PHY 742 Quantum Mechanics II
12-12:50 AM MWF Olin 103

Plan for Lecture 16

The Dirac equation –
Relativistic treatment of spin ½ particles
Ref: Chapter 16 of Professor Carlson's Textbook

1. Some simple concepts of special theory of relativity

2. Energy and momentum relationships

3. Dirac equation for a free particle

10	Mon: 02/07/2022	Chap. 11 (A-C)	Time evolution and Feynman path integrals	#9	02/09/2022
11	Wed: 02/09/2022	Chap. 11 (A-C)	Time evolution and Feynman path integrals	#10	02/11/2022
12	Fri: 02/11/2022	Chap. 15 A	Approximation methods for time evolution of quantum systems	#11	02/14/2022
13	Mon: 02/14/2022	Chap. 15	Approximate time evolution	#12	02/16/2022
14	Wed: 02/16/2022	Chap. 15	Fermi Golden Rule	#13	02/18/2022
15	Fri: 02/18/2022	Chap. 15	Matrix elements and selection rules		
	Mon: 02/21/2022	Chaps. (11-15)	Homework review & presentations		
16	Wed: 02/23/2022	Chap. 16	The Dirac equation	#14	02/25/2022
17	Fri: 02/25/2022	Chap. 16	The Dirac equation		

PHY 742 -- Assignment #14

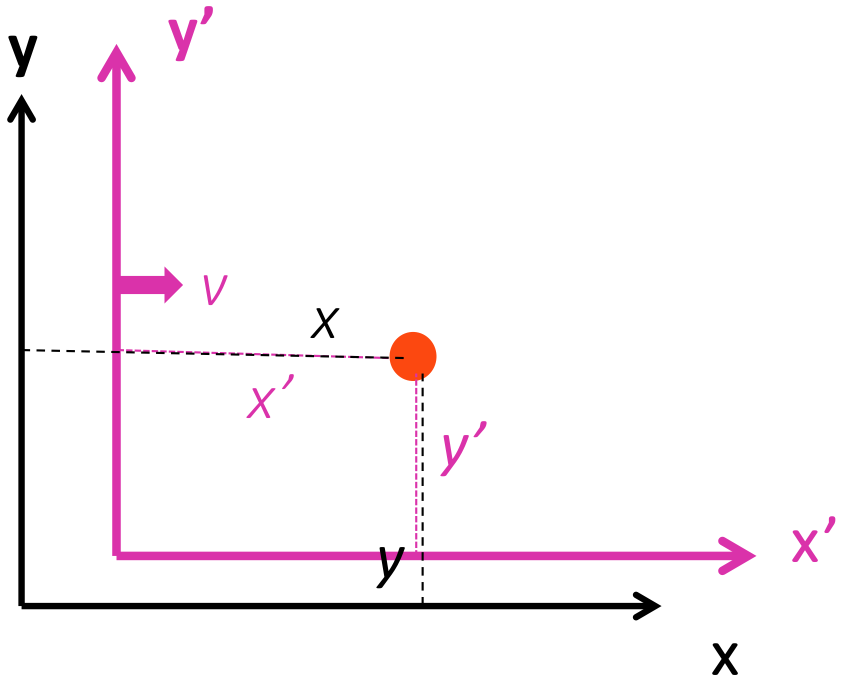
February 23, 2022

Start reading Chapter 16 in **Professor Carlson's QM textbook..**

1. Write the Hamiltonian operator for a spin 1/2 free particle according to the Dirac equation in its 4 x 4 form. Show that the eigenvalues and eigenvectors are consistent with the coupled two-component form that we discussed in class.

Notions of special relativity

- The basic laws of physics are the same in all frames of reference (at rest or moving at constant velocity).
- The speed of light in vacuum *c* is the same in all frames of reference.

