Advanced Quantum Mechanics

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2–8. A stable, spinless nucleus A of even parity has a spin-one, odd-parity excited state R whose only significant decay mode is $R \to A + \gamma$.
a) Assuming that excitations to intermediate states other than R are unimportant and ignoring the nuclear Thomson term (due to $\mathbf{A}\cdot\mathbf{A}$), show that the differential and total scattering cross sections of a γ-ray by the ground state A are given by

$$\frac{d\sigma}{d\Omega} = \frac{9}{16}\left(\frac{c}{\omega}\right)^2 (\boldsymbol{\epsilon}^{(\alpha)}\cdot\boldsymbol{\epsilon}^{(\alpha')})^2 \frac{\Gamma_R^2}{(E_R - E_A - \hbar\omega)^2 + (\Gamma_R^2/4)},$$

$$\sigma_{\text{tot}} = \frac{3}{2}(4\pi\lambda\!\!\!^{-2}) \frac{(\Gamma_R^2/4)}{(E_R - E_A - \hbar\omega)^2 + (\Gamma_R^2/4)},$$

where $\lambda\!\!\!^{-} = c/\omega$. (The second formula can be generalized to resonance scattering involving any multipole transition if we replace 3/2 by $(2J_R + 1)/[2(2J_A + 1)]$.)
b) Verify that the above expression for the total cross section is equal to $4\pi(c/\omega)\,\text{Im}f(\omega)$.
c) The nucleus C^{14} whose ground state is known to be a 0^+ state has an excited 1^- state (denoted by C^{14*}) 6.1 MeV above the ground state. The only decay mode of C^{14*} is known to be $C^{14} + \gamma$. Compute the total cross section at exact resonance and compare it with the cross section for nuclear Thomson scattering due to the C^{14} nucleus as a whole.

2–9. Assuming that $f(\omega)$ for the scattering of a high-energy photon by the hydrogen atom is given by the Thomson amplitude, derive the sum rule

$$2\pi^2 c r_0 = \int_0^\infty \sigma_{\text{tot}}(\omega)\, d\omega.$$

Show that within the framework of the approximations made in this chapter, the above sum rule is equivalent to the well-known Thomas-Reiche-Kuhn sum rule:‡

$$\sum_I \frac{2m\omega_{IA}}{\hbar} |\mathbf{x}_{IA}|^2 = 3.$$

2–10. Assuming the validity of perturbation theory, use the fixed-source neutral scalar theory of Problem 2–7 to obtain an expression for the probability of finding one virtual meson of energy $< \hbar\omega^{(\max)}$ around the nucleon.

2–11. Why is it legitimate to ignore Fig. 2–7(a) in estimating the Lamb shift?

‡Merzbacher (1961) p. 446.

CHAPTER 3

RELATIVISTIC QUANTUM MECHANICS OF SPIN-$\frac{1}{2}$ PARTICLES

3–1. PROBABILITY CONSERVATION IN RELATIVISTIC QUANTUM MECHANICS

It is almost traditional to start an exposition of the Dirac theory of spin-$\frac{1}{2}$ particles by discussing some of the "difficulties" of the Klein-Gordon theory. As we shall see later, there is actually nothing wrong in the Klein-Gordon equation if it is properly interpreted. We shall sketch, however, the usual arguments against the Klein-Gordon equation since they played an important historic role in the formulation of relativistic quantum mechanics.

In Schrödinger's wave mechanics we associate a complex-valued wave function ψ with a single particle, such that $|\psi|^2 d^3x$ gives the probability of finding the particle in a volume element d^3x. This interpretation is possible because the probability density P and the flux density $\mathbf{S}$ given by

$$P = |\psi|^2 > 0 \tag{3.1}$$

and

$$\mathbf{S} = -(i\hbar/2m)(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*) \tag{3.2}$$

satisfy the continuity equation

$$\partial P/\partial t + \boldsymbol{\nabla}\cdot\mathbf{S} = 0 \tag{3.3}$$

by virtue of the Schrödinger equation. Using Gauss' theorem, we also see that the integral over all space $\int P d^3x$ is a constant of the motion which can be set to unity by appropriately normalizing ψ.

If one is to construct a relativistic quantum mechanics in analogy with non-relativistic quantum mechanics, it may appear natural to impose the following requirements on the theory. First, with the relativistic wave function we must be able to construct bilinear forms which can be interpreted as the probability density and the flux density satisfying a continuity equation of type (3.3). The probability density we form must, of course, be positive definite. In addition, the special theory of relativity requires that P must be the fourth component of a four-vector density. To see this last point, we recall that $d^3x \to d^3x\sqrt{1 - (v/c)^2}$ under a Lorentz transformation because of the well-known Lorentz contraction of the volume element; if Pd^3x is to remain invariant, it is essential that P transforms like the fourth component of a four-vector $P \to P/\sqrt{1 - (v/c)^2}$. The continuity equation (3.3) takes the following covariant form,

$$(\partial/\partial x_\mu)s_\mu = 0, \tag{3.4}$$

where

$$s_\mu = (\mathbf{S}, icP). \tag{3.5}$$

Let us now see whether the relativistic quantum mechanics based on the Klein-Gordon equation satisfies the above requirements. Consider a four-vector density given by

$$s_\mu = A\left(\phi^* \frac{\partial \phi}{\partial x_\mu} - \frac{\partial \phi^*}{\partial x_\mu}\phi\right), \tag{3.6}$$

where ϕ is a solution to the free particle Klein-Gordon equation, and A is a multiplicative constant. The four-divergence of (3.6) vanishes,

$$\frac{\partial s_\mu}{\partial x_\mu} = A\left[\frac{\partial \phi^*}{\partial x_\mu}\frac{\partial \phi}{\partial x_\mu} - (\Box\phi^*)\phi + \phi^*\Box\phi - \frac{\partial \phi^*}{\partial x_\mu}\frac{\partial \phi}{\partial x_\mu}\right] = 0, \tag{3.7}$$

by virtue of the Klein-Gordon equation. For a Klein-Gordon particle moving at nonrelativistic velocities ($E \approx mc^2$),

$$\phi \sim \psi e^{-imc^2 t/\hbar}, \tag{3.8}$$

where ψ is the corresponding Schrödinger solution (cf. Problem 1-2). The components of s_μ are then given by

$$s_0 = -is_4 \approx (2imc/\hbar)A|\psi|^2, \qquad \mathbf{s} = A[\psi^*\nabla\psi - (\nabla\psi^*)\psi]. \tag{3.9}$$

If we set $A = -i\hbar/2m$, then $\mathbf{s}$ and s_0 are precisely the flux density and c times the probability density in the Schrödinger theory. Thus we obtain a four-vector current density from a solution of the Klein-Gordon equation with the following properties: (i) the current density satisfies the continuity equation, and (ii) the components of the current density coincide with the flux density and c times the probability density in the nonrelativistic limit.

So far everything appears satisfactory. There is, however, a difficulty in interpreting

$$P = \frac{i\hbar}{2mc^2}\left(\phi^*\frac{\partial \phi}{\partial t} - \frac{\partial \phi^*}{\partial t}\phi\right) \tag{3.10}$$

as the probability density. In the Schrödinger theory, in which the time derivative appears only linearly in the wave equation, the sign of the frequency is determined by the eigenvalue of the Hamiltonian operator. In contrast, because the Klein-Gordon equation is of second order in the time derivative, both $u(\mathbf{x})e^{-iEt/\hbar}$ and $u^*(\mathbf{x})e^{+iEt/\hbar}$ are equally good solutions for a given physical situation (cf. Problem 1-3). This means that P given by (3.10) can be positive or negative. We may arbitrarily omit all solutions of the form $u(\mathbf{x})e^{-iEt/\hbar}$ with $E < 0$. But this would be unjustified because solutions of the form $u(\mathbf{x})e^{-iEt/\hbar}$ with $E > 0$ *alone* do not form a complete set. It appears that we must either abandon the interpretation of (3.10) as the probability density or abandon the Klein-Gordon equation altogether.

Let us analyze the origin of this difficulty a little more closely. From the derivation of the continuity equation (3.7) we may infer that the appearance of the linear time derivative in s_0 is unavoidable so long as the wave function satisfies a partial differential equation *quadratic* in the time derivative. Perhaps we could avoid this difficulty if we wrote a relativistic wave equation *linear* in the time derivative.

In 1928, in what is undoubtedly one of the most significant papers in the physics of the twentieth century, P.A.M. Dirac succeeded in devising a relativistic wave equation starting with the requirement that the wave equation be linear in $\partial/\partial t$. Using his equation, known to us as the Dirac equation, he was able to construct a conserved four-vector density whose zeroth component is positive-definite. For this reason, from 1928 until 1934 the Dirac equation was considered to be the only correct wave equation in relativistic quantum mechanics.

In 1934 the Klein-Gordon equation was revived by W. Pauli and V. F. Weisskopf. Their proposal was that, up to a proportionality factor, s_μ given by (3.6) be interpreted as the charge-current density rather than as the probability-current density. As we saw in Section 1-3, an interpretation of this kind is reasonable in the classical field theory of a complex scalar field. The fact that the sign of s_0 changes when $u^*(\mathbf{x})e^{-iEt/\hbar}$ is substituted for $u(\mathbf{x})e^{iEt/\hbar}$ makes good sense if the *negative-energy* solution is interpreted as the wave function for a particle with *opposite electric charge* (cf. Problem 1-3).

