PHY 742 Quantum Mechanics II
12-12:50 PM MWF in Olin 103

Plan for Lecture 22

Quantization of the Electromagnetic fields

Read Chapter 17: Quantizing Electromagnetic Fields from Professor Carlson's.

1. Classical Hamiltonian for the electromagnetic fields
2. Quantum Hamiltonian for the electromagnetic fields
3. Photon eigenstates

	Mon: 03/21/2022		Project presentations I		
	Wed: 03/23/2022		Project presentations II		
21	Fri: 03/25/2022	Chap. 5 & 17	Quantization of the Electromagnetic Field	#17	03/28/2022
22	Mon: 03/28/2022	Chap. 17	Quantization of the Electromagnetic Field	#18	03/30/2022
23	Wed: 03/30/2022	Chap. 17	Quantization of the Electromagnetic Field		

PHY 742 -- Assignment #18

March 28, 2022

Continue reading Chapter 17 in **Professor Carlson's QM textbook.**.

1. Evaluate the 4 relationships between coherent states given on the last slide of Lecture 22 in order to check whether or not they are correct.

Gauber's coherent state: $$\left|c_\alpha\right\rangle \equiv \sum_{n=0}^{\infty} \frac{\alpha^n e^{-|\alpha|^2/2}}{\sqrt{n!}} \left|n\right\rangle$$

Here α represents a complex amplitude

It is possible to prove the following identies for the coherent states:

1. $\left\langle c_\alpha \middle| c_\alpha \right\rangle = 1$
2. $\left\langle c_\alpha \middle| a \middle| c_\alpha \right\rangle = \alpha$
3. $\left\langle c_\alpha \middle| a^\dagger \middle| c_\alpha \right\rangle = \alpha^*$
4. $\left|\left\langle c_\alpha \middle| c_\beta \right\rangle\right|^2 = e^{-|\alpha-\beta|^2}$

Electromagnetic field energy --

$$E_{\text{field}} = \frac{\epsilon_0}{2}\int d^3r \left(\left|\mathbf{E}(\mathbf{r},t)\right|^2 + c^2 \left|\mathbf{B}(\mathbf{r},t)\right|^2 \right)$$

In terms of the vector potential, using the Lorenz gauge with $\Phi = 0$:

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \qquad\qquad \mathbf{B} = \nabla \times \mathbf{A}$$

where $\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0$ and $\nabla \cdot \mathbf{A} = 0$

$$E_{\text{field}} = \frac{\epsilon_0}{2}\int d^3r \left(\left|\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t}\right|^2 + c^2 \left|\nabla \times \mathbf{A}(\mathbf{r},t)\right|^2 \right)$$

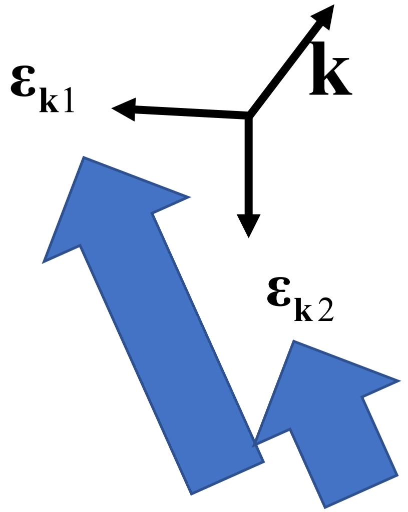
Plane wave solutions to electromagnetic waves in terms of vector potentials

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0 \qquad \nabla \cdot \mathbf{A} = 0$$

A pure plane wave takes the form

$$\mathbf{A}_{\mathbf{k}\sigma}(\mathbf{r},t) = A_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega_{\mathbf{k}} t} \qquad \omega_{\mathbf{k}} = |\mathbf{k}| c$$

$$\mathbf{k} \cdot \boldsymbol{\varepsilon}_{\mathbf{k}\sigma} = 0 \qquad \boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \cdot \boldsymbol{\varepsilon}_{\mathbf{k}\sigma'} = \delta_{\sigma\sigma'} \qquad \sigma = 1,2$$



3 mutually perpendicular vectors

polarization unit vectors.