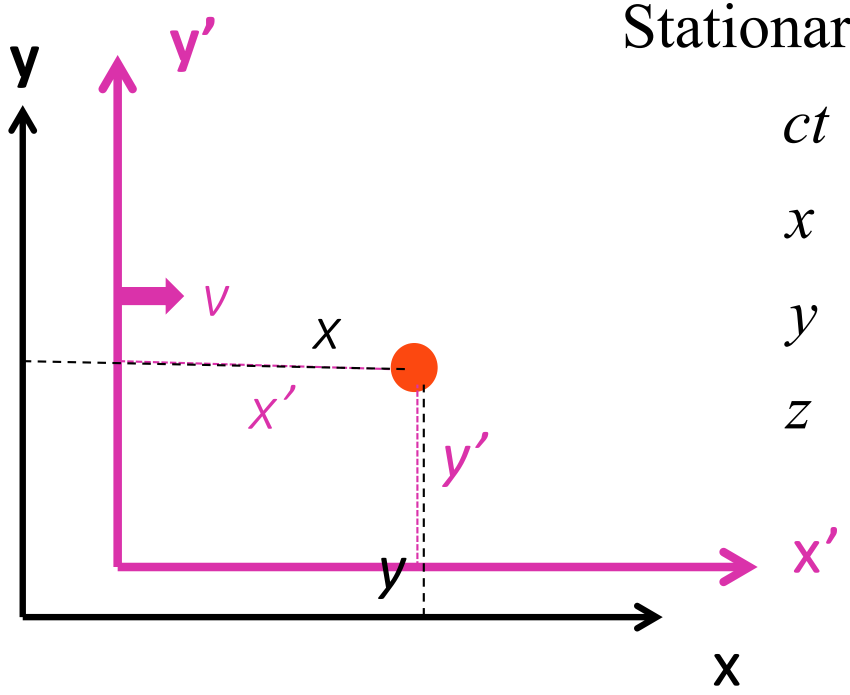


Lorentz transformations

Convenient notation :

$$\beta \equiv \frac{v}{c}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$



Stationary frame		Moving frame
ct	$=$	$\gamma(ct'+\beta x')$
x	$=$	$\gamma(x'+\beta ct')$
y	$=$	y'
z	$=$	z'

Lorentz transformations -- continued

$$\beta \equiv \frac{v}{c} \quad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\boldsymbol{\mathcal{L}} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{\mathcal{L}}^{-1} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \boldsymbol{\mathcal{L}} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} \qquad \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \boldsymbol{\mathcal{L}}^{-1} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Notice:

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2$$

Lorentz transformation of the velocity -- continued

Stationary frame		Moving frame
cdt	$=$	$\gamma(cdt'+\beta dx')$
dx	$=$	$\gamma(dx'+\beta cdt')$
dy	$=$	dy'
dz	$=$	dz'

Define: $u_x \equiv \frac{dx}{dt} \quad u_y \equiv \frac{dy}{dt} \quad u_z \equiv \frac{dz}{dt}$

$$u'_x \equiv \frac{dx'}{dt'} \quad u'_y \equiv \frac{dy'}{dt'} \quad u'_z \equiv \frac{dz'}{dt'}$$

$$\frac{dx}{dt} = \frac{\gamma(dx'+\beta cdt')}{\gamma(dt'+\beta dx'/c)} = \frac{u'_x+v}{1+vu'_x/c^2} = u_x$$

$$\frac{dy}{dt} = \frac{dy'}{\gamma(dt'+\beta dx'/c)} = \frac{u'_y}{\gamma(1+vu'_x/c^2)} = u_y$$

