PHY 742 Quantum Mechanics II
12-12:50 PM MWF Olin 103

Notes for Lecture 32

A short introduction to the quantum theory of superconductivity

Bardeen, Cooper, Schrieffer, Phys. Rev. 108, 1175 (1957)

1. Cooper pairs
2. Gap equation
3. Estimate of T_c

Some of the slides contain materials from the textbook, **Solid State Physics; Second Edition by Giuseppe Grosso and Giuseppe Pastori Parravicini (Academic Press, 2014)**

27	Fri: 04/08/2022		Multi electron atoms	#21	04/11/2022
28	Mon: 04/11/2022		Multi electron atoms	#22	04/18/2022
29	Wed: 04/13/2022		Multi electron atoms		
	Fri: 04/15/2022	*No class*	Holiday		
30	Mon: 04/18/2022		Hubbard model with multiple electrons	#23	04/22/2022
31	Wed: 04/20/2022		Hubbard model with multiple electrons		
32	Fri: 04/22/2022		BCS model of superconductivity		
33	Mon: 04/25/2022		BCS model of superconductivity		

PHYSICAL REVIEW VOLUME 108, NUMBER 5 DECEMBER 1, 1957

Theory of Superconductivity*

J. BARDEEN, L. N. COOPER,† AND J. R. SCHRIEFFER‡
Department of Physics, University of Illinois, Urbana, Illinois

(Received July 8, 1957)

A theory of superconductivity is presented, based on the fact that the interaction between electrons resulting from virtual exchange of phonons is attractive when the energy difference between the electrons states involved is less than the phonon energy, $\hbar\omega$. It is favorable to form a superconducting phase when this attractive interaction dominates the repulsive screened Coulomb interaction. The normal phase is described by the Bloch individual-particle model. The ground state of a superconductor, formed from a linear combination of normal state configurations in which electrons are virtually excited in pairs of opposite spin and momentum, is lower in energy than the normal state by amount proportional to an average $(\hbar\omega)^2$, consistent with the isotope effect. A mutually orthogonal set of excited states in one-to-one correspondence with those of the normal phase is obtained by specifying occupation of certain Bloch states and by using the rest to form a linear combination of virtual pair configurations. The theory yields a second-order phase transition and a Meissner effect in the form suggested by Pippard. Calculated values of specific heats and penetration depths and their temperature variation are in good agreement with experiment. There is an energy gap for individual-particle excitations which decreases from about $3.5kT_c$ at $T=0°$K to zero at T_c. Tables of matrix elements of single-particle operators between the excited-state superconducting wave functions, useful for perturbation expansions and calculations of transition probabilities, are given.

Some of you may wish to read the paper which is available from zsr.wfu.edu

https://doi.org/10.1103/PhysRev.108.1175

The Nobel Prize in Physics 1972

Photo from the Nobel Foundation archive.

John Bardeen

Prize share: 1/3

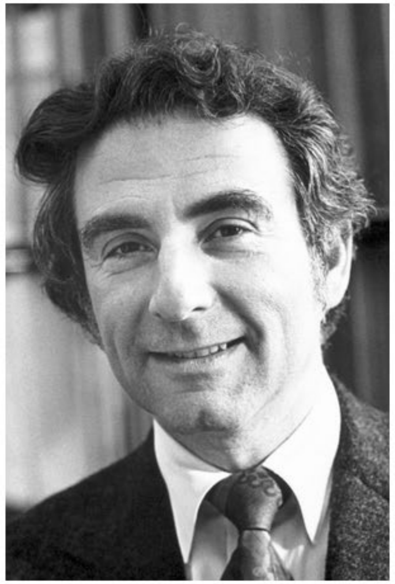


Photo from the Nobel Foundation archive.

Leon Neil Cooper

Prize share: 1/3

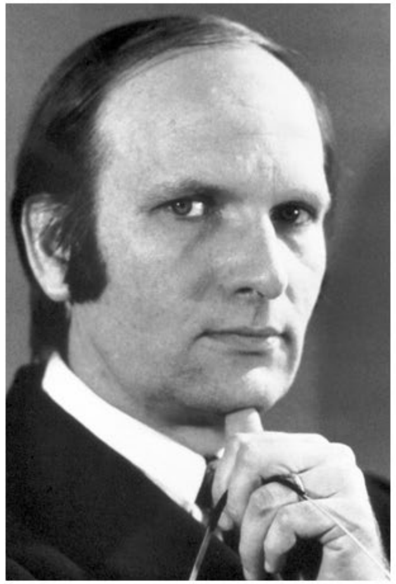


Photo from the Nobel Foundation archive.