The interpretation of s_μ as the charge-current density is even more satisfactory for a theory in which a solution to the Klein-Gordon equation is to be interpreted as a quantized field operator. In analogy with what we did for the complex scalar field in classical field theory we form a non-Hermitian field operator $\phi(\neq \phi^\dagger)$ such that

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}, \qquad \phi^\dagger = \frac{\phi_1 - i\phi_2}{\sqrt{2}}, \tag{3.11}$$

where ϕ_1 and ϕ_2 are Hermitian operators whose properties are given in Problem 2-3. Consider now a four-current *operator*

$$j_\mu = e\left(\phi^\dagger\frac{\partial \phi}{\partial x_\mu} - \frac{\partial \phi^\dagger}{\partial x_\mu}\phi\right). \tag{3.12}$$

It is easy to show (Problem 3-1) that the fourth component of j_μ has the property

$$\int (j_4/ic)\,d^3x = e\sum_{\mathbf{k}}(N_{\mathbf{k}}^{(+)} - N_{\mathbf{k}}^{(-)}), \tag{3.13}$$

where

$$N^{(\pm)} = a_{\mathbf{k}\pm}^\dagger a_{\mathbf{k}\pm}, \tag{3.14}$$

with

$$a_{\mathbf{k}\pm} = \frac{1}{\sqrt{2}}(a_{\mathbf{k}}^{(1)} \pm ia_{\mathbf{k}}^{(2)}), \qquad a_{\mathbf{k}\pm}^\dagger = \frac{1}{\sqrt{2}}(a_{\mathbf{k}}^{(1)} \mp ia_{\mathbf{k}}^{(2)}). \tag{3.15}$$

Physically, $N^{(+)}$ and $N^{(-)}$ are the number operators for the Klein-Gordon particle of charge e and for its antiparticle with charge $-e$. So the eigenvalue of the operator expression (3.13) is the total charge of the field. Usually the four-vector j_μ is regarded as the charge-current density operator.

As emphasized in Chapter 2, quantum field theory accommodates physical situations in which particles are created or annihilated. When there are processes like $\gamma \longrightarrow \pi^+ + \pi^-$ which take place in the Coulomb field of a nucleus, what is conserved is not the probability of finding a given particle integrated over all space but rather the total charge of the field given by the eigenvalue of (3.13).

Coming back to the original argument against the Klein-Gordon equation, we see that if we are to reject the Klein-Gordon equation on the ground that we cannot form a positive-definite probability density, we might as well give up the Maxwell theory which, as the reader may verify, cannot accommodate any conserved four-vector density bilinear in the electromagnetic field.

3–2. THE DIRAC EQUATION

Derivation of the Dirac equation. Even though the Klein-Gordon equation is quite satisfactory when properly interpreted, there is reason for rejecting it for the description of an electron. The Klein-Gordon equation cannot accommodate the spin-½ nature of the electron as naturally as the Dirac equation can. In this connection, let us first study how to incorporate the electron spin in nonrelativistic quantum mechanics.

In nonrelativistic quantum mechanics, in order to account for the interaction of the electron spin magnetic moment with the magnetic field, it is customary to add a term

$$H^{(\text{spin})} = -(e\hbar/2mc)\boldsymbol{\sigma}\cdot\mathbf{B} \tag{3.16}$$

to the usual Hamiltonian, as done originally by W. Pauli. This procedure appears somewhat artificial, especially if we subscribe to the philosophy that the only "fundamental" electromagnetic interactions are those which can be generated by the substitution $p_\mu \longrightarrow p_\mu - eA_\mu/c$. There is, however, a slightly less *ad hoc* way of introducing the spin magnetic moment interaction. In the usual wave-mechanical treatment of the electron, the kinetic energy operator in the absence of the vector potential is taken to be

$$H^{(\text{KE})} = \mathbf{p}^2/2m. \tag{3.17}$$

However, for a spin-½ particle we may just as well start with the expression

$$H^{(\text{KE})} = (\boldsymbol{\sigma}\cdot\mathbf{p})(\boldsymbol{\sigma}\cdot\mathbf{p})/2m. \tag{3.18}$$

This alternative form is indistinguishable from (3.17) for all practical purposes when there is no vector potential.‡ There is, however, a difference when we make the substitution $\mathbf{p} \rightarrow \mathbf{p} - e\mathbf{A}/c$. The expression (3.18) then becomes

$$\begin{aligned}\frac{1}{2m}\boldsymbol{\sigma}\cdot\left(\mathbf{p}-\frac{e\mathbf{A}}{c}\right)\boldsymbol{\sigma}\cdot\left(\mathbf{p}-\frac{e\mathbf{A}}{c}\right)&\\ = \frac{1}{2m}\left(\mathbf{p}-\frac{e\mathbf{A}}{c}\right)^2 + \frac{i}{2m}\boldsymbol{\sigma}\cdot\left[\left(\mathbf{p}-\frac{e\mathbf{A}}{c}\right)\times\left(\mathbf{p}-\frac{e\mathbf{A}}{c}\right)\right]&\\ = \frac{1}{2m}\left(\mathbf{p}-\frac{e\mathbf{A}}{c}\right)^2 - \frac{e\hbar}{2mc}\boldsymbol{\sigma}\cdot\mathbf{B},&\end{aligned} \tag{3.19}$$

where we have used

$$\mathbf{p}\times\mathbf{A} = -i\hbar(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\times\mathbf{p}. \tag{3.20}$$

(The operator $\mathbf{p}$ is assumed to act on everything that stands to the right; in contrast,

‡We recall that the formula $(\boldsymbol{\sigma}\cdot\mathbf{A})(\boldsymbol{\sigma}\cdot\mathbf{B}) = \mathbf{A}\cdot\mathbf{B} + i\boldsymbol{\sigma}\cdot(\mathbf{A}\times\mathbf{B})$ holds even if $\mathbf{A}$ and $\mathbf{B}$ are operators.

the $\boldsymbol{\nabla}$ operator in (3.20) acts only on $\mathbf{A}$.) Note that the spin magnetic moment generated in this way has the correct gyromagnetic ratio $g = 2$.‡

Our object is to derive a relativistic wave equation for a spin-½ particle. Just as we incorporated the electron spin into the nonrelativistic theory by using the kinetic energy operator (3.18), we can incorporate the electron spin into the general framework of relativistic quantum mechanics by taking the operator analog of the classical expression

$$(E^2/c^2) - \mathbf{p}^2 = (mc)^2 \tag{3.21}$$

to be

$$\left(\frac{E^{(\text{op})}}{c} - \boldsymbol{\sigma}\cdot\mathbf{p}\right)\left(\frac{E^{(\text{op})}}{c} + \boldsymbol{\sigma}\cdot\mathbf{p}\right) = (mc)^2, \tag{3.22}$$

where

$$E^{(\text{op})} = i\hbar\frac{\partial}{\partial t} = i\hbar c\frac{\partial}{\partial x_0}, \tag{3.23}$$

and $\mathbf{p} = -i\hbar\boldsymbol{\nabla}$ as before. This enables us to write a second-order equation (due to B. L. van der Waerden),

$$\left(i\hbar\frac{\partial}{\partial x_0} + \boldsymbol{\sigma}\cdot i\hbar\boldsymbol{\nabla}\right)\left(i\hbar\frac{\partial}{\partial x_0} - \boldsymbol{\sigma}\cdot i\hbar\boldsymbol{\nabla}\right)\phi = (mc)^2\phi, \tag{3.24}$$

for a free electron, where ϕ is now a *two-component* wave function.

We are interested in obtaining a wave equation of first order in the time derivative. Relativistic covariance suggests that the wave equation linear in $\partial/\partial t$ must be linear in $\boldsymbol{\nabla}$ also. An analogy with the Maxwell theory may now be helpful. The free-field D'Alembertian equation $\Box A_\mu = 0$ is a second-order equation, while the free-field Maxwell equation $(\partial/\partial x_\mu)\,F_{\mu\nu} = 0$ is a first-order equation. Note that $F_{\mu\nu}$ obtained by differentiating A_μ has more components than A_μ. This increase in the number of components is the price we have to pay when we work with the first-order equation.

Motivated by this analogy, we can define two *two-component* wave functions $\phi^{(R)}$ and $\phi^{(L)}$:

$$\phi^{(R)} = \frac{1}{mc}\left(i\hbar\frac{\partial}{\partial x_0} - i\hbar\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\right)\phi, \qquad \phi^{(L)} = \phi. \tag{3.25}$$

The total number of components has now been increased to four. The superscripts R and L come from the fact that as $m \longrightarrow 0$, $\phi^{(R)}$ and $\phi^{(L)}$, respectively, describe a right-handed (spin *parallel* to the momentum direction) and a left-handed (spin *antiparallel* to the momentum direction) state of the spin-½ particle, as we shall see later. The second-order equation (3.24) is now equivalent to two first-order equations

$$\begin{aligned}[i\hbar\boldsymbol{\sigma}\cdot\boldsymbol{\nabla} - i\hbar(\partial/\partial x_0)]\phi^{(L)} &= -mc\phi^{(R)},\\ [-i\hbar\boldsymbol{\sigma}\cdot\boldsymbol{\nabla} - i\hbar(\partial/\partial x_0)]\phi^{(R)} &= -mc\phi^{(L)}.\end{aligned} \tag{3.26}$$

‡Historically, all this was first obtained by working out the nonrelativistic limit of the Dirac theory, as we shall show in the next section. For this reason, most textbooks state that the $g = 2$ relation is a consequence of the Dirac theory. We have seen, however, that the $g = 2$ relation follows just as naturally from the nonrelativistic Schrödinger-Pauli theory if we start with the kinetic energy operator (3.18). This point was emphasized particularly by R. P. Feynman.

