PHY 742 Quantum Mechanics II
12-12:50 PM MWF in Olin 103

Notes for Lecture 24

Interaction of quantum electromagnetic fields with matter

Read Professor Carlson's textbook: Chapter 18. Photons and Atoms

1. Review of quantum theory of electromagnetism
2. Quantum treatment of the interaction of atoms and electromagnetic fields
3. Examples of atomic transitions
4. Some comments on lasers and masers

	Mon: 03/21/2022		Project presentations I		
	Wed: 03/23/2022		Project presentations II		
21	Fri: 03/25/2022	Chap. 5 & 17	Quantization of the Electromagnetic Field	#17	03/28/2022
22	Mon: 03/28/2022	Chap. 17	Quantization of the Electromagnetic Field	#18	03/30/2022
23	Wed: 03/30/2022	Chap. 17	Quantization of the Electromagnetic Field	#19	04/01/2022
24	Fri: 04/01/2022	Chap. 18	Absorption and emission of photons		

Summary of quantum electromagnetism

Previously, we derived the quantum electromagnetic Hamiltonian (omitting diverging term)

$$H_{\text{EM}} = \sum_{\mathbf{k}'\sigma'} \left(\hbar\omega_{\mathbf{k}'} a^{\dagger}_{\mathbf{k}'\sigma'} a_{\mathbf{k}'\sigma'} \right)$$

This is expressed in terms of operators $a_{\mathbf{k}\sigma}$ and $a^{\dagger}_{\mathbf{k}\sigma}$ operators for wavevector $\mathbf{k}$ and polarization σ.

With commutation relations: $\left[a_{\mathbf{k}\sigma}, a^{\dagger}_{\mathbf{k}'\sigma'} \right] = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'}$ $\quad \left[a_{\mathbf{k}\sigma}, a_{\mathbf{k}'\sigma'} \right] = 0$ $\quad \left[a^{\dagger}_{\mathbf{k}\sigma}, a^{\dagger}_{\mathbf{k}'\sigma'} \right] = 0$

It is convenient to define the photon number operator

$\mathbf{N}_{\mathbf{k}\sigma} \equiv a^{\dagger}_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma}$ with eigenvalues/eigenstates $\mathbf{N}_{\mathbf{k}\sigma} \left| n_{\mathbf{k}\sigma} \right\rangle = n_{\mathbf{k}\sigma} \left| n_{\mathbf{k}\sigma} \right\rangle$

Note that each wavevector $\mathbf{k}$ and polarization σ is independent (separable) in the EM Hamiltonian so that the system eigenstates are products of eigenstates for each mode:

$$\left| n_{\mathbf{k}_1\sigma_1} n_{\mathbf{k}_2\sigma_2} n_{\mathbf{k}_3\sigma_3} n_{\mathbf{k}_4\sigma_4} \ldots \right\rangle = \left| n_{\mathbf{k}_1\sigma_1} \right\rangle \left| n_{\mathbf{k}_2\sigma_2} \right\rangle \left| n_{\mathbf{k}_3\sigma_3} \right\rangle \left| n_{\mathbf{k}_4\sigma_4} \right\rangle \ldots.$$

Properties of the creation and annihilation operators:

$$a_{\mathbf{k}\sigma}\left|n_{\mathbf{k}\sigma}\right\rangle = \sqrt{n_{\mathbf{k}\sigma}}\left|n_{\mathbf{k}\sigma}-1\right\rangle$$

$$a^{\dagger}_{\mathbf{k}\sigma}\left|n_{\mathbf{k}\sigma}\right\rangle = \sqrt{n_{\mathbf{k}\sigma}+1}\left|n_{\mathbf{k}\sigma}+1\right\rangle$$

Quantum mechanical form of vector potential --

$$\mathbf{A}\left(\mathbf{r},t\right) = \sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t} + a^{\dagger}_{\mathbf{k}\sigma}e^{-\left(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t\right)}\right)$$

Note: We are assuming that the polarization vector is real. More generally there is a phase factor for each mode which we are ignoring at this moment.

