.2.4 $$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1-j_2-m_3}\Delta(j_1 j_2 j_3)\left((j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(j_3+m_3)!(j_3-m_3)!\right)^{\frac{1}{2}}$$
$$\times\sum_s \frac{(-1)^s}{s!(j_1+j_2-j_3-s)!(j_1-m_1-s)!(j_2+m_2-s)!(j_3-j_2+m_1+s)!(j_3-j_1-m_2+s)!},$$

where

34.2.5 $$\Delta(j_1 j_2 j_3) = \left(\frac{(j_1+j_2-j_3)!(j_1-j_2+j_3)!(-j_1+j_2+j_3)!}{(j_1+j_2+j_3+1)!}\right)^{\frac{1}{2}},$$

The quantities j_1, j_2, j_3 in the $3j$ symbol are called *angular momenta*. Either all of them are nonnegative integers, or one is a nonnegative integer and the other two are half-odd positive integers. They must form the sides of a triangle (possibly degenerate). They therefore satisfy the *triangle conditions*

34.2.1 $$|j_r - j_s| \leq j_t \leq j_r + j_s,$$

where r, s, t is any permutation of $1, 2, 3$. The corresponding *projective quantum numbers* m_1, m_2, m_3 are given by

34.2.2 $$m_r = -j_r, -j_r+1, \ldots, j_r-1, j_r, \qquad r = 1, 2, 3,$$

and satisfy

34.2.3 $$m_1 + m_2 + m_3 = 0.$$

Back to calculation of matrix element: $\left\langle f^0 \middle| \tilde{H}^1 \middle| I^0 \right\rangle = \left\langle f^0 \middle| -eFr\cos\theta \middle| I^0 \right\rangle$

H^0 eigenstates for H-like ion: $\left| I^0 = 1s \right\rangle = \left(\frac{Z^3}{a_0^3 \pi} \right)^{1/2} e^{-Zr/a_0}$

$$\left| f^0 = 2p_0 \right\rangle = \left(\frac{Z^3}{32 a_0^3 \pi} \right)^{1/2} \frac{Zr}{a_0} e^{-Zr/2a_0} \cos\theta$$

$$\left\langle f^0 \middle| \tilde{H}^1 \middle| I^0 \right\rangle = -eF \frac{Z^3}{a_0^3 \pi} \left(\frac{1}{32} \right)^{1/2} 2\pi \frac{2}{3} \int_0^\infty r^3 dr \frac{Zr}{a_0} e^{-3Zr/2a_0}$$

$$= -eF \frac{Z^3}{a_0^3 \pi} \left(\frac{1}{32} \right)^{1/2} 2\pi \frac{2}{3} \left(\frac{a_0}{Z} \right)^4 \int_0^\infty x^4 dx \; e^{-\frac{3}{2}x}$$

$$= -\frac{eFa_0}{\sqrt{2}Z} \frac{256}{243}$$

Summary of results for resonant transitions for H-like ion *1s→2p₀*

$$\mathscr{R}_{I\to f} \approx \frac{2\pi}{\hbar}\left|\left\langle f^0\middle|\tilde{H}^1\middle|I^0\right\rangle\right|^2 \delta\left(-\hbar\omega + E_f^0 - E_I^0\right)$$

$$\left\langle f^0\middle|\tilde{H}^1\middle|I^0\right\rangle = -\frac{eFa_0}{\sqrt{2}Z}\frac{256}{243}$$

$$\hbar\omega = E_f^0 - E_I^0 = \frac{3}{4}\frac{Z^2e^2}{8\pi\epsilon_0 a_0} = \;10.204\; Z^2 \text{ eV}$$

More general problem -- which transitions can occur?g

On Friday, we will discuss "selection rules" for transitions between spherically symmetric states in due to interaction with an electromagnetic field in the dipole approximation --

$$\left\langle f^0 \middle| \tilde{H}^1 \middle| I^0 \right\rangle = \left\langle f^0 \middle| -eF_0 r\cos\theta \middle| I^0 \right\rangle$$

Symmetry analysis of the matrix element finds non-trivial matrix elements for $\ell_f - \ell_I = \pm 1$ and $m_f - m_I = 0, \pm 1.$

Summary of results of 1st order theory for a time harmonic perturbation of the form:

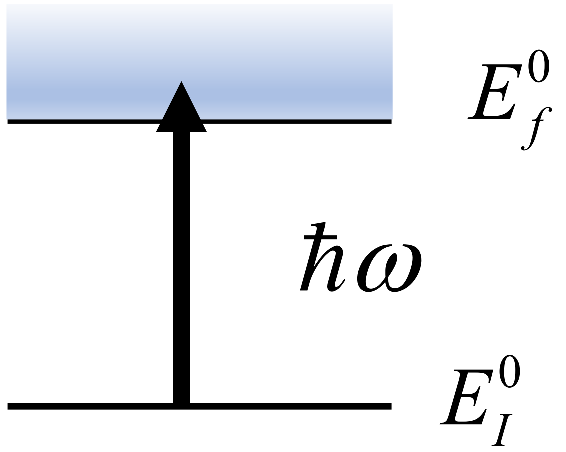
$$\text{Suppose that } H^1(t) = \tilde{H}^1 h(t)$$

$$\text{where} \qquad h(t) \equiv \begin{cases} 0 & \text{for } t < 0 \text{ and } t > T \\ 2\sin\omega t & \text{for } 0 < t < T \end{cases}$$

Fermi golden rule:

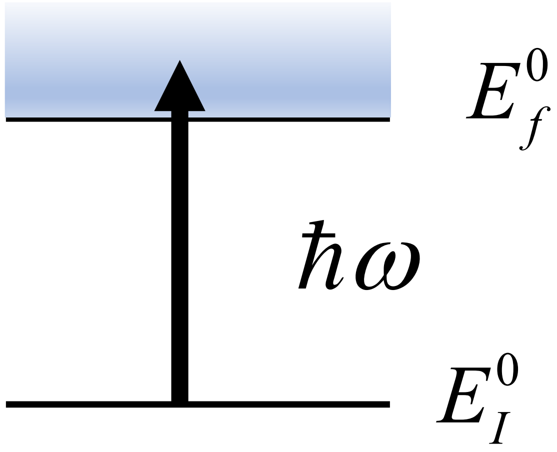
$$\mathcal{R}_{I\to f} = \frac{\left|k^1_{I\to f}(t)\right|^2}{T} \approx \frac{2\pi}{\hbar}\left|\left\langle f^0\right|\tilde{H}^1\left|I^0\right\rangle\right|^2 \delta\left(\hbar\left(\omega-\omega_{fi}\right)\right)$$

$$\text{where} \qquad \hbar\omega_{fi} \equiv E_f^0 - E_I^0$$



Treatment of the case when the initial state is bound and the final state is in the continuum spectrum --

Absorption of radiation in the case of photoemission of a H-like atom



Initial state: $$\left|I^0 = 1s\right\rangle = \left(\frac{Z^3}{a_0^3\pi}\right)^{1/2} e^{-Zr/a_0} \qquad E_I^0 = -\frac{Z^2e^2}{8\pi\epsilon_0 a_0}$$

It is convenient to approximate the final state as a plane wave (Born approximation)

$$\left|f^0\right\rangle \approx \mathcal{N} e^{i\mathbf{k}\cdot\mathbf{r}} \qquad \text{where } k = \sqrt{(2mE_f^0/\hbar^2)}$$