John Robert Schrieffer

The Nobel Prize in Physics 1972 was awarded jointly to John Bardeen, Leon Neil Cooper and John Robert Schrieffer "for their jointly developed theory of superconductivity, usually called the BCS-theory."

Overview of superconductivity --

Ref:D. Teplitz, editor, Electromagnetism – paths to research,
Plenum Press (1982); Chapter 1 written by Brian Schwartz and Sonia Frota-Pessoa

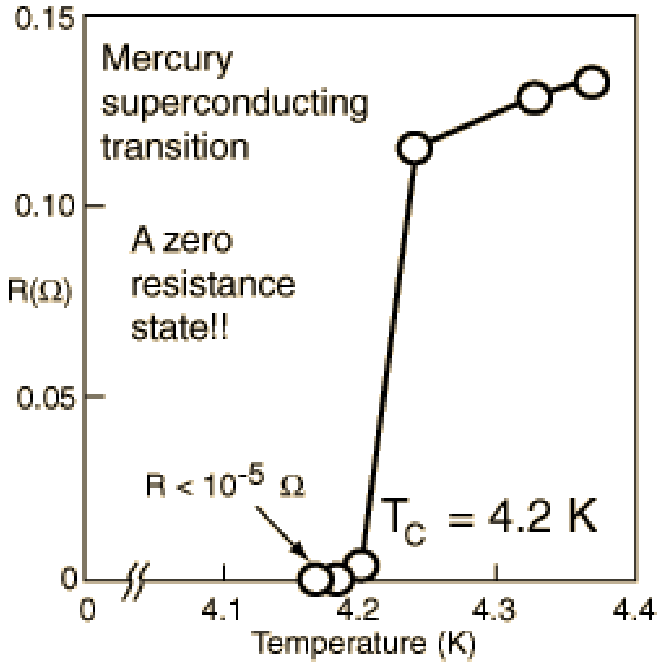
History:

1908 H. Kamerlingh Onnes successfully liquified He

1911 H. Kamerlingh Onnes discovered that Hg at 4.2 K has vanishing resistance

1957 Theory of superconductivity by Bardeen, Cooper, and Schrieffer

The surprising observation was that electrical resistivity abruptly dropped when the temperature of the material was lowered below a critical temperature T_c.



Some thoughts related to the statistical mechanics of Bose particles

For Bose particles, many particles can occupy the same state. This means that for non-interacting Bose particles, according to statistical mechanics, at very low temperature, it is possible for a macroscopic number of particles to occupy the lowest single particle state and produce a "Bose condensate". ^{4}He is not a good example of this, since the particles have significant interactions, but the superfluid behavior is logically related. A better example was demonstrated in 1995 with 2.5 x 10^{12} ^{87}Rb atoms cooled to 1.7 x 10^{-7}K.

For superconductivity, electrons are the particles. How is possible for Fermi particles to behave with Bose-like statistics?

Notion of a Cooper pair

Starting with a material with all the states filled up to the Fermi level, we focus attention on a pair of states which have a net attractive interaction *U*($\mathbf{r}_1$,$\mathbf{r}_2$):

$$\left[\frac{\mathbf{p}_1^2}{2m} + \frac{\mathbf{p}_2^2}{2m} + U(\mathbf{r}_1,\mathbf{r}_2)\right]\psi(\mathbf{r}_1\sigma_1,\mathbf{r}_2\sigma_2) = E\psi(\mathbf{r}_1\sigma_1,\mathbf{r}_2\sigma_2)$$

$$\psi(\mathbf{r}_1\sigma_1,\mathbf{r}_2\sigma_2) = \phi(\mathbf{r}_1,\mathbf{r}_2)\chi(\sigma_1,\sigma_2)$$

The thinking is that the interaction is related to lattice vibrations and had an energy $\hbar\omega_D$