Note that unless the particle is massless, these first-order equations couple $\phi^{(R)}$ and $\phi^{(L)}$ just as the Maxwell equations, also first-order equations, couple **E** and **B**.

Equation (3.26) is equivalent to the celebrated wave equation of Dirac (cf. Problem 3–5). To bring it to the form originally written by Dirac, we take the sum and the difference of (3.26). We then have

$$\begin{aligned} -i\hbar(\boldsymbol{\sigma}\cdot\nabla)(\phi^{(R)} - \phi^{(L)}) - i\hbar(\partial/\partial x_0)(\phi^{(L)} + \phi^{(R)}) &= -mc(\phi^{(L)} + \phi^{(R)}), \\ i\hbar(\boldsymbol{\sigma}\cdot\nabla)(\phi^{(L)} + \phi^{(R)}) + i\hbar(\partial/\partial x_0)(\phi^{(R)} - \phi^{(L)}) &= -mc(\phi^{(R)} - \phi^{(L)}), \end{aligned} \tag{3.27}$$

or, denoting the sum and the difference of $\phi^{(R)}$ and $\phi^{(L)}$ by ψ_A and ψ_B, we have

$$\begin{pmatrix} -i\hbar(\partial/\partial x_0) & -i\hbar\boldsymbol{\sigma}\cdot\nabla \\ i\hbar\boldsymbol{\sigma}\cdot\nabla & i\hbar(\partial/\partial x_0) \end{pmatrix}\begin{pmatrix}\psi_A \\ \psi_B\end{pmatrix} = -mc\begin{pmatrix}\psi_A \\ \psi_B\end{pmatrix}. \tag{3.28}$$

Defining a *four*-component wave function ψ by

$$\psi = \begin{pmatrix}\psi_A \\ \psi_B\end{pmatrix} = \begin{pmatrix}\phi^{(R)} + \phi^{(L)} \\ \phi^{(R)} - \phi^{(L)}\end{pmatrix}, \tag{3.29}$$

we can rewrite (3.28) more concisely as

$$\left(\boldsymbol{\gamma}\cdot\nabla + \gamma_4\frac{\partial}{\partial(ix_0)}\right)\psi + \frac{mc}{\hbar}\psi = 0, \tag{3.30}$$

or

$$\left(\gamma_\mu\frac{\partial}{\partial x_\mu} + \frac{mc}{\hbar}\right)\psi = 0, \tag{3.31}$$

where γ_μ with $\mu = 1, 2, 3, 4$ are 4×4 matrices given by

$$\gamma_k = \begin{pmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{pmatrix}, \qquad \gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{3.32}$$

which really mean

$$\gamma_3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \qquad \gamma_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \text{etc.}‡ \tag{3.33}$$

Equation (3.31) is the famous Dirac equation.§

‡We use the standard form of the 2×2 Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The symbol I stands for the 2×2 identity matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

§In deriving the Dirac equation we have not followed the path originally taken by Dirac. The first published account of the derivation given here is found in B. L. van der Waerden's 1932 book on group theory. The original approach of Dirac can be found in many books, for example, Rose (1961), pp. 39–44; Bjorken and Drell (1964), pp. 6–8.

We wish to emphasize that (3.31) is actually four differential equations that couple the four components of ψ represented by a single-column matrix

$$\psi = \begin{pmatrix}\psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4\end{pmatrix}. \tag{3.34}$$

A four-component object of this kind is known as a bispinor or, more commonly, as a *Dirac spinor*. If there is any confusion as to the real meaning of (3.31), the reader may write the matrix indices explicitly as follows:

$$\sum_{\beta=1}^{4}\left[\sum_{\mu=1}^{4}(\gamma_\mu)_{\alpha\beta}\frac{\partial}{\partial x_\mu} + \left(\frac{mc}{\hbar}\right)\delta_{\alpha\beta}\right]\psi_\beta = 0. \tag{3.35}$$

The fact that ψ has four components has nothing to do with the four-dimensional nature of space-time; ψ_β does *not* transform like a four-vector under a Lorentz transformation, as we shall see in Section 3–4.

The 4×4 matrices γ_μ we introduced are called the *gamma matrices* or the *Dirac matrices*. They satisfy the following anticommutation relations as we may readily verify:

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}. \tag{3.36a}$$

For instance,

$$\gamma_4^2 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}^2 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

$$\begin{aligned} \gamma_1\gamma_2 + \gamma_2\gamma_1 &= \begin{pmatrix} 0 & -i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \sigma_1\sigma_2 + \sigma_2\sigma_1 & 0 \\ 0 & \sigma_1\sigma_2 + \sigma_2\sigma_1 \end{pmatrix} = 0. \end{aligned} \tag{3.36b}$$

Moreover, each γ_μ is seen to be Hermitian,

$$\gamma_\mu^\dagger = \gamma_\mu, \tag{3.37}$$

and traceless.‡

Multiplying (3.30) by γ_4, we see that the Dirac equation can also be written in the Hamiltonian form

$$H\psi = i\hbar(\partial\psi/\partial t), \tag{3.38}$$

where

$$H = -ic\hbar\boldsymbol{\alpha}\cdot\nabla + \beta mc^2, \tag{3.39}$$

with

$$\beta = \gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \alpha_k = i\gamma_4\gamma_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}. \tag{3.40}$$

‡In the literature some people define gamma matrices that do not quite satisfy (3.36a) and (3.37). This is deplorable. Our notation agrees with that of Dirac's original paper and Pauli's *Handbuch* article. The alternative notation sometimes found in the literature is summarized in Appendix B.

The matrices $\boldsymbol{\alpha}$ and β satisfy‡

$$\{\alpha_k, \beta\} = 0, \qquad \beta^2 = 1, \qquad \{\alpha_k, \alpha_l\} = 2\delta_{kl}. \tag{3.41}$$

In Section 3–6, we shall make extensive use of the Hamiltonian formalism based on (3.39).

Conserved current. We shall now derive the differential law of current conservation. First let us define

$$\bar{\psi} = \psi^\dagger \gamma_4, \tag{3.42}$$

where $\bar{\psi}$ is called the adjoint spinor, as distinguished from the Hermitian conjugate spinor $\psi^\dagger$. Explicitly $\bar{\psi}$ as well as $\psi^\dagger$ can be represented by single-row matrices, that is, if

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \tag{3.43}$$

then

$$\begin{aligned} \psi^\dagger &= (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*), \\ \bar{\psi} &= (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*). \end{aligned} \tag{3.44}$$

To obtain the wave equation for $\bar{\psi}$ we start with the Hermitian conjugate of the Dirac equation,

$$\frac{\partial}{\partial x_k}\psi^\dagger \gamma_k + \frac{\partial}{\partial x_4^*}\psi^\dagger \gamma_4 + \frac{mc}{\hbar}\psi^\dagger = 0. \tag{3.45}$$

Multiplying (3.45) by γ_4 from the right, we obtain the *adjoint equation*

$$-\frac{\partial}{\partial x_\mu}\bar{\psi}\gamma_\mu + \frac{mc}{\hbar}\bar{\psi} = 0, \tag{3.46}$$

where we have used

$$\frac{\partial}{\partial x_4^*} = \frac{\partial}{\partial (ict)^*} = -\frac{\partial}{\partial x_4}, \tag{3.47}$$

and $\gamma_k\gamma_4 = -\gamma_4\gamma_k$. We now multiply the original Dirac equation (3.31) from the left by $\bar{\psi}$, the adjoint equation (3.46) from the right by ψ, and subtract. Then

$$(\partial/\partial x_\mu)(\bar{\psi}\gamma_\mu\psi) = 0. \tag{3.48}$$

Thus we see that

$$s_\mu = ic\bar{\psi}\gamma_\mu\psi = (c\psi^\dagger \boldsymbol{\alpha}\psi, ic\psi^\dagger\psi) \tag{3.49}$$

‡In addition to γ_μ, α_k, and β, one also finds in the literature

$$\rho_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \rho_2 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \qquad \text{and} \qquad \rho_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

We shall, however, not use the rho matrices in this book.

satisfies the continuity equation.‡ Using Green's theorem we see that

$$\int \bar{\psi}\gamma_4\psi d^3x = \int \psi^\dagger\psi d^3x = \text{const}, \tag{3.50}$$

which may be set to unity by appropriately normalizing ψ. Now, unlike (3.9),

$$\bar{\psi}\gamma_4\psi = \psi^\dagger\psi = \sum_{\beta=1}^{4} \psi_\beta^*\psi_\beta \tag{3.51}$$

is necessarily positive-definite. So we may be tempted to identify $\bar{\psi}\gamma_4\psi = \psi^\dagger\psi$ with the probability density as we did in the Schrödinger theory. With this interpretation,

$$s_k = ic\bar{\psi}\gamma_k\psi = c\psi^\dagger\alpha_k\psi \tag{3.52}$$

is identified with the flux density. For the covariance of the continuity equation, we must, of course, prove rigorously that s_μ indeed transforms like a four-vector; this will be done in Section 3–4.