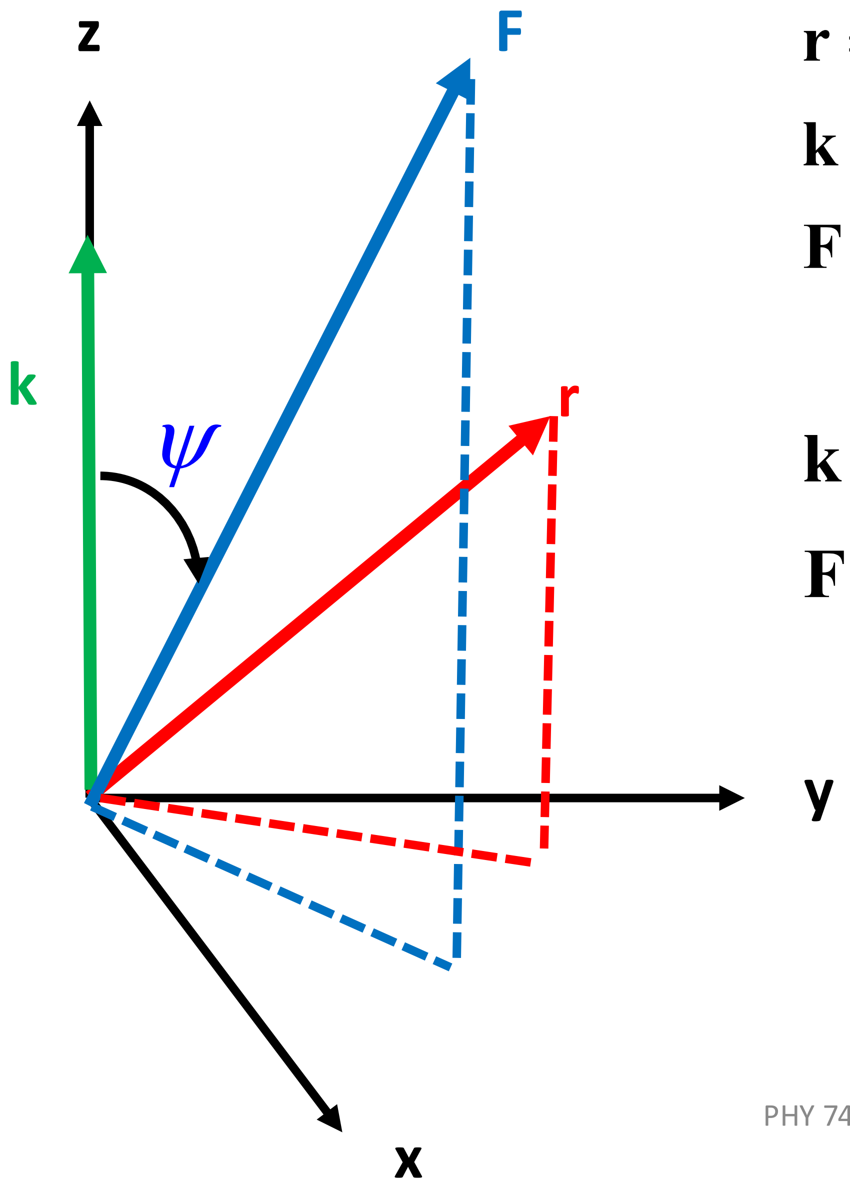
Transition rate:

$$\mathcal{R}_{I\to f} \approx \frac{2\pi}{\hbar}\left|\left\langle f^0\right|\tilde{H}^1\left|I^0\right\rangle\right|^2 \delta\left(-\hbar\omega + E_f^0 - E_I^0\right)$$

In this case, for a uniform electric field amplitude $\mathbf{F}$:

$$\tilde{H}^1(\mathbf{r}) = e\mathbf{F}\cdot\mathbf{r} \qquad \text{or} \qquad \tilde{H}'^1(\mathbf{r}) = \frac{e}{m\omega}\mathbf{F}\cdot\mathbf{p}$$

Convenient coordinate system --



$$\mathbf{r} = r\left(\sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}\right)$$