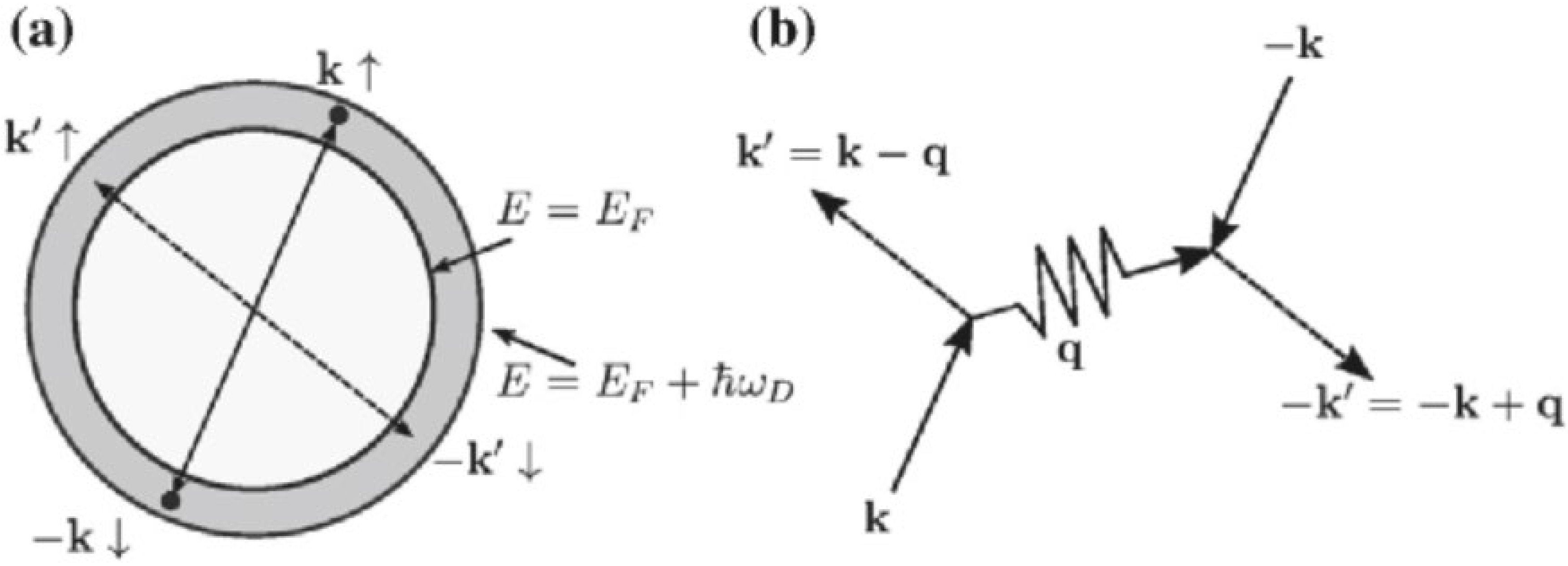


Figure 18.7 (a) Schematic representation of a *single* Cooper pair, added to the ground state of a free-electron gas. Two "extra electrons" in the pair state ($\mathbf{k}\uparrow$, $-\mathbf{k}\downarrow$) scatter freely to the pair states ($\mathbf{k}'\uparrow$, $-\mathbf{k}'\downarrow$), in the energy region $E_F < E_{\mathbf{k}},\ E_{\mathbf{k}'} < E_F + \hbar\omega_D$, where the phonon-mediated attractive interaction is operative, and form a bound Cooper pair. (b) Schematic representation of the scattering of two electrons with wavevectors ($\mathbf{k}$, $-\mathbf{k}$) into the state ($\mathbf{k}'$, $-\mathbf{k}'$) via the emission and subsequent absorption of a phonon of momentum $\hbar\mathbf{q}$.

Properties of pair wavefunction

$$\psi(\mathbf{r}_1\sigma_1, \mathbf{r}_2\sigma_2) = \phi(\mathbf{r}_1, \mathbf{r}_2)\chi(\sigma_1, \sigma_2)$$

Note: $\sigma = \alpha \equiv \uparrow$

$\sigma = \beta \equiv \downarrow$

Spin part:

$$\chi^{(S=0)} = \frac{1}{\sqrt{2}}[\alpha(1)\beta(2) - \beta(1)\alpha(2)].$$

or

$$\chi^{(S=1)} = \begin{cases} \alpha(1)\alpha(2), \\ \frac{1}{\sqrt{2}}[\alpha(1)\beta(2) + \beta(1)\alpha(2)], \\ \beta(1)\beta(2), \end{cases}$$

Note that:

$$\chi^{S=0}(\sigma_1, \sigma_2) = -\chi^{S=0}(\sigma_2, \sigma_1)$$

$$\Rightarrow \phi^{S=0}(\mathbf{r}_1, \mathbf{r}_2) = \phi^{S=0}(\mathbf{r}_2, \mathbf{r}_1)$$

Note that:

$$\chi^{S=1}(\sigma_1, \sigma_2) = \chi^{S=1}(\sigma_2, \sigma_1)$$

$$\Rightarrow \phi^{S=1}(\mathbf{r}_1, \mathbf{r}_2) = -\phi^{S=1}(\mathbf{r}_2, \mathbf{r}_1)$$

Here are some possible functional forms for the spatial part of the Cooper pair. Which of these are associated with S=0 and which are associated with S=1?

$$1. \quad \phi(\mathbf{r}_1, \mathbf{r}_2) = A\cos(k(\mathbf{r}_1 - \mathbf{r}_2))$$

$$2. \quad \phi(\mathbf{r}_1, \mathbf{r}_2) = A\sin(k(\mathbf{r}_1 - \mathbf{r}_2))$$

Properties of pair wavefunction – continued

Assume that the electron pair can be represented by a linear combination of plane wave states of wavevectors k and -k:

$$\phi(\mathbf{r}_1,\mathbf{r}_2) = \sum_{\mathbf{k}} g(\mathbf{k}) \frac{1}{V} e^{i\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)}$$

What is g(k)?