Quantum mechanical form of **A**, **E**, and **B** fields --

$$\mathbf{A}(\mathbf{r},t)=\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}+a^{\dagger}_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)$$

Electric field:

$$\mathbf{E}=-\frac{\partial\mathbf{A}}{\partial t}\quad\Rightarrow\mathbf{E}(\mathbf{r},t)=i\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}-a^{\dagger}_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)$$

Magnetic field:

$$\mathbf{B}=\nabla\times\mathbf{A}\quad\Rightarrow\mathbf{B}(\mathbf{r},t)=i\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\mathbf{k}\times\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}-a^{\dagger}_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)$$

Previously (Lecture 14), we considered a charged particle in the presence of a classical electromagnetic field characterized by vector potential A and scalar potential *U*:

Hamiltonian of particle and field: $$H(\mathbf{r},t)=\frac{1}{2m}\left(\mathbf{p}-q\mathbf{A}(\mathbf{r},t)\right)^2+V(\mathbf{r})+qU(\mathbf{r},t)$$

Zero order Hamiltonian of particle and field: $$H^0(\mathbf{r})=-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{r})$$

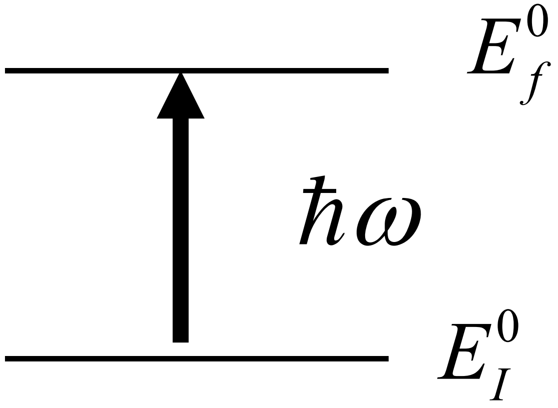
First order interaction Hamiltonian: $$H^1(\mathbf{r},t)=\frac{-q}{m}\mathbf{A}(\mathbf{r},t)\cdot\mathbf{p}+\frac{i\hbar q}{2m}\left(\nabla\cdot\mathbf{A}(\mathbf{r},t)\right)+qU(\mathbf{r},t)$$

Time dependent electric field: $$\mathbf{F}(\mathbf{r},t)=-\nabla U(\mathbf{r},t)-\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}$$

We used time dependent perturbation theory to analyze the effects of H^1

Fermi Golden rule for the rate of transition between states *I* and *f*:

$$\mathcal{R}_{I\to f} \approx \frac{2\pi}{\hbar}\left|\left\langle f^0\right|\tilde{H}^1\left|I^0\right\rangle\right|^2 \delta\left(-\hbar\omega + E_f^0 - E_I^0\right)$$



What is different about the quantum case?

1. Minor differences only for cases of very small or large EM fields?
2. New physics introduced?

What is different about the quantum case?

Using our quantum treatment, it is convenient to assume that the scalar field $U(\mathbf{r},t)=0$ and $\nabla\cdot\mathbf{A}(\mathbf{r},t)=0$ --

$$H_{\rm EM}=\sum_{\mathbf{k}'\sigma'}\left(\hbar\omega_{\mathbf{k}'}a^{\dagger}_{\mathbf{k}'\sigma'}a_{\mathbf{k}'\sigma'}\right)$$

Hamiltonian of system: $H(\mathbf{r},t)=\frac{1}{2m}\left(\mathbf{p}-q\mathbf{A}(\mathbf{r},t)\right)^2+V(\mathbf{r})+H_{\rm EM}$