For the pure plane wave, the following relations hold:

$$\frac{\partial \mathbf{A}_{\mathbf{k}\sigma}(\mathbf{r},t)}{\partial t} = -i\omega_{\mathbf{k}} A_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega_{\mathbf{k}} t}$$

$$\nabla \times \mathbf{A}_{\mathbf{k}\sigma}(\mathbf{r},t) = i\mathbf{k} \times A_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega_{\mathbf{k}} t}$$

General form of vector potential as a superposition of plane waves:

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{V}\sum_{\mathbf{k}\sigma}\mathbf{A}_{\mathbf{k}\sigma}(\mathbf{r},t) = \frac{1}{V}\sum_{\mathbf{k}\sigma} A_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}$$

Here V denotes the volume of the analysis system; different treatments put this factor in different ways.

Now we must evaluate the electromagnetic field energy --

$$E_{\text{field}} = \frac{\epsilon_0}{2}\int d^3r\left(\left|\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}\right|^2 + c^2\left|\nabla\times\mathbf{A}(\mathbf{r},t)\right|^2\right)$$

Because of the orthogonality of the plane waves, the result can be expressed as a sum over distinct plane wave modes:

$$E_{\text{field}} = \frac{\epsilon_0}{2V}\sum_{\mathbf{k}\sigma}\left|A_{\mathbf{k}\sigma}\right|^2\left(\omega_{\mathbf{k}}^2 + c^2\left|\mathbf{k}\right|^2\right)$$

Note that we can use the identity

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

Some details, with more care to use real functions --

$$\mathbf{A}(\mathbf{r},t)=\frac{1}{2V}\sum_{\mathbf{k}\sigma}\left(\mathbf{A}_{\mathbf{k}\sigma}(\mathbf{r},t)+\mathbf{A}^*_{\mathbf{k}\sigma}(\mathbf{r},t)\right)=\frac{1}{2V}\sum_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(A_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}+A^*_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)$$

Electromagnetic field energy --

$$E_{\text{field}}=\frac{\epsilon_0}{2V}\int d^3r\left(\left|\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}\right|^2+c^2\left|\nabla\times\mathbf{A}(\mathbf{r},t)\right|^2\right)$$

Note that the plane waves are distributed throughout the analysis volume

such that the following orthogonality holds. $\frac{1}{V}\int d^3r\; e^{i\mathbf{k}\cdot\mathbf{r}-i\mathbf{k}'\cdot\mathbf{r}}=\delta_{\mathbf{k}\mathbf{k}'}$

Also recall that $\omega_{\mathbf{k}}=|\mathbf{k}|c$ and average out all high frequency contributions

to the field energy -- $E_{\text{field}}=\frac{\epsilon_0}{4V}\sum_{\mathbf{k}\sigma}\left(A_{\mathbf{k}\sigma}A^*_{\mathbf{k}\sigma}+A^*_{\mathbf{k}\sigma}A_{\mathbf{k}\sigma}\right)\left(\omega_{\mathbf{k}}^2+c^2|\mathbf{k}|^2\right)$

$$E_{\text{field}}=\frac{\epsilon_0}{2V}\sum_{\mathbf{k}\sigma}\omega_{\mathbf{k}}^2\left(A_{\mathbf{k}\sigma}A^*_{\mathbf{k}\sigma}+A^*_{\mathbf{k}\sigma}A_{\mathbf{k}\sigma}\right)$$

In the next slide, we will "jump" to quantizing the electromagnetic field using the analogy of the harmonic oscillator Hamiltonian. In fact, the analogy has nothing to do with the physics of the harmonic oscillator other than their particle symmetry as Bose particles.