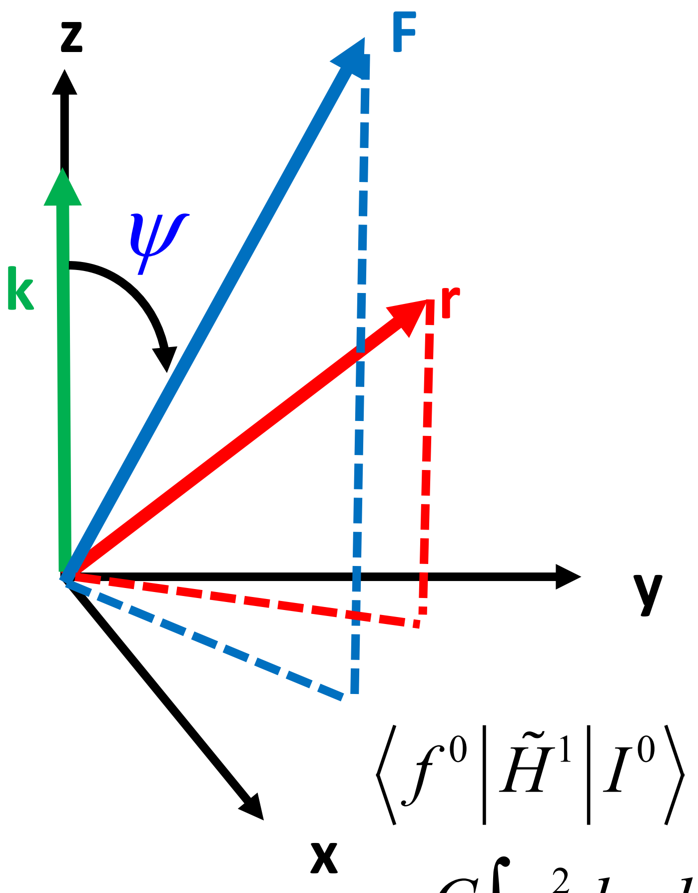
$$\mathbf{k} = k\hat{\mathbf{z}}$$

$$\mathbf{F} = F\left(\sin\psi\cos\chi\hat{\mathbf{x}} + \sin\psi\sin\chi\hat{\mathbf{y}} + \cos\psi\hat{\mathbf{z}}\right)$$

$$\mathbf{k}\cdot\mathbf{r} = kr\cos\theta$$

$$\mathbf{F}\cdot\mathbf{r} = Fr\left(\sin\psi\sin\theta\cos\left(\chi-\phi\right) + \cos\psi\cos\theta\right)$$

Approximate photoemission -- continued



$$\mathbf{r} = r\left(\sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}\right)$$

$$\mathbf{k} = k\hat{\mathbf{z}}$$

$$\mathbf{F} = F\left(\sin\psi\cos\chi\hat{\mathbf{x}} + \sin\psi\sin\chi\hat{\mathbf{y}} + \cos\psi\hat{\mathbf{z}}\right)$$

$$\mathbf{k}\cdot\mathbf{r} = kr\cos\theta$$

$$\mathbf{F}\cdot\mathbf{r} = Fr\left(\sin\psi\sin\theta\cos\left(\chi-\phi\right) + \cos\psi\cos\theta\right)$$

$$\left\langle f^0 \middle| \tilde{H}^1 \middle| I^0 \right\rangle = \left\langle f^0 \middle| e\mathbf{F}\cdot\mathbf{r} \middle| I^0 \right\rangle$$

$$= C\int r^2 dr\; d\cos\theta\; d\phi\; e^{ikr\cos\theta} e^{-Zr/a_0} r\left(\sin\psi\sin\theta\cos\left(\chi-\phi\right) + \cos\psi\cos\theta\right)$$

$$\text{where } C \equiv \left(\frac{Z^3}{a_0^3\pi}\right)^{1/2} \mathcal{N}eF$$

Approximate photoemission -- continued

Some details:

$$\int r^2 dr\ d\cos\theta\ d\phi\ e^{ikr\cos\theta} e^{-Zr/a_0} r\left(\sin\psi\sin\theta\cos\left(\chi-\phi\right)+\cos\psi\cos\theta\right)$$

$$=2\pi\int r^2 dr\ d\cos\theta\ e^{ikr\cos\theta} e^{-Zr/a_0} r\cos\psi\cos\theta$$

$$=\frac{4i\pi\cos\psi}{k^2}\int_0^\infty r dr\left(kr\cos(kr)-\sin(kr)\right)e^{-Zr/a_0}$$

$$=-32i\pi\cos\psi\frac{ka_0^5}{Z^5}\frac{1}{\left(1+k^2a_0^2/Z^2\right)^3}$$

$$\left\langle f^0\middle|e\mathbf{F}\cdot\mathbf{r}\middle|I^0\right\rangle=-32i\pi\left(\frac{Z^3}{a_0^3\pi}\right)^{1/2}\mathcal{N}eF\frac{ka_0^5}{Z^5}\frac{\cos\psi}{\left(1+k^2a_0^2/Z^2\right)^3}$$

Approximate photoemission -- continued

$$\mathcal{R}_{I\to f} \approx \frac{2\pi}{\hbar}\left|\left\langle f^0\right|\tilde{H}^1\left|I^0\right\rangle\right|^2 \delta\left(-\hbar\omega + E_f^0 - E_I^0\right)$$

$$\left\langle f^0\right|e\mathbf{F}\cdot\mathbf{r}\left|I^0\right\rangle = -32i\pi\left(\frac{Z^3}{a_0^3\pi}\right)^{1/2} \mathcal{N}eF\frac{ka_0^5}{Z^5}\frac{\cos\psi}{\left(1+k^2a_0^2/Z^2\right)^3}$$

Digression – In general, the full transition rate is determined by averaging over all initial states and summing over all final states.