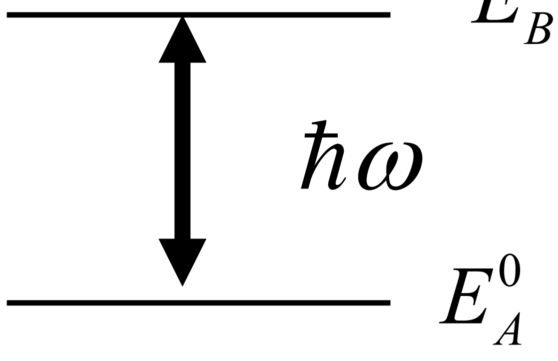
Hamiltonian of separate particle and EM systems: $H^0(\mathbf{r})=-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{r})+H_{\rm EM}$

First order interaction Hamiltonian: $H^1(\mathbf{r},t)=\frac{-q}{m}\mathbf{A}(\mathbf{r},t)\cdot\mathbf{p}$

Time dependent vector potential: $\mathbf{A}(\mathbf{r},t)=\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\left(a_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}+a^{\dagger}_{\mathbf{k}\sigma}e^{-(i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t)}\right)$

We can still use the Fermi Golden rule for transitions between two states of the zero order system $A^0 \leftrightarrow B^0$

$$\mathcal{R}_{A^0 \leftrightarrow B^0} \approx \frac{2\pi}{\hbar} \left| \left\langle B^0 \middle| \tilde{H}^1 \middle| A^0 \right\rangle \right|^2 \delta\left(\pm\hbar\omega \mp E_B^0 \pm E_A^0 \right)$$



Now the states $\left|A^0\right\rangle$ and $\left|B^0\right\rangle$ include both the eigenstates of the isolated particle and of the isolated EM system. For example we can denote

$$\left|A^0\right\rangle = \left|p_A; n_{\mathbf{k}_A\sigma_A}\right\rangle \qquad \left|B^0\right\rangle = \left|p_B; n_{\mathbf{k}_B\sigma_B}\right\rangle$$

In these terms the matrix elements can be evaluated --

$$\left\langle B^0 \middle| \tilde{H}^1 \middle| A^0 \right\rangle = -\frac{q}{m}\sqrt{\frac{\hbar}{2V\epsilon_0\omega_{\mathbf{k}}}} \times$$

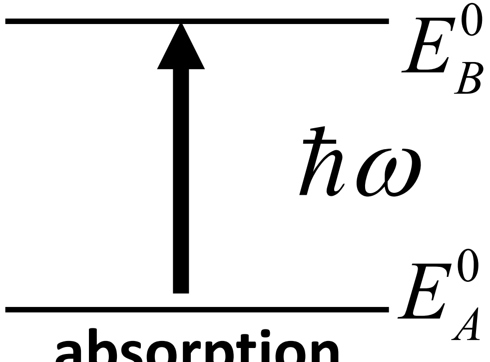
(Here we are suppressing the time dependence which should also be taken into account.)

$$\left(\sqrt{n_{\mathbf{k}_A\sigma_A}} \left\langle p_B \middle| e^{i\mathbf{k}\cdot\mathbf{r}} \mathbf{p}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \middle| p_A \right\rangle \left\langle n_{\mathbf{k}_B\sigma_B} \middle| n_{\mathbf{k}_A\sigma_A} - 1 \right\rangle + \sqrt{n_{\mathbf{k}_A\sigma_A} + 1} \left\langle p_B \middle| e^{-i\mathbf{k}\cdot\mathbf{r}} \mathbf{p}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \middle| p_A \right\rangle \left\langle n_{\mathbf{k}_B\sigma_B} \middle| n_{\mathbf{k}_A\sigma_A} + 1 \right\rangle \right)$$

$$\Rightarrow n_{\mathbf{k}_B\sigma_B} = n_{\mathbf{k}_A\sigma_A} \pm 1 \quad \text{corresponding to absorption or emission of a photon}$$

Some details -- assume $E_B^0 > E_A^0$

$$\mathcal{R}_{A^0 \to B^0} \approx \frac{2\pi}{\hbar} \left| \left\langle B^0 \middle| \tilde{H}^1 \middle| A^0 \right\rangle \right|^2 \delta\left(-\hbar\omega + E_B^0 - E_A^0\right)$$



