PHY 742 Quantum Mechanics II
12-12:50 PM MWF in Olin 103

Plan for Lecture 21

Quantization of the Electromagnetic fields

Review the "raising" and "lowering" operators presented in Professor Carlson's textbook in Chapter 5: The Harmonic Oscillators and Chapter 17: Quantizing Electromagnetic Fields.

1. **Review of the harmonic oscillator**
2. **Particle creation and annihilation operator formalism**
3. **Hamiltonian for the electromagnetic fields**

	Mon: 03/14/2022	No class	*APS March Meeting*	Prepare Project	
	Wed: 03/16/2022	No class	*APS March Meeting*	Prepare Project	
	Fri: 03/18/2022	No class	*APS March Meeting*	Prepare Project	
	Mon: 03/21/2022		Project presentations I		
	Wed: 03/23/2022		Project presentations II		
21	Fri: 03/25/2022	Chap. 5 & 17	Quantization of the Electromagnetic Field	#17	03/28/2022

PHY 742 -- Assignment #17

March 25, 2022

Read Chapters 5 and 17 in **Professor Carlson's QM textbook.**. In the following CQM refers to the textbook.

1. Using Eq. 5.3 of CQM to express the displacement operator X in terms of creation and annihilation operators, show that the following expectation value is correct: $\langle n|X^4|n\rangle = (\hbar/(2m\omega))^2(3+6n(n+1))$ for the $|n\rangle$ eigenstate of the harmonic oscillator Hamiltonian in the phonon number basis.

Review of the harmonic oscillator --

Why?

1. **We like harmonic oscillators?**
2. **All of physics can be mapped into harmonic oscillators?**
3. **Physicists only know how to solve harmonic oscillator problems?**
4. **Harmonic oscillators inspire a new way of thinking about quantum mechanics?**

One-dimensional harmonic oscillator

$$H\psi(x) = \left(\frac{P^2}{2m} + \frac{m\omega^2}{2} X^2 \right) \psi(x) = E\psi(x)$$

Define:

$$a = \left(\frac{m\omega}{2\hbar} \right)^{1/2} X + i \left(\frac{1}{2m\omega\hbar} \right)^{1/2} P$$

$$a^\dagger = \left(\frac{m\omega}{2\hbar} \right)^{1/2} X - i \left(\frac{1}{2m\omega\hbar} \right)^{1/2} P$$

Note that:

$$[a, a^\dagger] = 1$$

It follows that:

$$H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right) \qquad \psi \Rightarrow |n\rangle \qquad E \Rightarrow E_n = \hbar\omega \left(n + \frac{1}{2} \right)$$

Representation of the position and momentum operators in terms of the energy eigenstates of the harmonic oscillator:

$$X \leftrightarrow \left(\frac{\hbar}{2m\omega}\right)^{1/2} \begin{array}{c} \begin{array}{ccccc} n= 0 & 1 & 2 & 3 & \ldots\ldots \end{array} \\ \begin{bmatrix} 0 & 1^{1/2} & 0 & 0 & \cdots \\ 1^{1/2} & 0 & 2^{1/2} & 0 & \\ 0 & 2^{1/2} & 0 & 3^{1/2} & \\ 0 & 0 & 3^{1/2} & 0 & \\ \vdots & & & & \end{bmatrix} \end{array}$$

$$P \leftrightarrow i\left(\frac{m\omega\hbar}{2}\right)^{1/2} \begin{bmatrix} 0 & -1^{1/2} & 0 & 0 & \cdots \\ 1^{1/2} & 0 & -2^{1/2} & 0 & \\ 0 & 2^{1/2} & 0 & -3^{1/2} & \\ 0 & 0 & 3^{1/2} & 0 & \\ \vdots & & & & \end{bmatrix}$$

Representation of the raising and lowering operators in terms of the energy eigenstates of the harmonic oscillator:

$$
a^\dagger \leftrightarrow
\begin{array}{cc}
 & \begin{array}{cccc} n=0 & n=1 & n=2 & \cdots \end{array} \\
\begin{array}{c} n=0 \\ n=1 \\ n=2 \\ \cdot \\ \cdot \end{array} &
\begin{bmatrix}
0 & 0 & 0 & \cdots \\
1^{1/2} & 0 & 0 & \\
0 & 2^{1/2} & 0 & \\
0 & 0 & 3^{1/2} & \\
\vdots & & &
\end{bmatrix}
\end{array}
$$

$$
a \leftrightarrow
\begin{bmatrix}
0 & 1^{1/2} & 0 & 0 & \cdots \\
0 & 0 & 2^{1/2} & 0 & \\
0 & 0 & 0 & 3^{1/2} & \\
\vdots & & & &
\end{bmatrix}
$$

Summary of results

$$H\left|n\right\rangle = \hbar\omega\left(\frac{1}{2} + a^\dagger a\right)\left|n\right\rangle = \hbar\omega\left(\frac{1}{2} + n\right)\left|n\right\rangle$$

$$a\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle$$

$$a^\dagger\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle$$

Contributing to the discussion –

The creation and annihilation operators within the harmonic oscillator formalism seem to have been introduced by mathematical logic and found to have very interesting properties. In fact, as shown in Chapter 5, starting from the creation and annihilation operators, one can deduce the Harmonic Oscillator spectrum. These operators do not by themselves represent physical quantities and therefore do not "have" to be Hermitian. The matrix form of X and P in the basis of |n> is just one of many ways to represent these operators.