In our case, there is only one initial state, but a continuum of final states.

$$\mathcal{R}_I(\omega) = \sum_f \mathcal{R}_{I\to f} = \sum_f \frac{2\pi}{\hbar}\left|\left\langle f^0\right|\tilde{H}^1\left|I^0\right\rangle\right|^2 \delta\left(-\hbar\omega + E_f^0 - E_I^0\right)$$

Approximate photoemission -- continued

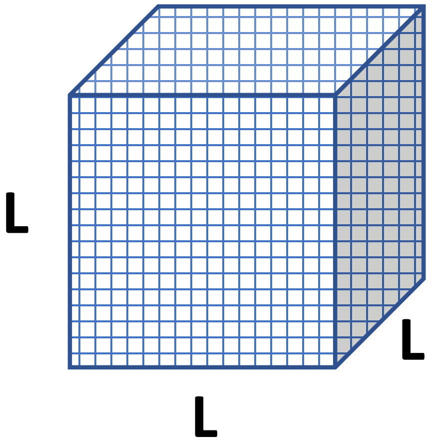
$$\mathcal{R}_I(\omega) = \sum_f \mathcal{R}_{I\to f} \quad = \sum_f \frac{2\pi}{\hbar}\left|\left\langle f^0 \middle| \tilde{H}^1 \middle| I^0 \right\rangle\right|^2 \delta\left(-\hbar\omega + E_f^0 - E_I^0\right)$$

Note that there are contributions when $E_f^0 = E_I^0 + \hbar\omega$

$$\left| f^0 \right\rangle \approx \mathcal{N} e^{i\mathbf{k}\cdot\mathbf{r}} \qquad \text{where } k = \sqrt{(2mE_f^0 / \hbar^2)}$$

Note that there are multiple values of k for each E_f^0

Digression – how can we count the number of plane waves?



Imagine that the plane waves associated with this box satisfy periodic boundary conditions. $\Psi(\mathbf{r}) = \Psi(\mathbf{r} + n_1 L\hat{\mathbf{x}} + n_2 L\hat{\mathbf{y}} + n_3 L\hat{\mathbf{z}}) = \mathcal{N} e^{i\mathbf{k}\cdot\mathbf{r}}$

This is only possible if $\mathbf{k} = \frac{2\pi}{L}\left(m_1\hat{\mathbf{x}} + m_2\hat{\mathbf{y}} + m_3\hat{\mathbf{z}}\right)$

Now we can count the number of final states --

$$\sum_f \rightarrow \sum_{\mathbf{k}} \rightarrow \left(\frac{L}{2\pi}\right)^3 \int d^3k = \frac{\mathcal{V}}{(2\pi)^3}\int d^3k$$

For consistency, we should normalize the plane waves within the box

$$\Rightarrow \mathcal{N} = \sqrt{\frac{1}{L^3}} = \sqrt{\frac{1}{\mathcal{V}}}$$

Approximate photoemission -- continued

$$\mathcal{R}_I(\omega) = \sum_f \mathcal{R}_{I\to f} \quad = \sum_f \frac{2\pi}{\hbar} \left| \left\langle f^0 \middle| \tilde{H}^1 \middle| I^0 \right\rangle \right|^2 \delta\left(-\hbar\omega + E_f^0 - E_I^0\right)$$

$$\left\langle f^0 \middle| e\mathbf{F}\cdot\mathbf{r} \middle| I^0 \right\rangle = -32i\pi \left(\frac{Z^3}{a_0^3\pi}\right)^{1/2} \left(\frac{1}{\mathcal{V}^{1/2}}\right) eF \frac{ka_0^5}{Z^5} \frac{\cos\psi}{\left(1+k^2a_0^2/Z^2\right)^3}$$