Max Planck 1858-1947

Historical importance of the formula for Blackbody radiation

A blackbody means an idealized opaque (non-reflective) material which can absorb and emit electromagnetic radiation. If the body has an equilibrium temperature T, the energy associated with the blackbody is <U>. Using statistical mechanics and the assumption of quantized electromagnetic radiation, Planck showed that the black body internal energy and its distribution is given by in terms of frequency *f*:

$$\langle U \rangle = \frac{Vh^4}{\pi^2 \hbar^3 c^3} \int df\, f^3 \frac{1}{e^{\beta hf} - 1} = \frac{8\pi V h}{c^3} \int_0^\infty df \frac{f^3}{e^{\beta hf} - 1}$$

Figure from:
An Introduction to Thermal Physics, by Daniel V. Schroeder (Addison Wesley, 2000 and now Oxford University Press)

Showing frequency distribution of blackbody radiation from the big bang.

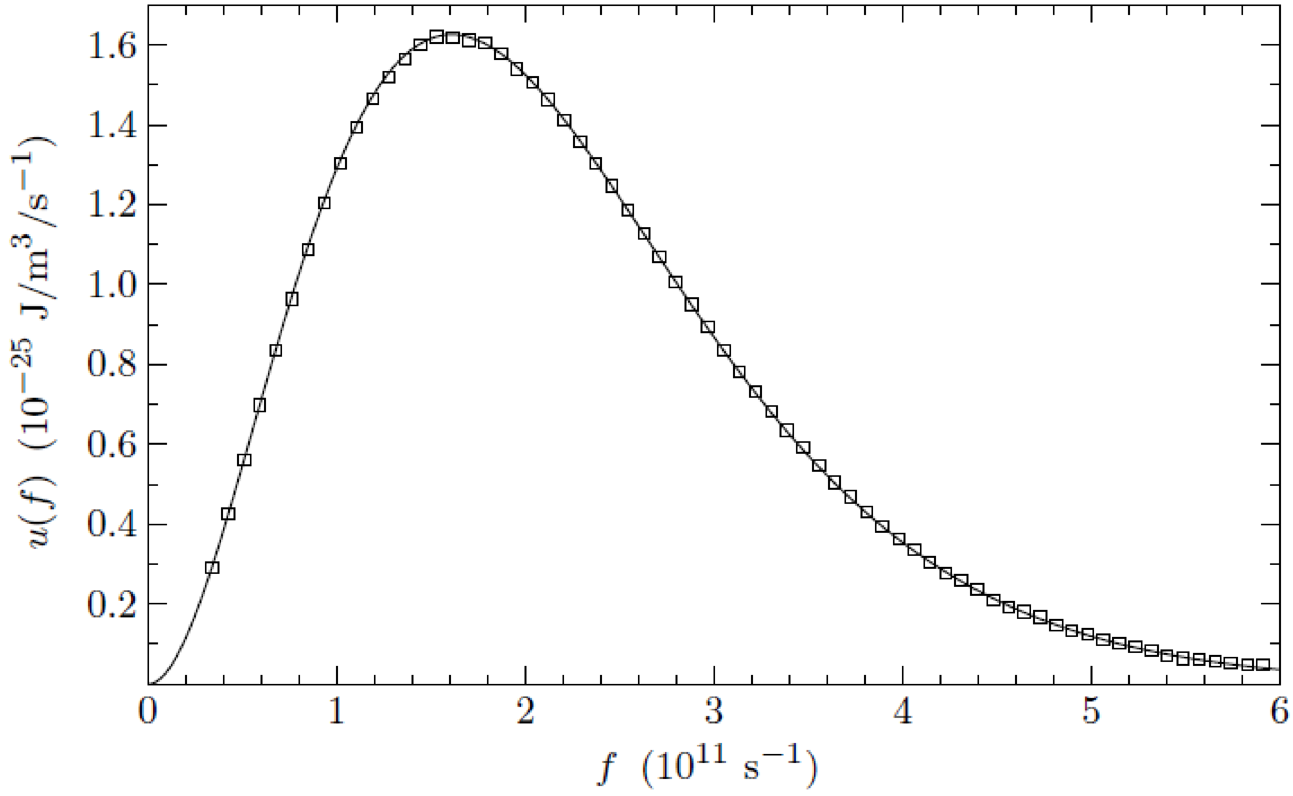


Figure 7.20. Spectrum of the cosmic background radiation, as measured by the *Cosmic Background Explorer* satellite. Plotted vertically is the energy density per unit frequency, in SI units. Note that a frequency of 3×10^{11} s^{-1} corresponds to a wavelength of $\lambda = c/f = 1.0$ mm. Each square represents a measured data point. The point-by-point uncertainties are too small to show up on this scale; the size of the squares instead represents a liberal estimate of the uncertainty due to systematic effects. The solid curve is the theoretical Planck spectrum, with the temperature adjusted to 2.735 K to give the best fit. From J. C. Mather et al., *Astrophysical Journal Letters* **354**, L37 (1990); adapted courtesy of NASA/GSFC and the COBE Science Working Group. Subsequent measurements from this experiment and others now give a best-fit temperature of 2.728 ± 0.002 K. Copyright ©2000, Addison-Wesley.

Electromagnetic field energy expression:

$$E_{\text{field}} = \frac{\epsilon_0}{2V} \sum_{\mathbf{k}\sigma} \omega_{\mathbf{k}}^2 \left(A_{\mathbf{k}\sigma} A_{\mathbf{k}\sigma}^* + A_{\mathbf{k}\sigma}^* A_{\mathbf{k}\sigma} \right)$$

Here $A_{\mathbf{k}\sigma}$ represents the amplitude of the vector potential.

Big leap --

Suppose that $A_{\mathbf{k}\sigma} \rightarrow C_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma}$ $\quad A_{\mathbf{k}\sigma}^* \rightarrow C_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma}^\dagger$

where $C_{\mathbf{k}\sigma}$ is a constant

$$E_{\text{field}} = \frac{\epsilon_0}{2V} \sum_{\mathbf{k}\sigma} \omega_{\mathbf{k}}^2 C_{\mathbf{k}\sigma}^2 \left(a_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma}^\dagger + a_{\mathbf{k}\sigma}^\dagger a_{\mathbf{k}\sigma} \right)$$

More leaping -- $\quad C_{\mathbf{k}\sigma} = \sqrt{\frac{V\hbar}{\epsilon_0 \omega_{\mathbf{k}}}}$

$$E_{\text{field}} = \frac{1}{2} \sum_{\mathbf{k}\sigma} \hbar\omega_{\mathbf{k}} \left(a_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma}^\dagger + a_{\mathbf{k}\sigma}^\dagger a_{\mathbf{k}\sigma} \right) = \sum_{\mathbf{k}\sigma} \hbar\omega_{\mathbf{k}} \left(a_{\mathbf{k}\sigma}^\dagger a_{\mathbf{k}\sigma} + \frac{1}{2} \right)$$

Here $a_{\mathbf{k}\sigma}$ and $a^{\dagger}_{\mathbf{k}\sigma}$ are "borrowed" from the Harmonic oscillator formalism.

Commutation relations: $\left[a_{\mathbf{k}\sigma}, a^{\dagger}_{\mathbf{k}'\sigma'}\right] = \delta_{\mathbf{kk}'}\delta_{\sigma\sigma'}$ $\quad \left[a_{\mathbf{k}\sigma}, a_{\mathbf{k}'\sigma'}\right] = 0$ $\quad \left[a^{\dagger}_{\mathbf{k}\sigma}, a^{\dagger}_{\mathbf{k}'\sigma'}\right] = 0$

$$H_{\text{field}} = \frac{1}{2}\sum_{\mathbf{k}\sigma}\hbar\omega_{\mathbf{k}}\left(a_{\mathbf{k}\sigma}a^{\dagger}_{\mathbf{k}\sigma} + a^{\dagger}_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma}\right) = \sum_{\mathbf{k}\sigma}\hbar\omega_{\mathbf{k}}\left(a^{\dagger}_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma} + \frac{1}{2}\right)$$