Note that:

$$g^{S=0}(\mathbf{k}) = g^{S=0}(-\mathbf{k})$$

$$g^{S=1}(\mathbf{k}) = -g^{S=1}(-\mathbf{k})$$

Note that the states composing Cooper pairs are supposed to exist in the energy range $E_F \leq E_{\mathbf{k}} \leq E_F + \hbar\omega_D$

Define Fourier transform of interaction potential:

$$U_{\mathbf{k}\mathbf{k}'} = \iint \frac{1}{V} e^{-i\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)} U(\mathbf{r}_1-\mathbf{r}_2) \frac{1}{V} e^{i\mathbf{k}'\cdot(\mathbf{r}_1-\mathbf{r}_2)} d\mathbf{r}_1 d\mathbf{r}_2$$
$$= \frac{1}{N\Omega} \int e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} U(\mathbf{r})\, d\mathbf{r}$$

V volume of sample composed of N unit cells

Ω volume of unit cell

Equation satisfied by pair amplitude functions:

$$(2E_{\mathbf{k}} - E) g(\mathbf{k}) + \sum_{\mathbf{k}'} U_{\mathbf{k}\mathbf{k}'} g(\mathbf{k}') = 0 \qquad E_F < E_{\mathbf{k}}, E_{\mathbf{k}'} < E_F + \hbar\omega_D$$

Cooper pair equations -- continued

$$(2E_{\mathbf{k}} - E)g(\mathbf{k}) + \sum_{\mathbf{k}'} U_{\mathbf{k}\mathbf{k}'} g(\mathbf{k}') = 0 \qquad E_F < E_{\mathbf{k}}, E_{\mathbf{k}'} < E_F + \hbar\omega_D$$

Simplified model for interaction:

$$U_{\mathbf{k}\mathbf{k}'} = -U_0/N$$

$$(2E_{\mathbf{k}} - E)g(\mathbf{k}) - U_0 \frac{1}{N} \sum_{\mathbf{k}'} g(\mathbf{k}') = 0 \qquad E_F < E_{\mathbf{k}}, E_{\mathbf{k}'} < E_F + \hbar\omega_D; \quad U_0 > 0.$$

In this approximation, for triplet states $\sum_{\mathbf{k}} g^{S=1}(\mathbf{k}) = 0$

$\Rightarrow$ Cooper pair states can only be singlet states

Cooper pair equations -- continued

Non-trivial solution for singlet state:

$$(2E_{\mathbf{k}} - E)g(\mathbf{k}) - U_0 \frac{1}{N} \sum_{\mathbf{k}'} g(\mathbf{k}') = 0$$

E_{pair}

Equation to determine eigenstate energy:

$$1 = U_0 \frac{1}{N} \sum_{\mathbf{k}} \frac{1}{2E_{\mathbf{k}} - E_{\text{pair}}} \qquad E_F < E_{\mathbf{k}} < E_F + \hbar\omega_D.$$

$$1 = U_0 \frac{1}{N} \int_{E_F}^{E_F+\hbar\omega_D} D_0(E) \frac{1}{2E - E_{\text{pair}}}\, dE,$$

n

Density of states (one electron basis)

How did the DOS come into the result?

Cooper pair equations -- continued

Using the Debye approximation:

$$1 = U_0 \, n_0 \int_{E_F}^{E_F + \hbar\omega_D} \frac{1}{2E - E_{\text{pair}}} \, dE = \frac{1}{2} U_0 \, n_0 \, \ln \frac{2E_F + 2\hbar\omega_D - E_{\text{pair}}}{2E_F - E_{\text{pair}}}$$

where $\mathrm{n}_0 \equiv \frac{D_0(E_F)}{N} = \frac{3}{4}\frac{Z}{E_F}$ denoting Fermi level DOS for a single spin

for simple metal of valence Z

Binding energy:

$$\Delta_b = 2E_F - E_{\text{pair}} = \hbar\omega_D \frac{e^{-1/U_0 \, n_0}}{\sinh[1/U_0 n_0]} \approx 2\hbar\omega_D \exp[-2/U_0 n_0].$$

Shows that a singlet Cooper pair is more stable than the independent particle system even for small U_0.

What justifies this conclusion?