Further comments --

The harmonic oscillator states clearly have an associated quantum number *n.* It is convenient to call *n* a "phonon number" for the moment. We will generalize this notion in the context of electromagnetic fields.

How does this beautiful formalism lead to the notion of creation and annihilation operators?

The phonon number eigenvalues take the values $n = 0, 1, 2, ...$

$a|0\rangle = 0 \quad a|1\rangle = |0\rangle \quad a|2\rangle = \sqrt{2}|1\rangle$ ... interpretation of a as annihilation operator

$a^\dagger|0\rangle = |1\rangle \quad a^\dagger|1\rangle = \sqrt{2}|2\rangle \quad a^\dagger|2\rangle = \sqrt{3}|3\rangle$ ... interpretation of $a^\dagger$ as creation operator

It follows that $|n\rangle = \frac{1}{\sqrt{(n!)}}\left(a^\dagger\right)^n |0\rangle$

➔**We can "create" any phonon state from the ground state with this operator.**

Extension of these ideas to multiple independent harmonic oscillator modes

$$\omega \Rightarrow \{\omega_1, \omega_2, \omega_3 ....\}$$

$$a \Rightarrow \{a_1, a_2, a_3 ....\}$$

$$a^\dagger \Rightarrow \{a_1^\dagger, a_2^\dagger, a_3^\dagger ....\}$$

Here *1,2,..i,j...* denotes an arbitrary index referencing distinct modes.

$$\text{Commutation relations: } \left[a_i, a_j\right] = 0$$

$$\text{Commutation relations: } \left[a_i^\dagger, a_j^\dagger\right] = 0$$

$$\text{Commutation relations: } \left[a_i, a_j^\dagger\right] = \delta_{ij}$$

This result means that for a multiphonon state $\left|n_1, n_2 .... n_i ... n_j ... n_N\right\rangle$, the action of the creation operator works as follows:

$$a_i^\dagger a_j^\dagger \left|n_1, n_2 .... n_i ... n_j ... n_N\right\rangle = \sqrt{n_i + 1}\sqrt{n_j + 1}\left|n_1, n_2 ....(n_i + 1)...(n_j + 1)... n_N\right\rangle$$

Later, we will see how this formalism has the capability of keeping track of symmetry/antisymmetry properties of multi particle systems.

Maxwell's equations

$$\text{Microscopic or vacuum form } (\mathbf{P}=0;\ \mathbf{M}=0):$$

$$\text{Coulomb's law:} \qquad \nabla\cdot\mathbf{E} = \rho/\varepsilon_0$$

$$\text{Ampere-Maxwell's law:} \qquad \nabla\times\mathbf{B} - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} = \mu_0\mathbf{J}$$

$$\text{Faraday's law:} \qquad \nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0$$

$$\text{No magnetic monopoles:} \qquad \nabla\cdot\mathbf{B} = 0$$

$$\Rightarrow c^2 = \frac{1}{\varepsilon_0\mu_0}$$

Recall the electromagnetic field energy --

$$E_{\text{field}} = \frac{\epsilon_0}{2}\int d^3r \left(\left|\mathbf{E}(\mathbf{r},t)\right|^2 + c^2\left|\mathbf{B}(\mathbf{r},t)\right|^2 \right)$$

It will be convenient to express Maxwell's equations and the electromagnetic field energy in terms of scalar and vector potentials:

$$\nabla\cdot\mathbf{B} = 0 \qquad \Rightarrow \quad \mathbf{B} = \nabla\times\mathbf{A}$$

$$\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0 \quad \Rightarrow \nabla\times\left(\mathbf{E} + \frac{\partial\mathbf{A}}{\partial t}\right) = 0 \quad \Rightarrow \mathbf{E} + \frac{\partial\mathbf{A}}{\partial t} = -\nabla\Phi \quad \Rightarrow \mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}$$

Far from sources, the remaining equations become:

$$\nabla\cdot\mathbf{E} = 0 \qquad \Rightarrow \quad \nabla^2\Phi + \frac{\partial\nabla\cdot\mathbf{A}}{\partial t} = 0$$

$$\nabla\times\mathbf{B} - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} = 0 \quad \Rightarrow \nabla\times\left(\nabla\times\mathbf{A}\right) + \frac{1}{c^2}\left(\frac{\partial\nabla\Phi}{\partial t} + \frac{\partial^2\mathbf{A}}{\partial t^2}\right) = 0$$