From the analogy of the Harmonic oscillator, the eigenstates of the EM Field Hamiltonian are integers $n_{\mathbf{k}\sigma}$:

$$H_{\text{field}}\left|n_{\mathbf{k}\sigma}\right\rangle = \sum_{\mathbf{k}'\sigma'}\hbar\omega_{\mathbf{k}'}\left(a^{\dagger}_{\mathbf{k}'\sigma'}a_{\mathbf{k}'\sigma'} + \frac{1}{2}\right)\left|n_{\mathbf{k}\sigma}\right\rangle = \left(\hbar\omega_{\mathbf{k}}n_{\mathbf{k}\sigma} + \sum_{\mathbf{k}'\sigma'}\frac{\hbar\omega_{\mathbf{k}'}}{2}\right)\left|n_{\mathbf{k}\sigma}\right\rangle$$

$$H^{\text{fixed}}_{\text{field}}\left|n_{\mathbf{k}\sigma}\right\rangle = \sum_{\mathbf{k}'\sigma'}\left(\hbar\omega_{\mathbf{k}'}a^{\dagger}_{\mathbf{k}'\sigma'}a_{\mathbf{k}'\sigma'}\right)\left|n_{\mathbf{k}\sigma}\right\rangle = \hbar\omega_{\mathbf{k}}n_{\mathbf{k}\sigma}\left|n_{\mathbf{k}\sigma}\right\rangle$$

Uncontrolled energy shift

Some additional comments on the "fixed" solution --

EM Field Hamiltonian acting on eigenstate $\left|n_{\mathbf{k}\sigma}\right\rangle$:

$$H_{\text{field}}\left|n_{\mathbf{k}\sigma}\right\rangle = \sum_{\mathbf{k}'\sigma'} \hbar\omega_{\mathbf{k}'}\left(a^{\dagger}_{\mathbf{k}'\sigma'}a_{\mathbf{k}'\sigma'} + \frac{1}{2}\right)\left|n_{\mathbf{k}\sigma}\right\rangle = \hbar\omega_{\mathbf{k}} n_{\mathbf{k}\sigma}\left|n_{\mathbf{k}\sigma}\right\rangle + \sum_{\mathbf{k}'\sigma'}\frac{\hbar\omega_{\mathbf{k}'}}{2}\left|n_{\mathbf{k}\sigma}\right\rangle$$

$$H^{\text{fixed}}_{\text{field}}\left|n_{\mathbf{k}\sigma}\right\rangle = \sum_{\mathbf{k}'\sigma'}\left(\hbar\omega_{\mathbf{k}'}a^{\dagger}_{\mathbf{k}'\sigma'}a_{\mathbf{k}'\sigma'}\right)\left|n_{\mathbf{k}\sigma}\right\rangle = \hbar\omega_{\mathbf{k}} n_{\mathbf{k}\sigma}\left|n_{\mathbf{k}\sigma}\right\rangle$$

Troublesome term

Comment: For the phonon case which served as our model, the notion of zero point motion makes physical sense. For the electromagnetic Hamiltonian the role of the equivalent concept is not quite clear (at least to me). We need to be careful when we see divergent energies to distinguish physical processes from mathematical issues.

Creation and annihilation operators:

$$a_{\mathbf{k}\sigma}\left|n_{\mathbf{k}\sigma}\right\rangle = \sqrt{n_{\mathbf{k}\sigma}}\left|n_{\mathbf{k}\sigma}-1\right\rangle$$

$$a^{\dagger}_{\mathbf{k}\sigma}\left|n_{\mathbf{k}\sigma}\right\rangle = \sqrt{n_{\mathbf{k}\sigma}+1}\left|n_{\mathbf{k}\sigma}+1\right\rangle$$

Quantum mechanical form of vector potential in real space --

$$\mathbf{A}(\mathbf{r},t) = \sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t} + a^{\dagger}_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)$$

Note: We are assuming that the polarization vector is real.