For the time being, we assume that a solution ψ to the Dirac equation is a single-particle wave function subject to the above interpretation of $\psi^\dagger\psi$ as the probability density. Toward the end of this chapter, however, we shall point out some of the difficulties associated with this interpretation. For a more satisfactory interpretation of the Dirac theory, it is necessary to quantize the Dirac field. This we shall do in Section 3–10.

Representation independence. In concluding this section we shall mention briefly the notion of the representation independence of the Dirac equation. Suppose somebody writes

$$\left(\gamma'_\mu \frac{\partial}{\partial x_\mu} + \frac{mc}{\hbar}\right)\psi' = 0, \tag{3.53}$$

where the only defining property of γ'_μ is that γ'_μ with $\mu = 1 \ldots 4$ forms a set of 4×4 matrices satisfying

$$\{\gamma'_\mu, \gamma'_\nu\} = 2\delta_{\mu\nu}. \tag{3.54}$$

Our assertion is that this equation is equivalent to the Dirac equation (3.31) where the matrices γ_μ are explicitly given by (3.32). Note that we do not assert that for a given physical situation ψ and ψ' are the same. Different sets of 4×4 matrices satisfying (3.54) are referred to as sets of the gamma matrices in different representations.

We can prove the representation independence of the Dirac equation by appealing to what is known as Pauli's fundamental theorem: Given two sets of 4×4 matrices satisfying

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \qquad \text{and} \qquad \{\gamma'_\mu, \gamma'_\nu\} = 2\delta_{\mu\nu}$$

‡All this can be readily proved even in the presence of the electromagnetic interaction generated by the substitution

$$-i\hbar\frac{\partial}{\partial x_\mu} \longrightarrow -i\hbar\frac{\partial}{\partial x_\mu} - \frac{eA_\mu}{c}$$

in (3.31).

with $\mu, \nu = 1 \ldots 4$, there exists a nonsingular 4×4 matrix S such that

$$S\gamma_\mu S^{-1} = \gamma'_\mu. \tag{3.55}$$

Moreover, S is unique up to a multiplicative constant. The proof of the theorem can be found in Appendix C. Assuming the validity of Pauli's theorem, we rewrite (3.53) as

$$\left(S\gamma_\mu S^{-1}\frac{\partial}{\partial x_\mu} + \frac{mc}{\hbar}\right) SS^{-1}\psi' = 0, \tag{3.56}$$

where S is the matrix that relates the set $\{\gamma'_\mu\}$ and the set $\{\gamma_\mu\}$ via a relation of the form (3.55). Multiplying by S^{-1} from the left, we obtain

$$\left(\gamma_\mu\frac{\partial}{\partial x_\mu} + \frac{mc}{\hbar}\right) S^{-1}\psi' = 0. \tag{3.57}$$

This is the same as the original Dirac equation with $S^{-1}\psi'$ as solution. In other words, (3.53) is equivalent to the Dirac equation, (3.31), and the wave functions ψ' and ψ are related by

$$\psi' = S\psi. \tag{3.58}$$

Let us consider the case where the γ'_μ are also Hermitian. By taking the Hermitian conjugate of (3.55) we see that S can be chosen to be unitary $S^\dagger = S^{-1}$. With a unitary S we see that expressions like the probability density and the flux density are the same:

$$\begin{aligned}\bar{\psi}'\gamma'_\mu\psi' &= \psi'^\dagger\gamma'_4\gamma'_\mu\psi' \\ &= \psi^\dagger S^\dagger S\gamma_4 S^{-1}S\gamma_\mu S^{-1}S\psi \\ &= \bar{\psi}\gamma_\mu\psi.\end{aligned} \tag{3.59}$$

Evidently all the physical consequences are the same regardless of whether we use (3.31) or (3.53). Note, however, that the wave functions for the same physical situation *look* different when different representations are used.‡

In practice three representations of the gamma matrices are found in the literature:

a) The standard (Dirac-Pauli) representation given explicitly by (3.32).

b) The Weyl representation in which not only γ_k but also γ_4 are off-diagonal (cf. Problem 3–5).

‡Actually we are already familiar with an analogous situation in the Pauli two-component theory. In nonrelativistic quantum mechanics it is customary to use the standard representation of the σ matrices in which σ_3 is diagonal. From the point of view of the commutation relations, however, one may as well work with what we may call "a nonconformist's representation,"

$$\sigma_1 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

The spin-up spinor (the spin in the positive z-direction) is then given by

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} \quad \text{rather than by} \quad \begin{pmatrix}1\\ 0\end{pmatrix}.$$

c) The Majorana representation in which the γ_k are purely real and γ_4 is purely imaginary, hence $\gamma_\mu(\partial/\partial x_\mu)$ is purely real.

In this book, whenever explicit forms of γ_μ or ψ are called for, only the standard (Dirac-Pauli) representation is used.

3–3. SIMPLE SOLUTIONS; NONRELATIVISTIC APPROXIMATIONS; PLANE WAVES

Large and small components. Before we study the behavior of Dirac's wave function ψ under Lorentz transformations, let us examine the kind of physics buried in the harmless-looking equation (3.31).

In the presence of electromagnetic couplings, the Dirac equation reads

$$\left(\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c}A_\mu\right)\gamma_\mu\psi + \frac{mc}{\hbar}\psi = 0, \tag{3.60}$$

where the usual replacement $-i\hbar(\partial/\partial x_\mu) \longrightarrow -i\hbar(\partial/\partial x_\mu) - eA_\mu/c$ is assumed to be valid. Assuming that A_μ is time independent, we let the time dependence of ψ be given by

$$\psi = \psi(\mathbf{x}, t)|_{t=0}e^{-iEt/\hbar} \tag{3.61}$$

(which, of course, means that ψ is an eigenfunction of $i\hbar\,\partial/\partial t$ with eigenvalue E). We can then write the coupled equations for the upper and lower two components, ψ_A and ψ_B, as follows (Eq. 3.28):

$$\begin{aligned}\left[\boldsymbol{\sigma}\cdot\left(\mathbf{p} - \frac{e\mathbf{A}}{c}\right)\right]\psi_B &= \frac{1}{c}(E - eA_0 - mc^2)\psi_A, \\ -\left[\boldsymbol{\sigma}\cdot\left(\mathbf{p} - \frac{e\mathbf{A}}{c}\right)\right]\psi_A &= -\frac{1}{c}(E - eA_0 + mc^2)\psi_B,\end{aligned} \tag{3.62}$$

where $A_\mu = (\mathbf{A}, iA_0)$ as before. Using the second equation, we can readily eliminate ψ_B in the first equation to obtain

$$\left[\boldsymbol{\sigma}\cdot\left(\mathbf{p} - \frac{e\mathbf{A}}{c}\right)\right]\left[\frac{c^2}{E - eA_0 + mc^2}\right]\left[\boldsymbol{\sigma}\cdot\left(\mathbf{p} - \frac{e\mathbf{A}}{c}\right)\right]\psi_A = (E - eA_0 - mc^2)\psi_A. \tag{3.63}$$

Up to now we have made no approximations. We now assume that

$$E \approx mc^2, \qquad |eA_0| \ll mc^2. \tag{3.64}$$

Defining the energy measured from mc^2 by

$$E^{(\mathrm{NR})} = E - mc^2, \tag{3.65}$$

we can make the following expansion:

$$\frac{c^2}{E - eA_0 + mc^2} = \frac{1}{2m}\left[\frac{2mc^2}{2mc^2 + E^{(\mathrm{NR})} - eA_0}\right] = \frac{1}{2m}\left[1 - \frac{E^{(\mathrm{NR})} - eA_0}{2mc^2} + \cdots\right]. \tag{3.66}$$

This can be regarded as an expansion in powers of $(v/c)^2$ since $E^{(\mathrm{NR})} - eA_0$ is roughly $[\mathbf{p} - (e\mathbf{A}/c)]^2/2m \approx mv^2/2$. Keeping only the leading term in (3.66), we

obtain

$$\frac{1}{2m}\boldsymbol{\sigma}\cdot\left(\mathbf{p}-\frac{e\mathbf{A}}{c}\right)\boldsymbol{\sigma}\cdot\left(\mathbf{p}-\frac{e\mathbf{A}}{c}\right)\psi_A=(E^{(\mathrm{NR})}-eA_0)\psi_A, \tag{3.67}$$

which, as we have already seen (cf. Eq. 3.19), becomes

$$\left[\frac{1}{2m}\left(\mathbf{p}-\frac{e\mathbf{A}}{c}\right)^2-\frac{e\hbar}{2mc}\boldsymbol{\sigma}\cdot\mathbf{B}+eA_0\right]\psi_A=E^{(\mathrm{NR})}\psi_A. \tag{3.68}$$

Thus to zeroth order in $(v/c)^2$, ψ_A is nothing more than the Schrödinger-Pauli two-component wave function in nonrelativistic quantum mechanics multiplied by $e^{-imc^2t/\hbar}$. Using the second expression of (3.62), we see that ψ_B is "smaller" than ψ_A by a factor of roughly $|\mathbf{p}-e(\mathbf{A}/c)|/2mc\approx v/2c$, provided that (3.64) is valid. For this reason with $E\sim mc^2$, ψ_A and ψ_B are respectively known as the *large* and *small components* of the Dirac wave function ψ.