$n_{\mathbf{k}_B\sigma_B} = n_{\mathbf{k}_A\sigma_A} - 1$

$n_{\mathbf{k}_A\sigma_A}$

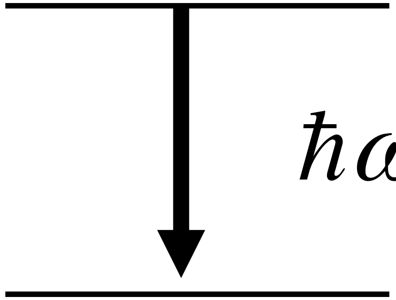
absorption

$$\mathcal{R}_{A^0 \to B^0} \approx \frac{2\pi}{\hbar} \left| \frac{q}{m} \sqrt{\frac{\hbar\, n_{\mathbf{k}_A\sigma_A}}{2V\epsilon_0\omega_{\mathbf{k}}}} \left\langle p_B \middle| e^{i\mathbf{k}\cdot\mathbf{r}} \mathbf{p} \cdot \boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \middle| p_A \right\rangle \right|^2 \delta\left(-\hbar\omega + E_B^0 - E_A^0\right)$$

$$= \frac{\pi q^2}{m^2 V \epsilon_0 \omega_{\mathbf{k}}} n_{\mathbf{k}_A\sigma_A} \left| \left\langle p_B \middle| e^{i\mathbf{k}\cdot\mathbf{r}} \mathbf{p} \cdot \boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \middle| p_A \right\rangle \right|^2 \delta\left(-\hbar\omega + E_B^0 - E_A^0\right)$$

More details -- assume $E_B^0 > E_A^0$

$$\mathcal{R}_{B^0 \to A^0} \approx \frac{2\pi}{\hbar} \left| \left\langle A^0 \middle| \tilde{H}^1 \middle| B^0 \right\rangle \right|^2 \delta\left(\hbar\omega + E_A^0 - E_B^0 \right)$$



$E_B^0 \quad n_{\mathbf{k}_B \sigma_B} = n_{\mathbf{k}_A \sigma_A} + 1$

$\hbar\omega$

$E_A^0 \quad n_{\mathbf{k}_A \sigma_A}$

emission

$$\mathcal{R}_{B^0 \to A^0} \approx \frac{2\pi}{\hbar} \left| \frac{q}{m} \sqrt{\frac{\hbar \left(n_{\mathbf{k}_A \sigma_A} + 1 \right)}{2V \epsilon_0 \omega_{\mathbf{k}}}} \left\langle p_A \middle| e^{i\mathbf{k}\cdot\mathbf{r}} \mathbf{p} \cdot \boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \middle| p_B \right\rangle \right|^2 \delta\left(\hbar\omega + E_A^0 - E_B^0 \right)$$

$$= \frac{\pi q^2}{m^2 V \epsilon_0 \omega_{\mathbf{k}}} \left(n_{\mathbf{k}_A \sigma_A} + 1 \right) \left| \left\langle p_A \middle| e^{i\mathbf{k}\cdot\mathbf{r}} \mathbf{p} \cdot \boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \middle| p_B \right\rangle \right|^2 \delta\left(\hbar\omega + E_A^0 - E_B^0 \right)$$

What is different about the quantum case?

Classical EM field

- **Matrix element depends on atomic selection rules**
- **Matrix element depends on EM field intensity**

Quantum EM field

- **Matrix element depends on atomic selection rules**
- **Matrix element depends on photon eigenstates; absorption different from emission**
- **Possibility of spontaneous emission**

Summary of results --

$$\text{Absorption: } \mathcal{R}_{A^0\rightarrow B^0} \approx \frac{\pi q^2}{m^2 V\epsilon_0 \omega_{\mathbf{k}}} n_{\mathbf{k}_A\sigma_A} \left|\left\langle p_B \left| e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{p}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\right| p_A\right\rangle\right|^2 \delta\left(-\hbar\omega + E_B^0 - E_A^0\right)$$