Variational determination of the ground-state wavefunction in the BCS model

Second quantization

$$\left\{c_{\mathbf{k}\sigma}, c_{\mathbf{k}'\sigma'}\right\} = \left\{c^\dagger_{\mathbf{k}\sigma}, c^\dagger_{\mathbf{k}'\sigma'}\right\} = 0, \qquad \left\{c_{\mathbf{k}\sigma}, c^\dagger_{\mathbf{k}'\sigma'}\right\} = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'}.$$

Note that the Cooper pair singlet state can be written

$$\psi(\mathbf{r}_1\sigma_1, \mathbf{r}_2\sigma_2) = \sum_{\mathbf{k}} g(\mathbf{k}) \frac{1}{\sqrt{2}} \frac{1}{V} \left[e^{i\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)} \alpha(1)\beta(2) - e^{-i\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)} \beta(1)\alpha(2) \right]$$

$$\equiv \sum_{\mathbf{k}} g(\mathbf{k})\, c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} |0\rangle,$$

Variational determination of the ground-state wavefunction in the BCS model -- continued

$$H = \underbrace{\sum_{\mathbf{k}} E_{\mathbf{k}} \left(c^{\dagger}_{\mathbf{k}\uparrow} c_{\mathbf{k}\uparrow} + c^{\dagger}_{-\mathbf{k}\downarrow} c_{-\mathbf{k}\downarrow} \right)}_{\text{independent particle states}} + \underbrace{\sum_{\mathbf{k}\mathbf{k}'} U_{\mathbf{k}\mathbf{k}'} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}}_{\text{Cooper-pair interaction}}.$$

Consider a ground state wavefunction of the form

$$\boxed{|\Psi_S\rangle = \prod_{\mathbf{k}} \left(u_{\mathbf{k}} + v_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} \right) |0\rangle},$$

$$u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2 = 1$$

probability amplitude for forming Cooper pair

Variational determination of the ground-state wavefunction in the BCS model -- continued

Need to minimize the expectation value:

$$\boxed{W_S = \langle \Psi_S | H_{\rm BCS} | \Psi_S \rangle}$$

$$\boxed{H_{\rm BCS} = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \left(c^{\dagger}_{\mathbf{k}\uparrow} c_{\mathbf{k}\uparrow} + c^{\dagger}_{-\mathbf{k}\uparrow} c_{-\mathbf{k}\uparrow} \right) + \sum_{\mathbf{k}\mathbf{k}'} U_{\mathbf{k}\mathbf{k}'} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}}$$

$$\varepsilon_{\mathbf{k}} = E_{\mathbf{k}} - \mu = (\hbar^2 \mathbf{k}^2/2m) - \mu$$

Here μ is essentially E_F.

After some algebra:

$$W_S = \langle \Psi_S | H_{\rm BCS} | \Psi_S \rangle = 2 \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} v_{\mathbf{k}}^2 + \sum_{\mathbf{k}\mathbf{k}'} U_{\mathbf{k}\mathbf{k}'} u_{\mathbf{k}} v_{\mathbf{k}} u_{\mathbf{k}'} v_{\mathbf{k}'}$$

Variational determination of the ground-state wavefunction in the BCS model -- continued

Convenient transformation:

$$\begin{cases} u_{\mathbf{k}} = \cos\theta_{\mathbf{k}} \\ v_{\mathbf{k}} = \sin\theta_{\mathbf{k}} \end{cases} \Longrightarrow \quad \sin 2\theta_{\mathbf{k}} = 2u_{\mathbf{k}}v_{\mathbf{k}}; \quad \cos 2\theta_{\mathbf{k}} = u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2.$$

$$W_S = 2\sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \sin^2\theta_{\mathbf{k}} + \frac{1}{4}\sum_{\mathbf{kk}'} U_{\mathbf{kk}'} \sin 2\theta_{\mathbf{k}} \sin 2\theta_{\mathbf{k}'}$$

$$\frac{\partial W_S}{\partial \theta_{\mathbf{k}}} = 0 \quad \Longrightarrow \quad 2\varepsilon_{\mathbf{k}} \sin 2\theta_{\mathbf{k}} + \sum_{\mathbf{k}'} U_{\mathbf{kk}'} \cos 2\theta_{\mathbf{k}} \sin 2\theta_{\mathbf{k}'} = 0$$

$$\Longrightarrow \quad 2\varepsilon_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} + \sum_{\mathbf{k}'} U_{\mathbf{kk}'} \left(u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2\right) u_{\mathbf{k}'} v_{\mathbf{k}'} = 0$$

Define: $$\Delta_{\mathbf{k}} = -\sum_{\mathbf{k}'} U_{\mathbf{kk}'} u_{\mathbf{k}'} v_{\mathbf{k}'}.$$