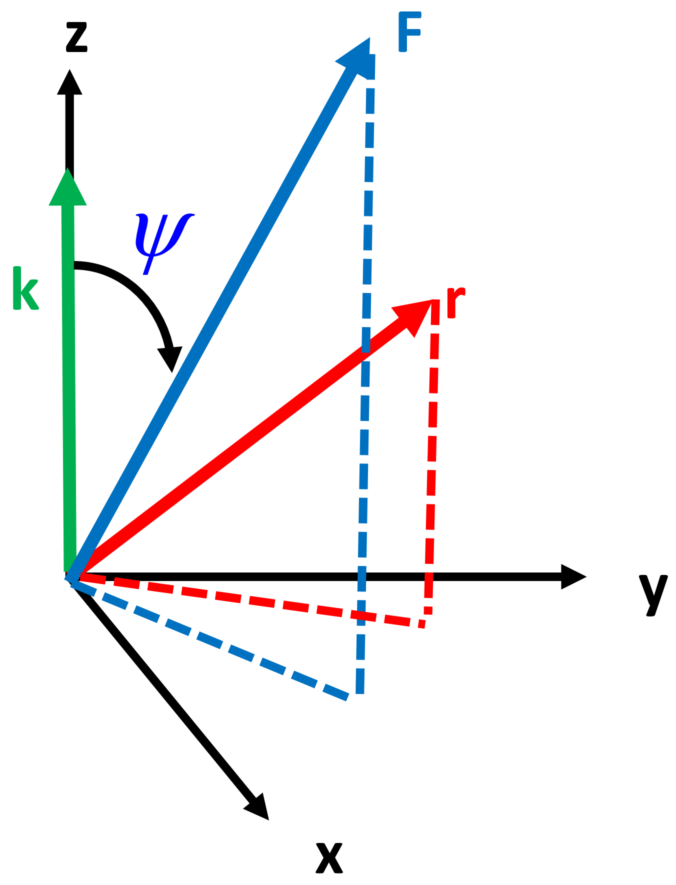
$$\mathcal{R}_I(\omega) = \frac{2\pi}{\hbar} \frac{\mathcal{V}}{(2\pi)^3} \int d^3k \left| 32i\pi \left(\frac{Z^3}{a_0^3\pi}\right)^{1/2} \left(\frac{1}{\mathcal{V}^{1/2}}\right) eF \frac{ka_0^5}{Z^5} \frac{\cos\psi}{\left(1+k^2a_0^2/Z^2\right)^3} \right|^2 \delta\left(\frac{\hbar^2k^2}{2m} - E_I^0 - \hbar\omega\right)$$

Writing $d^3k = k^2dk\ d\Omega_k$

$$\frac{\mathcal{R}_I(\omega)}{d\Omega_k} = \frac{2\pi}{\hbar} \frac{\mathcal{V}}{(2\pi)^3} \int k^2dk \left| 32i\pi \left(\frac{Z^3}{a_0^3\pi}\right)^{1/2} \left(\frac{1}{\mathcal{V}^{1/2}}\right) eF \frac{ka_0^5}{Z^5} \frac{\cos\psi}{\left(1+k^2a_0^2/Z^2\right)^3} \right|^2 \delta\left(\frac{\hbar^2k^2}{2m} - E_I^0 - \hbar\omega\right)$$

$$\frac{\mathcal{R}_I(\omega)}{d\Omega_k} = \frac{256mk^3e^2F^2a_0^7/Z^7}{\pi\hbar^3\left(1+k^2a_0^2/Z^2\right)^6}\cos^2\psi \qquad \text{where } \frac{\hbar^2k^2}{2m} = E_I^0 + \hbar\omega$$

Summary of results --



$$\frac{\mathcal{R}_I(\omega)}{d\Omega_k} = \frac{256 m k^3 e^2 F^2 a_0^7 / Z^7}{\pi \hbar^3 \left(1 + k^2 a_0^2 / Z^2\right)^6} \cos^2\psi$$

$$\text{where } \frac{\hbar^2 k^2}{2m} = E_I^0 + \hbar\omega$$

$$E_I^0 = -\frac{Z^2 e^2}{8\pi\epsilon_0 a_0} = -\frac{\hbar^2}{2 m a_0^2 / Z^2}$$

$$a_0 = \frac{4\pi\epsilon_0 \hbar^2}{m e^2}$$