Happens only if $n_{\mathbf{k}_A\sigma_A} > 0$

$$\text{Emission: } \mathcal{R}_{B^0\rightarrow A^0} \approx \frac{\pi q^2}{m^2 V\epsilon_0 \omega_{\mathbf{k}}} \left(n_{\mathbf{k}_A\sigma_A} + 1\right) \left|\left\langle p_A \left| e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{p}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\right| p_B\right\rangle\right|^2 \delta\left(\hbar\omega + E_A^0 - E_B^0\right)$$

Can happens even if $n_{\mathbf{k}_A\sigma_A} = 0$

Lasers and Masers were developed to make use of the relationship between absorption and emission of EM radiation

Rev. Mod. Phys. 99, S263 (1999)

Laser physics: Quantum controversy in action

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We summarize the history and discuss quantum subtleties of maser/laser physics from early days until the present. [S0034-6861(99)03302-4]

Phys. Rev. 159, 208 (1967)

PHYSICAL REVIEW VOLUME 159, NUMBER 2 10 JULY 1967

Quantum Theory of an Optical Maser.* I. General Theory

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(Received 9 February 1967)

A quantum statistical analysis of an optical maser is presented in generalization of the recent semiclassical theory of Lamb. Equations of motion for the density matrix of the quantized electromagnetic field are derived. These equations describe the irreversible dynamics of the laser radiation in all regions of operation (above, below, and at threshold). Nonlinearities play an essential role in this problem. The diagonal equations of motion for the radiation are found to have an apparent physical interpretation. At steady state, these equations may be solved via detailed-balance considerations to yield the photon statistical distribution $\rho_{n,n}$. The resulting distribution has a variance which is significantly larger than that for coherent light. The off-diagonal elements of the radiation density matrix describe the effects of phase diffusion in general and provide the spectral profile $|E(\omega)|^2$ as a special case. A detailed discussion of the physics involved in this paper is given in the concluding sections. The theory of the laser adds another example to the short list of solved problems in irreversible quantum statistical mechanics.

The Nobel Prize in Physics 1964

Photo from the Nobel Foundation archive.

Charles Hard Townes

Prize share: 1/2

Photo from the Nobel Foundation archive.

Nicolay Gennadiyevich Basov

Prize share: 1/4

Photo from the Nobel Foundation archive.

Aleksandr Mikhailovich Prokhorov

Prize share: 1/4

The Nobel Prize in Physics 1964 was divided, one half awarded to Charles Hard Townes, the other half jointly to Nicolay Gennadiyevich Basov and Aleksandr Mikhailovich Prokhorov "for fundamental work in the field of quantum electronics, which has led to the construction of oscillators and amplifiers based on the maser-laser principle."

Summarizing thoughts --

Quantum electromagnetic Hamiltonian (omitting diverging term)

$$H_{\mathrm{EM}} = \sum_{\mathbf{k}'\sigma'} \left(\hbar\omega_{\mathbf{k}'} a^{\dagger}_{\mathbf{k}'\sigma'} a_{\mathbf{k}'\sigma'} \right)$$

Note that omitting diverging terms is not completely correct.

These diverging terms reappear when we evaluate the variance of the E or B fields for a pure eigenstate of the EM Hamiltonian and is thought to be related to the notion of vacuum fluctuations.

We eluded to the fact the electric and magnetic quantum fields do not commute.

The derivation is complicated and gives the following result:

$$[E_x(\mathbf{r},t), B_y(\mathbf{r}',t)] = -i\frac{\hbar c^2}{\mu_0}\frac{\partial}{\partial z}\delta(\mathbf{r}-\mathbf{r}')$$

More summarizing thoughts --

$$\text{Glauber's} \quad \text{coherent state:} \quad \left|c_\alpha\right\rangle \equiv \sum_{n=0}^{\infty} \frac{\alpha^n e^{-|\alpha|^2/2}}{\sqrt{n!}} \left|n\right\rangle \quad \text{based on a single mode } n \rightarrow n_{\mathbf{k}\sigma}$$

These coherent states are an example of linear combinations of pure EM eigenstates that represent features of observed electromagnetic properties.