Variational determination of the ground-state wavefunction in the BCS model -- continued

In terms of the "gap parameter" the variational equations become:

$$\boxed{2\varepsilon_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} - \Delta_{\mathbf{k}}\left(u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2\right) = 0}.$$

$$u_{\mathbf{k}}^2 = \frac{1}{2}\left[1 + \frac{\varepsilon_{\mathbf{k}}}{\sqrt{\varepsilon_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2}}\right] \quad \text{and} \quad v_{\mathbf{k}}^2 = \frac{1}{2}\left[1 - \frac{\varepsilon_{\mathbf{k}}}{\sqrt{\varepsilon_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2}}\right].$$

$$\boxed{\Delta_{\mathbf{k}} = -\frac{1}{2}\sum_{\mathbf{k}'} U_{\mathbf{k}\mathbf{k}'} \frac{\Delta_{\mathbf{k}'}}{\sqrt{\varepsilon_{\mathbf{k}'}^2 + \Delta_{\mathbf{k}'}^2}}}$$

Variational determination of the ground-state wavefunction in the BCS model -- continued

Simplified model

$$U_{\mathbf{k}\mathbf{k}'} = \begin{cases} -U_0/N & \text{if } |\varepsilon_{\mathbf{k}}|, |\varepsilon_{\mathbf{k}'}| < \hbar\omega_D \quad (U_0 > 0), \\ 0 & \text{otherwise}, \end{cases}$$

$$\Delta_{\mathbf{k}} = \begin{cases} \Delta_0 & \text{if } |\varepsilon_{\mathbf{k}}| < \hbar\omega_D, \\ 0 & \text{otherwise}. \end{cases}$$

$$1 = \frac{1}{2} U_0 \frac{1}{N} \sum_{\mathbf{k}'} \frac{1}{\sqrt{\varepsilon_{\mathbf{k}'}^2 + \Delta_0^2}} \qquad \text{with} \qquad -\hbar\omega_D < \varepsilon_{\mathbf{k}'} < \hbar\omega_D.$$

Using DOS:

$$1 = \frac{1}{2} U_0 n_0 \int_{-\hbar\omega_D}^{\hbar\omega_D} \frac{d\varepsilon}{\sqrt{\varepsilon^2 + \Delta_0^2}}.$$

Variational determination of the ground-state wavefunction in the BCS model -- continued

$$1 = \frac{1}{2}U_0 n_0 \int_{-\hbar\omega_D}^{\hbar\omega_D} \frac{d\varepsilon}{\sqrt{\varepsilon^2 + \Delta_0^2}} = U_0 n_0 \sinh^{-1}\frac{\hbar\omega_D}{\Delta_0}$$

Solving for the gap parameter:

$$\Delta_0 = \frac{\hbar\omega_D}{\sinh(1/U_0 n_0)} \approx 2\hbar\omega_D \exp[-1/U_0 n_0]$$

Estimating the ground state energy of the superconducting state:

$$W_S - W_N = 2\sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} v_{\mathbf{k}}^2 + \sum_{\mathbf{k}\mathbf{k}'} U_{\mathbf{k}\mathbf{k}'} u_{\mathbf{k}} v_{\mathbf{k}} u_{\mathbf{k}'} v_{\mathbf{k}'} - 2\sum_{\mathbf{k}}^{k<k_F} \varepsilon_{\mathbf{k}}.$$

Estimating the ground state energy of the superconducting state – continued

Using the variational solution and integrating the DOS:

$$W_S - W_N = D_0(E_F)\int_{-\hbar\omega_D}^{\hbar\omega_D}\left(\varepsilon - \frac{2\varepsilon^2+\Delta_0^2}{2\sqrt{\varepsilon^2+\Delta_0^2}}\right)d\varepsilon - D_0(E_F)\int_{-\hbar\omega_D}^{0} 2\varepsilon\, d\varepsilon.$$

$$W_S - W_N = D_0(E_F)\left[-\hbar\omega_D\sqrt{\hbar^2\omega_D^2+\Delta_0^2}+\hbar^2\omega_D^2\right].$$

$$\approx -\frac{1}{2}D_0(E_F)\Delta_0^2$$

Effects of temperature:

Thermal average of Cooper pair operator:

$$a_{\mathbf{k}} = \left\langle c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} \right\rangle_T$$

Define

$$c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} = a_{\mathbf{k}} + \left(c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} - a_{\mathbf{k}} \right),$$

average

fluctuations

Modified Gap relationship

$$\Delta_{\mathbf{k}} = -\sum_{\mathbf{k}'} U_{\mathbf{k}\mathbf{k}'} u_{\mathbf{k}'} v_{\mathbf{k}'} [1 - 2f(w_{\mathbf{k}'})] \quad \text{with} \quad w_{\mathbf{k}} = \sqrt{\varepsilon_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2}.$$