Approximate Hamiltonian for an electrostatic problem. We shall now study the consequences of *keeping* the second term in (3.66). For simplicity, let us treat the case $\mathbf{A}=0$. The equation we must work with is

$$H_A^{(\mathrm{NR})}\psi_A=E^{(\mathrm{NR})}\psi_A, \tag{3.69}$$

where

$$H_A^{(\mathrm{NR})}=(\boldsymbol{\sigma}\cdot\mathbf{p})\frac{1}{2m}\left(1-\frac{E^{(\mathrm{NR})}-eA_0}{2mc^2}\right)(\boldsymbol{\sigma}\cdot\mathbf{p})+eA_0. \tag{3.70}$$

At first sight it might appear that we can regard (3.69) as the time-independent Schrödinger equation for ψ_A. There are, however, three difficulties with this interpretation. First, if we are working to order $(v/c)^2$, ψ_A no longer satisfies the normalization requirement because the probabilistic interpretation of the Dirac theory requires that

$$\int(\psi_A^\dagger\psi_A+\psi_B^\dagger\psi_B)\,d^3x=1, \tag{3.71}$$

where ψ_B is already of the order of v/c. Second, by expanding (3.70) it is easy to see that $H_A^{(\mathrm{NR})}$ contains a non-Hermitian term $i\hbar\mathbf{E}\cdot\mathbf{p}$. Third, (3.69) is not an eigenvalue equation for the energy since $H_A^{(\mathrm{NR})}$ contains $E^{(\mathrm{NR})}$ itself.

To overcome these difficulties, let us first note that the normalization requirement (3.71) can be written as

$$\int\psi_A^\dagger\left(1+\frac{\mathbf{p}^2}{4m^2c^2}\right)\psi_A\,d^3x\approx 1 \tag{3.72}$$

to order $(v/c)^2$ since, according to the second expression of (3.62),

$$\psi_B\approx\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{2mc}\psi_A. \tag{3.73}$$

This suggests that we should work with a new *two-component* wave function Ψ defined by

$$\Psi=\Omega\psi_A, \tag{3.74}$$

where

$$\Omega=1+(\mathbf{p}^2/8m^2c^2). \tag{3.75}$$

With this choice, Ψ is normalized to order $(v/c)^2$ since

$$\int\Psi^\dagger\Psi\,d^3x\approx\int\psi_A^\dagger[1+(\mathbf{p}^2/4m^2c^2)]\psi_A\,d^3x, \tag{3.76}$$

where we have used (3.71). Multiplying (3.69) from the left by $\Omega^{-1}=1-(\mathbf{p}^2/8m^2c^2)$, we obtain

$$\Omega^{-1}H_A^{(\mathrm{NR})}\Omega^{-1}\Psi=E^{(\mathrm{NR})}\Omega^{-2}\Psi. \tag{3.77}$$

Explicitly, to order $(v/c)^2$ we have

$$\left[\frac{\mathbf{p}^2}{2m}+eA_0-\left\{\frac{\mathbf{p}^2}{8m^2c^2},\left(\frac{\mathbf{p}^2}{2m}+eA_0\right)\right\}-\frac{(\boldsymbol{\sigma}\cdot\mathbf{p})}{2m}\left(\frac{E^{(\mathrm{NR})}-eA_0}{2mc^2}\right)(\boldsymbol{\sigma}\cdot\mathbf{p})\right]\Psi$$
$$=E^{(\mathrm{NR})}\left(1-\frac{\mathbf{p}^2}{4m^2c^2}\right)\Psi; \tag{3.78}$$

or, writing $E^{(\mathrm{NR})}\,\mathbf{p}^2$ as $\frac{1}{2}\{E^{(\mathrm{NR})},\mathbf{p}^2\}$, we have

$$\left[\frac{\mathbf{p}^2}{2m}+eA_0-\frac{\mathbf{p}^4}{8m^3c^2}+\frac{1}{8m^2c^2}(\{\mathbf{p}^2,(E^{(\mathrm{NR})}-eA_0)\}-2(\boldsymbol{\sigma}\cdot\mathbf{p})(E^{(\mathrm{NR})}-eA_0)(\boldsymbol{\sigma}\cdot\mathbf{p}))\right]\Psi$$
$$=E^{(\mathrm{NR})}\Psi. \tag{3.79}$$

In general, for any pair of operators A and B, we have

$$\{A^2,B\}-2ABA=[A,[A,B]]. \tag{3.80}$$

This very useful formula can be employed to simplify (3.79) where we set $\boldsymbol{\sigma}\cdot\mathbf{p}=A$ and $E^{(\mathrm{NR})}-eA_0=B$. Using

$$[\boldsymbol{\sigma}\cdot\mathbf{p},(E^{(\mathrm{NR})}-eA_0)]=-ie\hbar\boldsymbol{\sigma}\cdot\mathbf{E}, \tag{3.81}$$

and

$$[\boldsymbol{\sigma}\cdot\mathbf{p},-ie\hbar\boldsymbol{\sigma}\cdot\mathbf{E}]=-e\hbar^2\nabla\cdot\mathbf{E}-2e\hbar\boldsymbol{\sigma}\cdot(\mathbf{E}\times\mathbf{p}), \tag{3.82}$$

both of which are valid since $\nabla A_0=-\mathbf{E}$ and $\nabla\times\mathbf{E}=0$, we finally obtain

$$\left[\frac{\mathbf{p}^2}{2m}+eA_0-\frac{\mathbf{p}^4}{8m^3c^2}-\frac{e\hbar\boldsymbol{\sigma}\cdot(\mathbf{E}\times\mathbf{p})}{4m^2c^2}-\frac{e\hbar^2}{8m^2c^2}\nabla\cdot\mathbf{E}\right]\Psi=E^{(\mathrm{NR})}\Psi. \tag{3.83}$$

This equation is free of the difficulties mentioned earlier and can therefore be regarded as the Schrödinger equation for the two-component wave function.

The physical significance of each term in (3.83) will now be given. The first two need no explanation. The third term is due to the relativistic correction to the kinetic energy as seen from the expansion

$$\sqrt{(mc^2)^2+|\mathbf{p}|^2c^2}-mc^2=|\mathbf{p}|^2/2m-(|\mathbf{p}|^4/8m^3c^2)+\cdots \tag{3.84}$$

The fourth term represents the spin interaction of the moving electron with the electric field. Crudely speaking, we say this arises because the moving electron "sees" an apparent magnetic field given by $\mathbf{E}\times(\mathbf{v}/c)$. Naively, we expect in this way $-(e\hbar/2mc)\,\boldsymbol{\sigma}\cdot[\mathbf{E}\times(\mathbf{v}/c)]$, which is just twice the fourth term of (3.83). That this argument is incomplete was shown within the framework of classical electrodynamics two years before the advent of the Dirac theory by L. H. Thomas, who argued that a more careful treatment which would take into account the energy associated with the precession of the electron spin would result in reduction by

a factor of two, in agreement with the fourth term of (3.83).‡ Hence the fourth term of (3.83) is called the Thomas term. For a central potential

$$eA_0 = V(r), \tag{3.85}$$

we obtain

$$-\frac{e\hbar}{4m^2c^2}\boldsymbol{\sigma}\cdot(\mathbf{E}\times\mathbf{p}) = -\frac{\hbar}{4m^2c^2}\left(-\frac{1}{r}\frac{dV}{dr}\right)\boldsymbol{\sigma}\cdot(\mathbf{x}\times\mathbf{p}) = \frac{1}{2m^2c^2}\frac{1}{r}\frac{dV}{dr}\mathbf{S}\cdot\mathbf{L} \tag{3.86}$$

with $\mathbf{S} = \hbar\boldsymbol{\sigma}/2$. Thus the well-known spin-orbit force in atomic physics, represented by (3.86), is an automatic consequence of the Dirac theory.

As for the last term of (3.83) we note that $\nabla\cdot\mathbf{E}$ is just the charge density. For the hydrogen atom where $\nabla\cdot\mathbf{E} = -e\delta^{(3)}(\mathbf{x})$, it gives rise to an energy shift

$$\int \frac{e^2\hbar^2}{8m^2c^2}\delta^{(3)}(\mathbf{x})\,|\psi^{(\text{Schrö})}|^2 d^3x = \frac{e^2\hbar^2}{8m^2c^2}\,|\psi^{(\text{Schrö})}_{(\mathbf{x})}|^2\Big|_{\mathbf{x}=0}, \tag{3.87}$$

which is nonvanishing only for the s states [in contrast to (3.86) which affects all but the s states]. The last term of (3.83) was first studied in detail by C. G. Darwin; hence it is called the Darwin term. We shall postpone the physical interpretations of the Darwin term until Section 3–7.