Quantum mechanical form of vector potential --

$$\mathbf{A}(\mathbf{r},t) = \sum_{\mathbf{k}\sigma} \sqrt{\frac{\hbar}{2V\epsilon_0\omega_\mathbf{k}}} \boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \left(a_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_\mathbf{k}t} + a^\dagger_{\mathbf{k}\sigma} e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_\mathbf{k}t)} \right)$$

Electric field:

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \quad \Rightarrow \mathbf{E}(\mathbf{r},t) = i\sum_{\mathbf{k}\sigma} \sqrt{\frac{\hbar\omega_\mathbf{k}}{2V\epsilon_0}} \boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \left(a_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_\mathbf{k}t} - a^\dagger_{\mathbf{k}\sigma} e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_\mathbf{k}t)} \right)$$

Magnetic field:

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \Rightarrow \mathbf{B}(\mathbf{r},t) = i\sum_{\mathbf{k}\sigma} \sqrt{\frac{\hbar}{2V\epsilon_0\omega_\mathbf{k}}} \mathbf{k} \times \boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \left(a_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_\mathbf{k}t} - a^\dagger_{\mathbf{k}\sigma} e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_\mathbf{k}t)} \right)$$

$$\mathbf{E}(\mathbf{r},t) = i\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t} - a^{\dagger}_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)$$

$$\mathbf{B}(\mathbf{r},t) = i\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\mathbf{k}\times\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t} - a^{\dagger}_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)$$

What is the expectation value of the E field for a pure eigenstate |n> of the electromagnetic Hamiltonian?

1. **A complex (non zero) number**
2. **Zero**
3. **Infinity**

What is the expectation value of the B field for a pure eigenstate |n> of the electromagnetic Hamiltonian?

1. **A complex (non zero) number**
2. **Zero**
3. **Infinity**

➔In fact, these are non-trivial questions

At this point, we might wonder how the classical and quantum pictures of the EM field can be reconciled --

An interesting picture comes from a particular linear combination of quantum states of a single mode ($k\sigma$) arising for example in a laser

How does a quantum mechanical E or B field exist? Consider a linear combination of pure photon states --

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PHOTON CORRELATIONS*

Roy J. Glauber
Lyman Laboratory, Harvard University, Cambridge, Massachusetts
(Received 27 December 1962)

In 1956 Hanbury Brown and Twiss[1] reported that the photons of a light beam of narrow spectral width have a tendency to arrive in correlated pairs. We have developed general quantum mechanical methods for the investigation of such correlation effects and shall present here results for the distribution of the number of photons counted in an incoherent beam. The fact that photon correlations are enhanced by narrowing the spectral bandwidth has led to a prediction[2] of large-scale correlations to be observed in the beam of an optical maser. We shall indicate that this prediction is misleading and follows from an inappropriate model of the maser beam. In considering these problems we shall outline a method of describing the photon field which appears particularly well suited to the discussion of experiments performed with light beams, whether coherent or incoherent.

The correlations observed in the photoionization processes induced by a light beam were given a simple semiclassical explanation by Purcell,[3] who made use of the methods of microwave noise theory. More recently, a number of papers have been written examining the correlations in considerably greater detail. These papers[2,4-6] retain the assumption that the electric field in a light beam can be described as a classical Gaussian stochastic process. In actuality, the behavior of the photon field is considerably more

Gauber's coherent state: $\left| c_\alpha \right\rangle \equiv \sum_{n=0}^{\infty} \frac{\alpha^n e^{-|\alpha|^2/2}}{\sqrt{n!}} \left| n \right\rangle$

Here α represents a complex amplitude

It is possible to prove the following identies for the coherent states:

1. $\left\langle c_\alpha \middle| c_\alpha \right\rangle = 1$

2. $\left\langle c_\alpha \middle| a \middle| c_\alpha \right\rangle = \alpha$

3. $\left\langle c_\alpha \middle| a^\dagger \middle| c_\alpha \right\rangle = \alpha^*$

4. $\left| \left\langle c_\alpha \middle| c_\beta \right\rangle \right|^2 = e^{-|\alpha-\beta|^2}$