Fermi-Dirac distribution

$$f(E) = \frac{1}{e^{\beta E} + 1} \implies 1 - 2f(E) = \tanh \frac{\beta E}{2},$$

Modified Gap equation

$$\boxed{\Delta_{\mathbf{k}} = -\frac{1}{2} \sum_{\mathbf{k}'} U_{\mathbf{k}\mathbf{k}'} \frac{\Delta_{\mathbf{k}'}}{\sqrt{\varepsilon_{\mathbf{k}'}^2 + \Delta_{\mathbf{k}'}^2}} \tanh \frac{\beta \sqrt{\varepsilon_{\mathbf{k}'}^2 + \Delta_{\mathbf{k}'}^2}}{2}}.$$

Simplified model

$$U_{\mathbf{kk'}} = \begin{cases} -U_0/N & \text{if } |\varepsilon_{\mathbf{k}}|, |\varepsilon_{\mathbf{k'}}| < \hbar\omega_D \quad (U_0 > 0), \\ 0 & \text{otherwise}, \end{cases}$$

$$1 = \frac{1}{2} U_0 n_0 \int_{-\hbar\omega_D}^{\hbar\omega_D} \frac{d\varepsilon}{\sqrt{\varepsilon^2 + \Delta^2}} \tanh \frac{\beta\sqrt{\varepsilon^2 + \Delta^2}}{2},$$

Determine the critical temperature such that $\Delta(T_c) = 0$:

$$1 = U_0 n_0 \int_0^{\hbar\omega_D} \frac{1}{\varepsilon} \tanh \frac{\varepsilon}{2k_B T_c} \, d\varepsilon,$$

Evaluation of the integral

$$I(a) = \int_0^a \frac{1}{x} \tanh x \, dx = \left[\tanh x \; \ln x\right]_0^a - \int_0^a \ln x \frac{1}{\cosh^2 x} dx$$

$$(\text{for } a \gg 1) \approx \ln a - \int_0^\infty \ln x \frac{1}{\cosh^2 x} dx = \ln a + \ln \frac{4\gamma}{\pi} \quad \gamma = 1.78107\ldots$$

If $\frac{\hbar\omega_D}{2kT_c} >> 1:$

$$\int_0^{\hbar\omega_D/2k_B T_c} \frac{1}{x} \tanh x \, dx = \ln\left(\frac{2\gamma}{\pi}\frac{\hbar\omega_D}{k_B T_c}\right) \approx \ln \frac{1.13\hbar\omega_D}{k_B T_c} = \frac{1}{U_0 n_0}.$$

Then, in the weak coupling limit $U_0 n_0 \ll 1$ and $\hbar\omega_D/k_B T_c \gg 1$, we have

$$k_B T_c = 1.13\, \hbar\omega_D \exp[-1/U_0 n_0].$$

Numerical evaluation of integral:

$$1 = \frac{1}{2} U_0 n_0 \int_{-\hbar\omega_D}^{\hbar\omega_D} \frac{d\varepsilon}{\sqrt{\varepsilon^2 + \Delta^2}} \tanh \frac{\beta\sqrt{\varepsilon^2 + \Delta^2}}{2}.$$

$$1 = U_0 n_0 \int_0^{\hbar\omega_D} \frac{1}{\varepsilon} \tanh \frac{\varepsilon}{2k_B T_c} d\varepsilon,$$