Summary of velocity relationships

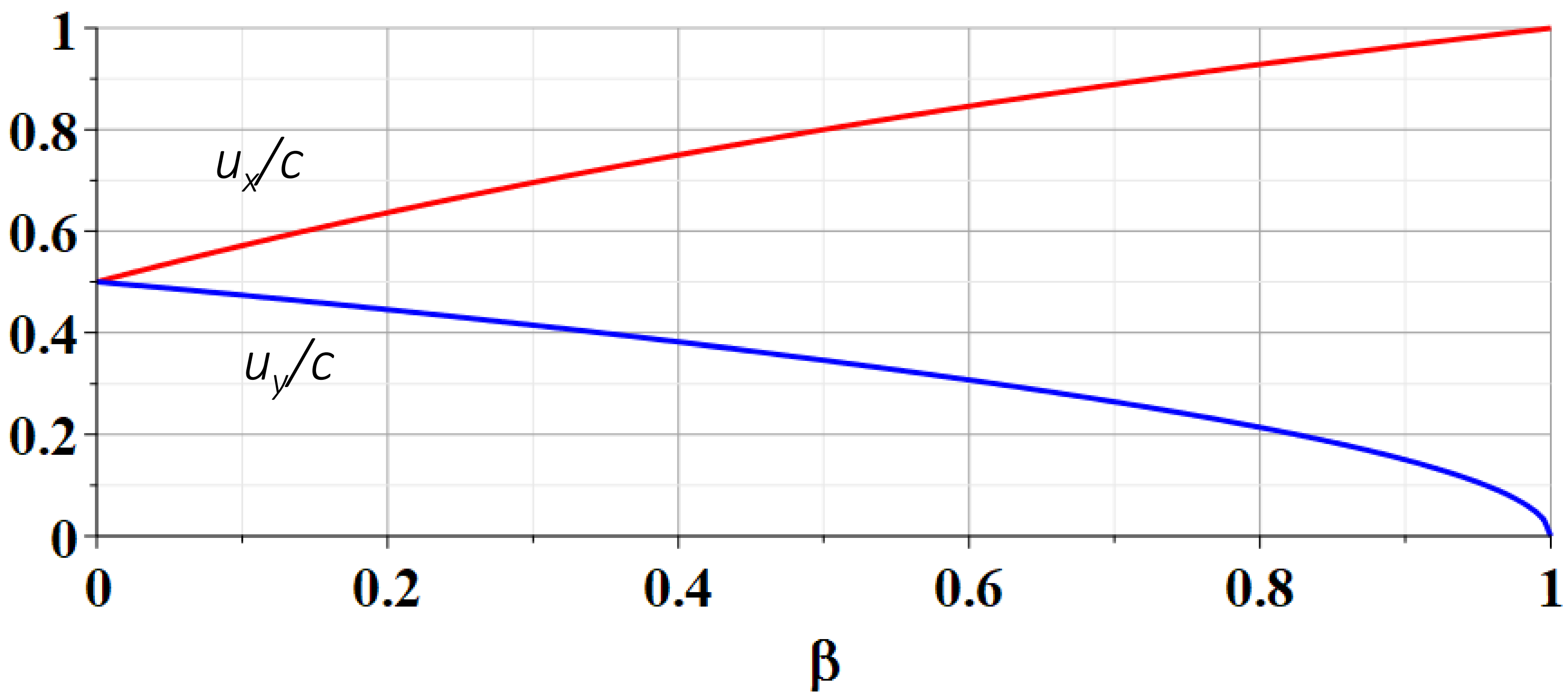
$$u_x = \frac{u'_x + v}{1 + vu'_x / c^2}$$

$$u_y = \frac{u'_y}{\gamma\left(1 + vu'_x / c^2\right)} \equiv \frac{u'_y}{\gamma_v\left(1 + vu'_x / c^2\right)}$$

$$u_z = \frac{u'_z}{\gamma\left(1 + vu'_x / c^2\right)} \equiv \frac{u'_z}{\gamma_v\left(1 + vu'_x / c^2\right)}$$

Example of velocity variation with β: $\beta \equiv \frac{v}{c}$

($u'_x/c = u'_y/c = 0.5$)



Velocity transformations continued:

Consider: $u_x = \frac{u'_x + v}{1 + vu'_x / c^2} \qquad u_y = \frac{u'_y}{\gamma_v \left(1 + vu'_x / c^2\right)} \qquad u_z = \frac{u'_z}{\gamma_v \left(1 + vu'_x / c^2\right)}.$

Note that $\gamma_u \equiv \frac{1}{\sqrt{1-(u/c)^2}} = \frac{1 + vu'_x / c^2}{\sqrt{1-(u'/c)^2}\sqrt{1-(v/c)^2}} = \gamma_v \gamma_{u'} \left(1 + vu'_x / c^2\right)$