Further manipulations of Maxwell's equations in terms of scalar and vector potentials --

$$\nabla \cdot \mathbf{E} = 0 \qquad \Rightarrow \nabla^2 \Phi + \frac{\partial \nabla \cdot \mathbf{A}}{\partial t} = 0$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = 0 \quad \Rightarrow \nabla \times \left(\nabla \times \mathbf{A}\right) + \frac{1}{c^2}\left(\frac{\partial \nabla \Phi}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) = 0$$

$$\Rightarrow \nabla\left(\nabla \cdot \mathbf{A}\right) - \nabla^2 \mathbf{A} + \frac{1}{c^2}\left(\frac{\partial \nabla \Phi}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) = 0$$

$$\Rightarrow \left(\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \nabla \underbrace{\left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right)}_{\textbf{zero in Lorenz gauge}} = 0$$

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = 0 \qquad \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0$$

Equations within the Lorenz gauge --

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0 \qquad \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = 0$$

It is further convenient to seek solutions with $\Phi \equiv 0 \quad \Rightarrow \nabla \cdot \mathbf{A} = 0$

$$\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Note that this is one of many possible choices and it turns out to be convenient.

Electromagnetic field energy for this choice --

$$E_{\text{field}} = \frac{\epsilon_0}{2}\int d^3r\left(\left|\mathbf{E}(\mathbf{r},t)\right|^2 + c^2\left|\mathbf{B}(\mathbf{r},t)\right|^2\right)$$

$$= \frac{\epsilon_0}{2}\int d^3r\left(\left|\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}\right|^2 + c^2\left|\nabla\times\mathbf{A}(\mathbf{r},t)\right|^2\right)$$

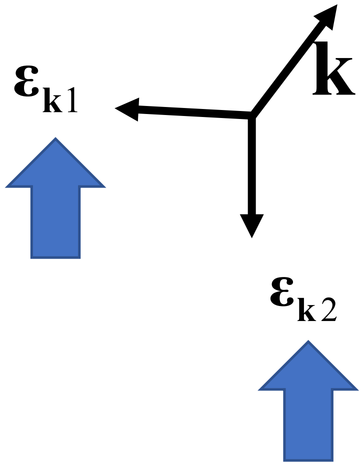
Plane wave solutions to electromagnetic waves in terms of vector potentials

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0 \qquad \nabla \cdot \mathbf{A} = 0$$

A pure plane wave takes the form

$$\mathbf{A}_{\mathbf{k}\sigma}(\mathbf{r},t) = A_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega_{\mathbf{k}} t} \qquad \omega_{\mathbf{k}} = |\mathbf{k}|c$$

$$\mathbf{k}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} = 0 \qquad \boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}\sigma'} = \delta_{\sigma\sigma'}$$



3 mutually perpendicular vectors

These are unit polarization vectors.

For the pure plane wave, the following relations hold:

$$\frac{\partial \mathbf{A}_{\mathbf{k}\sigma}(\mathbf{r},t)}{\partial t} = -i\omega_{\mathbf{k}} A_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega_{\mathbf{k}} t}$$

$$\nabla \times \mathbf{A}_{\mathbf{k}\sigma}(\mathbf{r},t) = i\mathbf{k} \times A_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega_{\mathbf{k}} t}$$

General form of vector potential as a superposition of plane waves:

$$\mathbf{A}(\mathbf{r},t)=\frac{1}{V}\sum_{\mathbf{k}\sigma}\mathbf{A}_{\mathbf{k}\sigma}(\mathbf{r},t)=\frac{1}{V}\sum_{\mathbf{k}\sigma}A_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}$$

Here V denotes the volume of the analysis system; different treatments put this factor in different ways.

Now we must evaluate the electromagnetic field energy --

$$E_{\text{field}}=\frac{\epsilon_0}{2}\int d^3r\left(\left|\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}\right|^2+c^2\left|\nabla\times\mathbf{A}(\mathbf{r},t)\right|^2\right)$$

Because of the orthogonality of the plane waves, the result can be expressed as a sum over distinct plane wave modes:

$$E_{\text{field}}=\frac{\epsilon_0}{2V}\sum_{\mathbf{k}\sigma}\left|A_{\mathbf{k}\sigma}\right|^2\left(\omega_{\mathbf{k}}^2+c^2\left|\mathbf{k}\right|^2\right)$$

Note that we can use the identity

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

Up to now, we have treated classical electromagnetic waves.
Next time, we will consider the quantum treatment of the electromagnetic field energy ↔ electromagnetic Hamiltonian