Using the third, fourth, and fifth terms of (3.83) as the perturbation Hamiltonian and the wave functions for the hydrogen atom in nonrelativistic quantum mechanics as the unperturbed wave functions, we can compute the lowest-order relativistic correction to the energy levels of the hydrogen atom. Since this calculation based on first-order time-independent perturbation theory is straightforward, we present only the results:§

$$\Delta E = -\left(\frac{e^2}{4\pi\hbar c}\right)^2\left(\frac{e^2}{8\pi a_0}\right)\frac{1}{n^3}\left(\frac{1}{j+\frac{1}{2}} - \frac{3}{4n}\right), \tag{3.88}$$

which is to be added to the unperturbed energy levels

$$E^{(0)} = (e^2/8\pi a_0 n^2). \tag{3.89}$$

The sum $E^{(0)} + \Delta E$ correctly describes the fine structure of the hydrogen atom to order $(\frac{1}{137})^2$ times the Rydberg energy $(e^2/8\pi a_0)$. Note that states with the same n but different j (for example, $2p_{1/2} - 2p_{3/2}$) are now split. On the other hand, states with the same n and the same j (for example, $2s_{1/2} - 2p_{1/2}$) are still degenerate. This degeneracy persists even in the exact treatment of the Coulomb potential problem, as we shall see in Section 3–8.‖

‡For the derivation of the Thomas factor in classical electrodynamics consult Jackson (1962), pp. 364–368.

§See, for example, Bethe and Salpeter (1957), pp. 59–61.

‖In practice it is of little interest to work out an expansion more accurate than (3.88) using only the Dirac equation with $V = -e^2/(4\pi r)$, because other effects such as the Lamb shift (discussed in the last chapter) and the hyperfine structure (to be discussed in Section 3–8) become more significant.

Free particles at rest. Let us now turn our attention to problems where exact solutions to the Dirac equation are possible. The simplest solvable problem deals with a free particle. We will first demonstrate that each component of the four-component wave function satisfies the Klein-Gordon equation if the particle is free. Although the validity of this statement is rather obvious if we go back to $\phi^{(L)}$ and $\phi^{(R)}$, we prove it by starting with the Dirac equation (3.31). Multiplying (3.31) from the left by $\gamma_\nu\,(\partial/\partial x_\nu)$, we have

$$\frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x_\mu}\gamma_\nu\gamma_\mu\psi - \left(\frac{mc}{\hbar}\right)^2\psi = 0. \tag{3.90}$$

Adding to (3.90) the same equation written in a form in which the summation indices μ and ν are interchanged, we obtain

$$\frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x_\mu}(\gamma_\nu\gamma_\mu + \gamma_\mu\gamma_\nu)\psi - 2\left(\frac{mc}{\hbar}\right)^2\psi = 0, \tag{3.91}$$

which reduces to

$$\Box\psi - (mc/\hbar)^2\psi = 0 \tag{3.92}$$

by virtue of the anticommutation relation of the gamma matrices (3.35). Note that (3.92) is to be understood as four separate *uncoupled* equations for each component of ψ. Because of (3.92), the Dirac equation admits a free-particle solution of the type

$$\psi \sim u(\mathbf{p})\exp\left[i(\mathbf{p}\cdot\mathbf{x}/\hbar) - i(Et/\hbar)\right] \tag{3.93}$$

with

$$E = \pm\sqrt{|\mathbf{p}|^2c^2 + m^2c^4}, \tag{3.94}$$

where $u(\mathbf{p})$ is a *four*-component spinor independent of $\mathbf{x}$ and t. Note that (3.93) is a simultaneous eigenfunction of $-i\hbar\nabla$ and $i\hbar\,(\partial/\partial t)$ with eigenvalues $\mathbf{p}$ and E, respectively.

For a particle at rest ($\mathbf{p} = 0$), the equation we must solve is

$$\gamma_4\frac{\partial}{\partial(ict)}\psi = -\frac{mc}{\hbar}\psi. \tag{3.95}$$

In accordance with (3.93) and (3.94), we first try the time-dependence $e^{-imc^2t/\hbar}$. Equation (3.95) then becomes

$$-\frac{imc^2}{ic\hbar}\begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix}\begin{pmatrix} u_A(0) \\ u_B(0)\end{pmatrix} = -\frac{mc}{\hbar}\begin{pmatrix} u_A(0) \\ u_B(0)\end{pmatrix}, \tag{3.96}$$

which is satisfied only if the lower two-component spinor $u_B(0)$ vanishes. But, using a similar argument, we see that (3.95) can be satisfied equally well by the time-dependence $e^{+imc^2t/\hbar}$ provided that the *upper* two-component spinor $u_A(0)$ vanishes. As in the Pauli theory, the nonvanishing two-component spinors can be taken as

$$\begin{pmatrix} 1 \\ 0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1\end{pmatrix}.$$

So there are four independent solutions to (3.95):

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}e^{-imc^2t/\hbar},\qquad \begin{pmatrix}0\\1\\0\\0\end{pmatrix}e^{-imc^2t/\hbar},$$
$$\begin{pmatrix}0\\0\\1\\0\end{pmatrix}e^{+imc^2t/\hbar},\qquad \begin{pmatrix}0\\0\\0\\1\end{pmatrix}e^{+imc^2t/\hbar}. \tag{3.97}$$

If we insist on the interpretation that $i\hbar(\partial/\partial t)$ is the Hamiltonian operator, the first two are "positive-energy" solutions while the last two are "negative-energy" solutions. Note that the eigenvalues of the Hamiltonian operator are $\pm mc^2$, depending on whether the eigenvalues of $\gamma_4 = \beta$ are ± 1; this also follows directly from the expression for the Hamiltonian (3.39). In Section 3–9 we shall show that the existence of negative-energy solutions is intimately related to the fact that the Dirac theory can accommodate a positron.

Earlier in this section we showed that in the nonrelativistic limit $E \approx mc^2$, the upper two-component spinor ψ_A coincides with the Schrödinger wave function apart from $e^{-imc^2t/\hbar}$. Therefore the first solution in (3.97) is the Dirac wave function for a particle at rest with spin *up*, since the eigenvalue of σ_3 is $+1$ when applied to $\begin{pmatrix}1\\0\end{pmatrix}$. This leads us to define a 4×4 matrix

$$\Sigma_3 = \frac{\gamma_1\gamma_2 - \gamma_2\gamma_1}{2i} = \begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix}, \tag{3.98}$$

whose eigenvalue is to be interpreted as the spin component in the positive z-direction in units of $\hbar/2$. This interpretation will be justified in later parts of this chapter where we shall show that

a) the operator $\Sigma_3/2$ is the infinitesimal generator of a rotation about the z-axis acting on the space-time independent part of the Dirac wave function; and that
b) for a central force problem the sum of $\mathbf{x} \times \mathbf{p}$ and $\hbar\boldsymbol{\Sigma}/2$ is indeed a constant of the motion, to be identified with the total angular momentum, where Σ_k is defined as it was in (3.98), that is,

$$\Sigma_k = \frac{\gamma_i\gamma_j - \gamma_j\gamma_i}{2i} = \begin{pmatrix}\sigma_k & 0\\ 0 & \sigma_k\end{pmatrix} \qquad (ijk) \text{ cyclic.}‡ \tag{3.99}$$

Evidently the solutions (3.97) are eigenfunctions of Σ_3 with eigenvalues $+1$ for the first and third and -1 for the second and fourth. Thus for each sign of the energy there are two independent solutions corresponding to two spin states, as expected from the spin-½ nature of the particle described by the Dirac equation.

‡The notation "(ijk) cyclic" stands for $(i, j, k) = (1, 2, 3)$, $(2, 3, 1)$, or $(3, 1, 2)$.

It has sometimes been stated that we need a $2 \times 2 = 4$ component wave function in the relativistic electron theory to take into account the two energy states and the two spin states of the electron for given $\mathbf{p}$. This argument is incomplete (if not incorrect). To see this we recall that although the wave function in the Klein-Gordon theory is a single-component wave function, it can accommodate the two energy states of a spin-zero particle. Furthermore, the two-component second-order equation (3.24) of Waerden, which we started with, is a perfectly valid equation for the electron; it can accommodate the two energy and the two spin states of the electron just as the Dirac equation can.‡ In this connection we note an important difference between a differential equation linear in $\partial/\partial t$, such as the diffusion (heat) equation or the Schrödinger equation, and a differential equation quadratic in the time derivative, such as the wave equation (for a vibrating string) or the Klein-Gordon equation. When we solve the problem of the temperature distribution of a potato placed in boiling water, it is sufficient to specify initially only the temperature of the inside region of the potato; we do *not* need to know *how fast* the potato is warming up. On the other hand, if we want to predict the time development of a vibrating string for $t > 0$, it is *not* sufficient to know only the displacement of the string at $t = 0$; we must also know how fast the various parts of the string are moving at $t = 0$. In general, then, when solving a partial differential equation quadratic in the time derivative, we must specify as initial conditions both the function and its time derivative. However, when solving a partial differential equation linear in the time derivative, we need specify *only* the function itself. Thus in the Dirac theory if we know ψ at $t = 0$, we can predict the time development of ψ for $t > 0$. By contrast, the two-component equation (3.24) of Waerden is second order in the time derivative; so we must know both ϕ and $\partial\phi/\partial t$ at $t = 0$ to predict the future development of ϕ. For an energy eigenstate, specifying $\partial\phi/\partial t$ (in addition to ϕ) amounts to specifying the sign of the energy. The number of independent components we must specify is *four* whether we use the four-component Dirac equation or the two-component Waerden equation.