$\Rightarrow \gamma_u c = \gamma_v \left(\gamma_{u'} c + \beta_v \gamma_{u'} u'_x\right)$

$\Rightarrow \gamma_u u_x = \gamma_v \left(\gamma_{u'} u'_x + \gamma_{u'} v\right) = \gamma_v \left(\gamma_{u'} u'_x + \beta_v \gamma_{u'} c\right)$

$\Rightarrow \gamma_u u_y = \gamma_{u'} u'_y \qquad \gamma_u u_z = \gamma_{u'} u'_z$

Velocity 4-vector: $\begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix} = \mathcal{L}_u \begin{pmatrix} \gamma_{u'} c \\ \gamma_{u'} u'_x \\ \gamma_{u'} u'_y \\ \gamma_{u'} u'_z \end{pmatrix}$

Some details:

$$\gamma_u = \gamma_v \gamma_{u'} \left(1 + v u'_x / c^2\right) \qquad \Rightarrow \left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{u'^2}{c^2}\right) = \left(1 - \frac{u^2}{c^2}\right)\left(1 + \frac{u_x v}{c^2}\right)^2$$

where $\quad u_x = \dfrac{u'_x + v}{1 + v u'_x / c^2} \qquad u_y = \dfrac{u'_y}{\gamma_v \left(1 + v u'_x / c^2\right)} \qquad u_z = \dfrac{u'_z}{\gamma_v \left(1 + v u'_x / c^2\right)}.$

$$\left(\frac{u_x^2}{c^2} + \frac{u_y^2}{c^2} + \frac{u_z^2}{c^2}\right)\left(1 + \frac{u_x v}{c^2}\right)^2 = \left(\frac{u'_x}{c} + \frac{v}{c}\right)^2 + \left(\frac{u'^2_y}{c^2} + \frac{u'^2_z}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)$$

$$\frac{u^2}{c^2}\left(1 + \frac{u_x v}{c^2}\right)^2 = \frac{u'^2}{c^2}\left(1 - \frac{v^2}{c^2}\right) + \left(1 + \frac{u_x v}{c^2}\right)^2 - \left(1 - \frac{v^2}{c^2}\right)$$

$$\Rightarrow \left(1 - \frac{u^2}{c^2}\right)\left(1 + \frac{u_x v}{c^2}\right)^2 = \left(1 - \frac{u'^2}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)$$

Significance of 4-velocity vector: $\begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix}$

Introduce the "rest" mass *m* of particle characterized by velocity **u**:

$$mc\begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix} = \begin{pmatrix} \gamma_u mc^2 \\ \gamma_u mu_x c \\ \gamma_u mu_y c \\ \gamma_u mu_z c \end{pmatrix} = \begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix}$$

Properties of energy-moment 4-vector:

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \mathcal{L}\begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} \qquad \begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} = \mathcal{L}^{-1}\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} \qquad \text{Note: } E^2 - p^2c^2 = E'^2 - p'^2c^2$$

Properties of Energy-momentum 4-vector -- continued

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \begin{pmatrix} \gamma_u mc^2 \\ \gamma_u m u_x c \\ \gamma_u m u_y c \\ \gamma_u m u_z c \end{pmatrix}$$

$$\text{Note: } E^2 - p^2c^2 = \frac{(mc^2)^2}{1-\beta_u{}^2}\left(1 - \left(\frac{u_x}{c}\right)^2 - \left(\frac{u_y}{c}\right)^2 - \left(\frac{u_z}{c}\right)^2\right) = (mc^2)^2 = E'^2 - p'^2 c^2$$