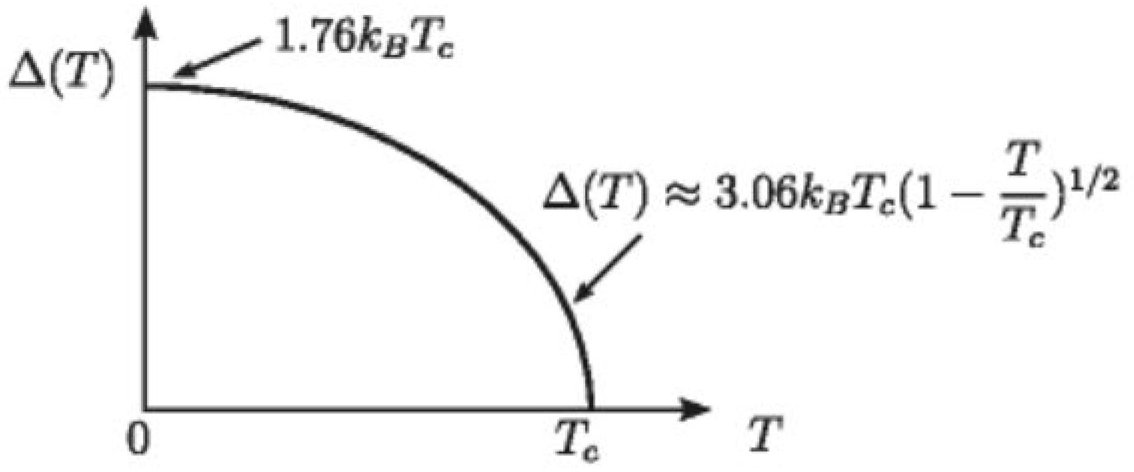


Figure 18.12 Behavior of the energy gap parameter $\Delta(T)$ for a superconductor in the BCS theory and in weak coupling limit.

Estimation of critical magnetic field (BCS paper)

$$H_c^2/8\pi = F_n - F_s,$$

Free energy of normal state

Free energy of superconducting state

After some approximations, etc.:

$$\frac{H_c^2}{8\pi} = N(0)(\hbar\omega)^2\left\{\left[1+\left(\frac{\epsilon_0}{\hbar\omega}\right)^2\right]^{\frac{1}{2}}-1\right\} - \frac{\pi^2}{3}N(0)(kT)^2$$

$$\times\left\{1-\beta^2\int_0^\infty d\epsilon\left[\frac{2\epsilon^2+\epsilon_0^2}{E}\right]f(\beta E)\right\}. \quad (3.38)$$

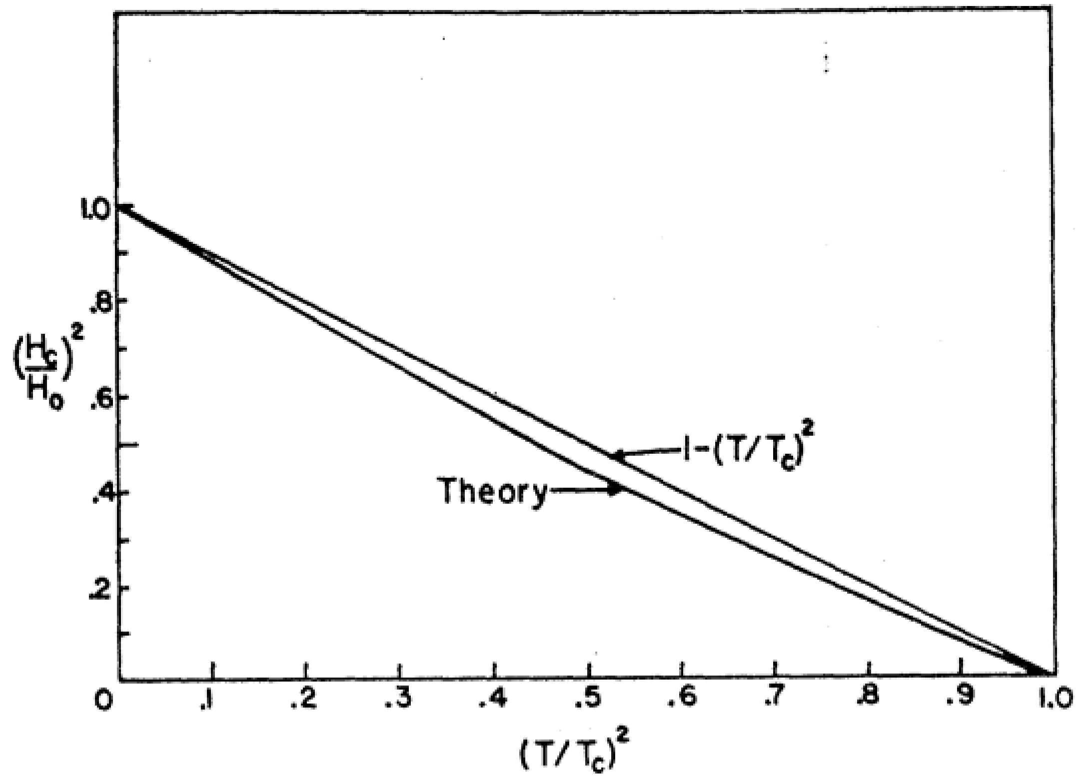


FIG. 2. Ratio of the critical field to its value at $T=0°K$ *vs* $(T/T_c)^2$. The upper curve is the $1-(T/T_c)^2$ law of the Gorter-Casimir theory and the lower curve is the law predicted by the theory in the weak-coupling limit. Experimental values generally lie between the two curves.

What do you think?

1. **Full of admiration**
2. **Full of disgust**
3. **Not full**
4. **Want to read more**
5. **Never want to see this again**