Plane-wave solutions. Now let us return to the free-particle problem, this time with $\mathbf{p} \neq 0$. Substituting

$$\psi = \begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix} = \begin{pmatrix}u_A(\mathbf{p})\\ u_B(\mathbf{p})\end{pmatrix}\exp\left(i\mathbf{p}\cdot\frac{\mathbf{x}}{\hbar} - i\frac{Et}{\hbar}\right) \tag{3.100}$$

into (3.62) with $A_0 = \mathbf{A} = 0$, we obtain

$$u_A(\mathbf{p}) = \frac{c}{E - mc^2}(\boldsymbol{\sigma}\cdot\mathbf{p})u_B(\mathbf{p}), \qquad u_B(\mathbf{p}) = \frac{c}{E + mc^2}(\boldsymbol{\sigma}\cdot\mathbf{p})u_A(\mathbf{p}). \tag{3.101}$$

‡One of the reasons that the two-component formalism based on (3.24) is not used so extensively as the Dirac theory is that a solution to (3.24) with the electromagnetic interaction added has a rather complicated transformation property under parity (cf. Problem 3–5). Nevertheless, the two-component equation can be used for solving practical problems in quantum electrodynamics just as the Dirac equation can, as shown by R. P. Feynman and L. M. Brown.

For $E = \sqrt{|\mathbf{p}|^2c^2 + m^2c^4} > 0$ we may try, apart from a normalization constant,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

for $u_A(\mathbf{p})$. We can easily find the lower two-component spinor $u_B(\mathbf{p})$ by using the second equation of (3.101) if we recall that

$$\boldsymbol{\sigma}\cdot\mathbf{p} = \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix}. \tag{3.102}$$

In this way we get two independent solutions for $E > 0$,

$$u^{(1)}(\mathbf{p}) = N\begin{pmatrix} 1 \\ 0 \\ p_3c/(E + mc^2) \\ (p_1 + ip_2)c/(E + mc^2) \end{pmatrix} \quad \text{and} \quad u^{(2)}(\mathbf{p}) = N\begin{pmatrix} 0 \\ 1 \\ (p_1 - ip_2)c/(E + mc^2) \\ -p_3c/(E + mc^2) \end{pmatrix}, \tag{3.103}$$

where the normalization constant N is to be determined later. For $E = -\sqrt{|\mathbf{p}|^2c^2 + m^2c^4} < 0$, we may start with the lower two-component spinor u_B set to

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

so that in the case where $\mathbf{p} = 0$, ψ reduces to the third and fourth solutions of (3.97). Using the first of (3.101) to obtain the upper spinor u_A, we have

$$u^{(3)}(\mathbf{p}) = N\begin{pmatrix} -p_3c/(|E| + mc^2) \\ -(p_1 + ip_2)c/(|E| + mc^2) \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u^{(4)}(\mathbf{p}) = N\begin{pmatrix} -(p_1 - ip_2)c/(|E| + mc^2) \\ p_3c/(|E| + mc^2) \\ 0 \\ 1 \end{pmatrix}. \tag{3.104}$$

Since

$$u^{(r)}(\mathbf{p}) \exp\left[i\frac{\mathbf{p}\cdot\mathbf{x}}{\hbar} - \frac{iEt}{\hbar}\right]$$

satisfies the free-field Dirac equation, it is evident that each free-particle spinor $u^{(r)}$ with $r = 1, \ldots, 4$ satisfies

$$(i\gamma\cdot p + mc)u^{(r)}(\mathbf{p}) = 0 \tag{3.105}$$

with $\gamma\cdot p = \gamma_\mu p_\mu$, $p = (\mathbf{p}, iE/c)$ regardless of whether $E > 0$ or $E < 0$. This can, of course, be checked by direct substitution.

As shown earlier, the four independent free-particle solutions with $\mathbf{p} = 0$ written in the form (3.97) are eigenspinors of the 4×4 matrix Σ_3. This is not true for the free-particle solutions we have written for the case where $\mathbf{p} \neq 0$, as we can directly verify by applying Σ_3 to (3.103) and (3.104). But suppose we choose the z-axis in the direction of momentum $\mathbf{p}$ so that $p_1 = p_2 = 0$. We then see that $u^{(r)}$ with $r = 1, \ldots, 4$ *are* eigenspinors of Σ_3 with eigenvalues of $+1$, -1, $+1$, -1, respectively. In general, although free-particle plane-wave solutions can always be chosen

so that they are eigenfunctions of $\Sigma\cdot\hat{\mathbf{p}}$ (where $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$), it is not possible to choose solutions in such a way that they are eigenfunctions of $\Sigma\cdot\hat{\mathbf{n}}$ with an *arbitrary* unit vector $\hat{\mathbf{n}}$.‡ As we shall see in Section 3–5, this peculiarity of the plane-wave solution in the Dirac theory stems from the fact that the operator $\Sigma\cdot\hat{\mathbf{n}}$ does not commute with the free-particle Hamiltonian unless $\hat{\mathbf{n}} = \pm\hat{\mathbf{p}}$ or $\mathbf{p} = 0$. The operator $\Sigma\cdot\hat{\mathbf{p}}$ which can be diagonalized simultaneously with the free-particle Hamiltonian is called the *helicity operator*. The eigenstates of helicity with eigenvalues $+1$ and -1 are referred to, respectively, as the *right-handed state* (spin parallel to motion) and the *left-handed state* (spin opposite to motion).

It is easy to see that for a *given fixed* $\mathbf{p}$, the free-particle spinors $u^{(r)}(\mathbf{p})$ given by (3.103) and (3.104) with $r = 1, \ldots, 4$ are orthogonal to each other;

$$u^{(r)\dagger}(\mathbf{p})u^{(r')}(\mathbf{p}) = 0 \quad \text{for} \quad r \neq r'. \tag{3.106}$$

For the normalization of $u^{(r)}$, two conventions are found in the literature:

(a)
$$u^{(r)\dagger}(\mathbf{p})u^{(r)}(\mathbf{p}) = 1, \tag{3.107}$$

which implies

$$[1 + (c^2|\mathbf{p}|^2)/(|E| + mc^2)^2]N^2 = 1, \tag{3.108}$$

hence

$$N = \sqrt{(|E| + mc^2)/2|E|}; \tag{3.109}$$

(b)
$$u^{(r)\dagger}(\mathbf{p})u^{(r)}(\mathbf{p}) = |E|/mc^2, \tag{3.110}$$

which means

$$N = \sqrt{(|E| + mc^2)/2mc^2}. \tag{3.111}$$

The second normalization convention which says that $u^\dagger u$ transforms like the zeroth component of a four-vector appears somewhat artificial at this stage. However, we shall see in the next section that this convention is quite natural from the relativistic point of view. Throughout this book we shall use the normalization condition given by the second form (3.110).

To summarize, the normalized plane-wave solutions for given $\mathbf{p}$ are:

$$\psi = \sqrt{\frac{mc^2}{EV}}\, u^{(1 \text{ or } 2)}(\mathbf{p}) \exp\left[i\mathbf{p}\cdot\frac{\mathbf{x}}{\hbar} - i\frac{Et}{\hbar}\right], \tag{3.112}$$

for $E = \sqrt{|\mathbf{p}|^2c^2 + m^2c^4} > 0$, and

$$\psi = \sqrt{\frac{mc^2}{|E|V}}\, u^{(3 \text{ or } 4)}(\mathbf{p}) \exp\left[i\frac{\mathbf{p}\cdot\mathbf{x}}{\hbar} + \frac{i|E|t}{\hbar}\right], \tag{3.113}$$

---

‡This situation is in sharp contrast to the nonrelativistic Pauli theory in which *any* space-time independent two-component spinor can be regarded as an eigenspinor of $\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector in *some* direction, that is,

$$(\boldsymbol{\sigma}\cdot\hat{\mathbf{n}})\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Assuming that the spinor is normalized, all we have to do is set $a = \cos(\theta_0/2)e^{-i\phi_0/2}$ and $b = \sin(\theta_0/2)e^{+i\phi_0/2}$, where θ_0 and ϕ_0 characterize the orientation of the unit vector along which the spin component is sharp.

for $E = -\sqrt{|\mathbf{p}|^2c^2 + m^2c^4} < 0$, where

$$u^{(1 \text{ or } 2)}(\mathbf{p}) = \sqrt{(E+mc^2)/2mc^2}\left[\begin{pmatrix} 1 \\ 0 \\ p_3c/(E+mc^2) \\ (p_1+ip_2)c/(E+mc^2) \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \\ (p_1-ip_2)c/(E+mc^2) \\ -p_3c/(E+mc^2) \end{pmatrix}\right], \tag{3.114}$$

for $E > 0$, and

$$u^{(3 \text{ or } 4)}(\mathbf{p}) = \sqrt{(|E|+mc^2)/2mc^2}\left[\begin{pmatrix} -p_3c/(|E|+mc^2) \\ -(p_1+ip_2)c/(|E|+mc^2) \\ 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} -(p_1-ip_2)c/(|E|+mc^2) \\ p_3c/(|E|+mc^2) \\ 0 \\ 1 \end{pmatrix}\right], \tag{3.115}$$

for $E < 0$. The square root factors in (3.112) and (3.113) merely compensate for $|E|^2/mc^2$ in (3.110), so that

$$\int_V \psi^\dagger\psi d^3x = 1. \tag{3.116}$$

As $V \to \infty$, the allowed values of E form a continuous spectrum. For positive-energy free-particle solutions, $mc^2 \leq E < \infty$, whereas for negative-energy solutions, $-\infty < E \leq -mc^2$, as shown in Fig. 3–1.