Notion of "rest energy": For $\mathbf{p} \equiv 0, \quad E = mc^2$

Define kinetic energy: $E_K \equiv E - mc^2 = \sqrt{p^2c^2 + m^2c^4} - mc^2$

Non-relativistic limit: If $\frac{p}{mc} << 1, \quad E_K = mc^2\left(\sqrt{1 + \left(\frac{p}{mc}\right)^2} - 1\right)$

$$\approx \frac{p^2}{2m} \qquad \text{for } \frac{p}{mc} << 1$$

Summary of relativistic energy relationships

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \begin{pmatrix} \gamma_u mc^2 \\ \gamma_u m u_x c \\ \gamma_u m u_y c \\ \gamma_u m u_z c \end{pmatrix}$$

$$E = \sqrt{p^2c^2 + m^2c^4} = \gamma_u mc^2$$

$$\text{Check: } \sqrt{p^2c^2 + m^2c^4} = mc^2\sqrt{\gamma_u{}^2\beta_u{}^2 + 1} = \gamma_u mc^2$$

$$\text{Example: } \quad \text{for an electron} \quad mc^2 = 0.5 \text{ MeV}$$

$$\text{for} \quad E = 200 \text{ GeV}$$

$$\gamma_u = \frac{E}{mc^2} = 4\times 10^5$$

$$\beta_u = \sqrt{1 - \frac{1}{\gamma_u{}^2}} \approx 1 - \frac{1}{2\gamma_u{}^2} \approx 1 - 3\times 10^{-12}$$

How do these relationships effect quantum mechanics?

Focusing on treatment of Fermi particles

Non-relativistic mechanics

$$E = \frac{\mathbf{p}^2}{2m}$$

$$\Downarrow$$

$$i\hbar \frac{\partial}{\partial t}\Psi = H\Psi$$

Relativistic mechanics

$$E^2 - \mathbf{p}^2 c^2 = \left(mc^2\right)^2$$

$$\Downarrow \quad \text{(with some license)}$$

$$\left(E - \mathbf{p}\cdot\boldsymbol{\sigma}c\right)\left(E + \mathbf{p}\cdot\boldsymbol{\sigma}c\right) = \left(mc^2\right)^2$$

$$\Downarrow$$

$$\left(i\hbar \frac{\partial}{\partial t} - \mathbf{p}\cdot\boldsymbol{\sigma}c\right)\left(i\hbar \frac{\partial}{\partial t} + \mathbf{p}\cdot\boldsymbol{\sigma}c\right)\Psi = \left(mc^2\right)^2 \Psi$$

The Nobel Prize in Physics 1933

Photo from the Nobel Foundation archive.

Erwin Schrödinger

Prize share: 1/2

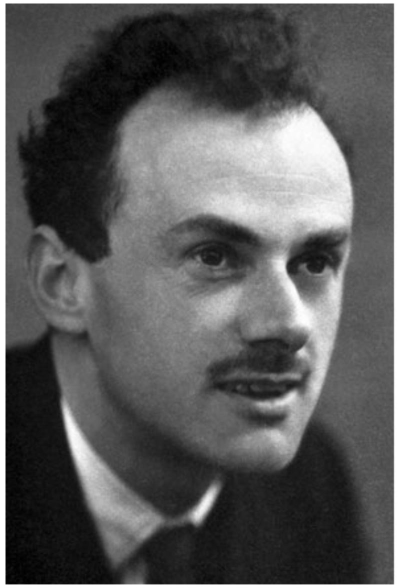


Photo from the Nobel Foundation archive.

Paul Adrien Maurice Dirac

Prize share: 1/2

The Nobel Prize in Physics 1933 was awarded jointly to Erwin Schrödinger and Paul Adrien Maurice Dirac "for the discovery of new productive forms of atomic theory."

Dirac's contribution --

During the intense period of 1925-26 quantum theories were proposed that accurately described the energy levels of electrons in atoms. These equations needed to be adapted to Albert Einstein's theory of relativity, however. In 1928 Paul Dirac formulated a fully relativistic quantum theory. The equation gave solutions that he interpreted as being caused by a particle equivalent to the electron, but with a positive charge. This particle, the positron, was later confirmed through experiments.

Digression on Pauli matrices (see Eq. 7.17 in your textbook)

$$\boldsymbol{\sigma} = \sigma_x \hat{\mathbf{x}} + \sigma_y \hat{\mathbf{y}} + \sigma_z \hat{\mathbf{z}}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note that $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$(\boldsymbol{\sigma} \cdot \mathbf{p})^2 = p^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Relativistic relationships – continued

Ref: J. J. Sakurai, Advanced Quantum Mechanics

$$\left(i\hbar\frac{\partial}{\partial t}-\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\left(i\hbar\frac{\partial}{\partial t}+\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\Psi=\left(mc^2\right)^2\Psi$$

➔**here, Ψ is a 2-component wavefunction**

Let $\Psi\equiv\Psi^L$ with $\left(i\hbar\frac{\partial}{\partial t}+\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\Psi^L\equiv mc^2\Psi^R$

Factored equations:

$$\left(i\hbar\frac{\partial}{\partial t}+\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\Psi^L=mc^2\Psi^R$$

$$\left(i\hbar\frac{\partial}{\partial t}-\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\Psi^R=mc^2\Psi^L$$

Relativistic relationships – continued

Ref: J. J. Sakurai, Advanced Quantum Mechanics

Factored equations:

$$\left(i\hbar\frac{\partial}{\partial t}+\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\Psi^{L}=mc^{2}\Psi^{R}$$

$$\left(i\hbar\frac{\partial}{\partial t}-\mathbf{p}\cdot\boldsymbol{\sigma}c\right)\Psi^{R}=mc^{2}\Psi^{L}$$

Dirac's rearrangement: $\varphi^{U}=\Psi^{R}+\Psi^{L}$

$$\varphi^{L}=\Psi^{R}-\Psi^{L}$$

$$\begin{pmatrix} i\hbar\frac{\partial}{\partial t} & -\mathbf{p}\cdot\boldsymbol{\sigma}c \\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -i\hbar\frac{\partial}{\partial t} \end{pmatrix}\begin{pmatrix}\varphi^{U}\\ \varphi^{L}\end{pmatrix}=mc^{2}\begin{pmatrix}\varphi^{U}\\ \varphi^{L}\end{pmatrix}$$

Relativistic relationships – continued

$$\begin{pmatrix} i\hbar\frac{\partial}{\partial t} & -\mathbf{p}\cdot\boldsymbol{\sigma}c \\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -i\hbar\frac{\partial}{\partial t} \end{pmatrix}\begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix} = mc^2 \begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix}$$

Further rearrangements:

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix} = \begin{pmatrix} mc^2 & \mathbf{p}\cdot\boldsymbol{\sigma}c \\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -mc^2 \end{pmatrix}\begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix}$$

$$\Psi = \begin{pmatrix} \varphi^U \\ \varphi^L \end{pmatrix}$$

$$H = \begin{pmatrix} mc^2 & \mathbf{p}\cdot\boldsymbol{\sigma}c \\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -mc^2 \end{pmatrix}$$

⬇

$$i\hbar\frac{\partial}{\partial t}\Psi = H\Psi$$

➔**here, Ψ is a 4-component wavefunction**

Four component wavefunction of free Fermi particle

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\varphi^U\\ \varphi^L\end{pmatrix}=\begin{pmatrix}mc^2 & \mathbf{p}\cdot\boldsymbol{\sigma}c\\ \mathbf{p}\cdot\boldsymbol{\sigma}c & -mc^2\end{pmatrix}\begin{pmatrix}\varphi^U\\ \varphi^L\end{pmatrix}$$

$$\text{Assume}\quad\begin{pmatrix}\varphi^U\\ \varphi^L\end{pmatrix}=\begin{pmatrix}\chi^U(\mathbf{k})\\ \chi^L(\mathbf{k})\end{pmatrix}e^{i\mathbf{k}\cdot\mathbf{r}-iEt/\hbar}$$

$$\Rightarrow \chi^U(\mathbf{k})=\frac{\hbar\mathbf{k}\cdot\boldsymbol{\sigma}c}{E-mc^2}\chi^L(\mathbf{k})$$

$$\chi^L(\mathbf{k})=\frac{\hbar\mathbf{k}\cdot\boldsymbol{\sigma}c}{E+mc^2}\chi^U(\mathbf{k})$$

$$E^2=\hbar^2c^2\mathbf{k}^2+m^2c^4$$

$$E=\pm\sqrt{\hbar^2c^2\mathbf{k}^2+m^2c^4}$$

Pauli matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\mathbf{k} \cdot \boldsymbol{\sigma} = \begin{pmatrix} k_z & k_x - ik_y \\ k_x + ik_y & -k_z \end{pmatrix}$$

$$\chi^U(\mathbf{k}) = \frac{\hbar \mathbf{k} \cdot \boldsymbol{\sigma} c}{E - mc^2} \chi^L(\mathbf{k}) \qquad \chi^L(\mathbf{k}) = \frac{\hbar \mathbf{k} \cdot \boldsymbol{\sigma} c}{E + mc^2} \chi^U(\mathbf{k})$$

Positive energy solutions: $E = \sqrt{\hbar^2 c^2 \mathbf{k}^2 + m^2 c^4}$

$$\chi^U_\uparrow(\mathbf{k}) = \mathcal{N} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi^L_\uparrow(\mathbf{k}) = \mathcal{N} \begin{pmatrix} \kappa_z \\ \kappa_+ \end{pmatrix} \qquad \kappa_z \equiv \frac{\hbar k_z c}{E + mc^2}$$

$$\chi^U_\downarrow(\mathbf{k}) = \mathcal{N} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \chi^L_\downarrow(\mathbf{k}) = \mathcal{N} \begin{pmatrix} \kappa_- \\ -\kappa_z \end{pmatrix} \qquad \kappa_\pm \equiv \frac{\hbar k_\pm c}{E + mc^2}$$

Pauli matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\mathbf{k}\cdot\boldsymbol{\sigma} = \begin{pmatrix} k_z & k_x - ik_y \\ k_x + ik_y & -k_z \end{pmatrix}$$

$$\chi^U(\mathbf{k}) = \frac{\hbar\mathbf{k}\cdot\boldsymbol{\sigma}c}{E - mc^2}\chi^L(\mathbf{k}) \qquad \chi^L(\mathbf{k}) = \frac{\hbar\mathbf{k}\cdot\boldsymbol{\sigma}c}{E + mc^2}\chi^U(\mathbf{k})$$

Negative energy solutions: $E = -\sqrt{\hbar^2 c^2 \mathbf{k}^2 + m^2 c^4}$

$$\chi^U_\uparrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} \kappa_z \\ \kappa_+ \end{pmatrix} \qquad \chi^L_\uparrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \kappa_z \equiv \frac{\hbar k_z c}{E - mc^2}$$

$$\chi^U_\downarrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} \kappa_- \\ -\kappa_z \end{pmatrix} \qquad \chi^L_\downarrow(\mathbf{k}) = \mathcal{N}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \kappa_\pm \equiv \frac{\hbar k_\pm c}{E - mc^2}$$

What does this all mean?

Positive energy solutions: $E = \sqrt{\hbar^2 c^2 \mathbf{k}^2 + m^2 c^4}$

$$\chi_\uparrow^U(\mathbf{k}) = \mathcal{N}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_\uparrow^L(\mathbf{k}) = \mathcal{N}\begin{pmatrix} \kappa_z \\ \kappa_+ \end{pmatrix} \qquad \kappa_z \equiv \frac{\hbar k_z c}{E + mc^2}$$

$$\chi_\downarrow^U(\mathbf{k}) = \mathcal{N}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \chi_\downarrow^L(\mathbf{k}) = \mathcal{N}\begin{pmatrix} \kappa_- \\ -\kappa_z \end{pmatrix} \qquad \kappa_\pm \equiv \frac{\hbar k_\pm c}{E + mc^2}$$

For $\hbar c|\mathbf{k}| \ll mc^2$ $\quad E \approx mc^2 + \frac{\hbar^2 |\mathbf{k}|^2}{2m}$

$$\chi_\uparrow^U(\mathbf{k}) = \mathcal{N}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_\uparrow^L(\mathbf{k}) \approx \mathcal{N}\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\chi_\downarrow^U(\mathbf{k}) = \mathcal{N}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \chi_\downarrow^L(\mathbf{k}) \approx \mathcal{N}\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$