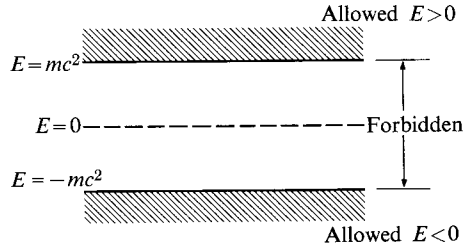

Fig. 3–1. The allowed values of E for free particles.

When the particles are not free, problems for which the Dirac equation can be solved exactly are rather scarce. In Section 3–8 we shall treat the classical problem of the electron in the Coulomb potential for which exact solutions can be found. Another problem that can be solved exactly is that of an electron in a uniform magnetic field, which is left as an exercise (Problem 3–2).

3–4. RELATIVISTIC COVARIANCE

Lorentz transformations and rotations. Before we establish the relativistic covariance of the Dirac equation, we shall review briefly the properties of Lorentz transformations. Since a Lorentz transformation is essentially a rotation in Minkowski space, let us first examine ordinary rotations in the usual three-dimensional space.

Consider a rotation through an angle ω, of the coordinate system in the 1-2 plane about the third axis. The sense of rotation is that of a right-handed screw advancing in the positive x_3-direction if $\omega > 0$. A point described by (x_1, x_2, x_3) in the old coordinate system is described in the new (primed) coordinate system by (x_1', x_2', x_3') where

$$x_1' = x_1\cos\omega + x_2\sin\omega, \qquad x_2' = -x_1\sin\omega + x_2\cos\omega. \tag{3.117}$$

Meanwhile, when we perform a Lorentz transformation along the x_1-direction such that the primed system moves with velocity $v = \beta c$, $(\mathbf{x}, ix_0)$ and $(\mathbf{x}', ix_0')$ are related by

$$x_1' = \frac{x_1}{\sqrt{1-\beta^2}} - \frac{\beta x_0}{\sqrt{1-\beta^2}}, \qquad x_0' = -\frac{\beta x_1}{\sqrt{1-\beta^2}} + \frac{x_0}{\sqrt{1-\beta^2}}, \tag{3.118}$$

or

$$x_1' = x_1\cosh\chi + ix_4\sinh\chi, \qquad x_4' = -ix_1\sinh\chi + x_4\cosh\chi, \tag{3.119}$$

where we have set

$$\tanh\chi = \beta. \tag{3.120}$$

Since the cosine and sine of an *imaginary* angle $i\chi$ are $\cosh\chi$ and $i\sinh\chi$, the above Lorentz transformation may be viewed as a rotation in Minkowski space by an angle $i\chi$ "in" the 1-4 plane, just as the transformation (3.117) represents a rotation in the 1-2 plane by a *real* angle ω. For both three-dimensional rotations and Lorentz transformations we have

$$a_{\mu\nu}a_{\lambda\nu} = \delta_{\mu\lambda}, \tag{3.121}$$

where $a_{\mu\nu}$ is defined by

$$x_\mu' = a_{\mu\nu}x_\nu. \tag{3.122}$$

From now on, the term Lorentz transformation will be used both for a three-dimensional rotation of the form (3.117) and for a "pure" Lorentz transformation of the form (3.119). At this stage we shall consider only those coordinate transformations which can be obtained by successive applications of a transformation that differs infinitesimally from the identity transformation. For example, (3.117) can be obtained by successively compounding an infinitesimal rotation

$$x_1' = x_1 + \delta\omega x_2, \qquad x_2' = -\delta\omega x_1 + x_2, \tag{3.123}$$

where $\delta\omega$ is the infinitesimal angle of rotation. This means that we restrict our considerations to the cases where

$$\det(a_{\mu\nu}) = 1, \qquad a_{44} > 0. \tag{3.124}$$

Covariance of the Dirac equation. What is meant by the covariance of the Dirac equation under Lorentz transformations? First of all, if someone, working with the primed system, were to formulate a relativistic electron theory, his first-order wave equation would *look* like the Dirac equation. Second, there exists an explicit

prescription that relates $\psi(x)$ and $\psi'(x')$, where $\psi(x)$ and $\psi'(x')$ are the wave functions corresponding to a given physical situation viewed in the unprimed and the primed system, respectively. Third, using the foregoing prescription, we must be able to show that the "Dirac-like" equation in the primed system not only looks like the Dirac equation in the unprimed system but actually is equivalent to it.

We take the point of view that the gamma matrices are introduced merely as useful short-hand devices that enable us to keep track of how the various components of ψ are coupled to each other. Hence the explicit forms of the gamma matrices are assumed to be unchanged under Lorentz transformations.‡ Note, in particular, that the gamma matrices *themselves* are not to be regarded as components of a four-vector even though, as we shall show in a moment, $\bar{\psi}\gamma_\mu\psi$ does transform like a four-vector. On the other hand, we expect that for a given physical situation the wave functions in different Lorentz frames are no longer the same.

With this point of view, if the Dirac equation is to look the same in the primed system, we must have

$$\gamma_\mu \frac{\partial}{\partial x'_\mu}\psi'(x') + \frac{mc}{\hbar}\psi'(x') = 0. \tag{3.125}$$

Note that the gamma matrices themselves are *not* primed. The question we must now ask is: How are ψ and ψ' related? An analogy with electrodynamics may be helpful here. When we perform a Lorentz transformation without a simultaneous change in the gauge, the components of $A_\mu(x)$ and $A'_\mu(x')$ are related by

$$A'_\mu(x') = a_{\mu\nu}A_\nu(x), \tag{3.126}$$

where $a_{\mu\nu}$ given by (3.122) depends only on the nature of the transformation and is independent of the space-time coordinates. Similarly, we may assume that the prescription that relates $\psi(x)$ and $\psi'(x')$ is a linear one that can be written as

$$\psi'(x') = S\psi(x), \tag{3.127}$$

where S is a 4×4 matrix which depends only on the nature of the Lorentz transformation and is completely independent of $\mathbf{x}$ and t. We rewrite (3.125), using $\partial/\partial x'_\mu = a_{\mu\nu}(\partial/\partial x_\nu)$ [which was proved in Chapter 1 (cf. Eq. 1.21)] as follows:

$$\gamma_\mu a_{\mu\nu}\frac{\partial}{\partial x_\nu}S\psi + \frac{mc}{\hbar}S\psi = 0 \tag{3.128}$$

or, multiplying S^{-1} from the left, we have

$$S^{-1}\gamma_\mu S a_{\mu\nu}\frac{\partial}{\partial x_\nu}\psi + \frac{mc}{\hbar}\psi = 0. \tag{3.129}$$

We now ask: Is (3.129) equivalent to the Dirac equation? The answer to this question is affirmative provided that we can find an S that satisfies

$$S^{-1}\gamma_\mu S\, a_{\mu\nu} = \gamma_\nu \tag{3.130}$$

‡This means that we are considering a Lorentz transformation not accompanied by a simultaneous change in the representation of the gamma matrices.

or, multiplying by $a_{\lambda\nu}$ and summing over ν, we obtain

$$S^{-1}\gamma_\lambda S = \gamma_\nu a_{\lambda\nu}, \tag{3.131}$$

where we have used (3.121).‡ The problem of demonstrating the relativistic covariance of the Dirac equation is now reduced to that of finding an S that satisfies (3.131).

We shall first treat the case of an ordinary rotation in three dimensions which we refer to as a "pure rotation." In looking for an S appropriate for the transformation (3.117), we recall that for a particle at rest the upper two-component spinor u_A is identical with the corresponding two-component spinor in the nonrelativistic Pauli theory. So in this particular case we know from nonrelativistic quantum mechanics how u_A transforms under three-dimensional rotations: if u_A is the Pauli spinor in the unprimed system, then

$$\left(\cos\frac{\omega}{2} + i\sigma_3 \sin\frac{\omega}{2}\right) \tag{3.132}$$

times u_A is the Pauli spinor corresponding to the same physical situation seen in the primed system defined by (3.117).§ Since the form of S is independent of whether it acts on the "at-rest" spinor or a more general wave function, it is natural to try for S the 4×4 analog of (3.132) given by

$$\begin{aligned} S_{\text{rot}} &= \cos\frac{\omega}{2} + i\Sigma_3 \sin\frac{\omega}{2} \\ &= \cos\frac{\omega}{2} + \gamma_1\gamma_2 \sin\frac{\omega}{2}, \end{aligned} \tag{3.133}$$

which is now assumed to act on a four-component wave function. Since $(\gamma_1\gamma_2)^2 = \gamma_1\gamma_2\gamma_1\gamma_2 = -\gamma_1^2\gamma_2^2 = -1$, S_{rot}^{-1} is given by

$$S_{\text{rot}}^{-1} = \cos\frac{\omega}{2} - \gamma_1\gamma_2 \sin\frac{\omega}{2}. \tag{3.134}$$

‡Note that the matrices $a_{\lambda\nu}$ rearrange the coordinates $\mathbf{x}$ and x_0, whereas the gamma matrices and the 4×4 matrix S rearrange the components of ψ. Although $a_{\mu\nu}$ and γ_μ are both 4×4 matrices, they are in entirely different spaces. The relation (3.131), for instance, really means

$$\sum_{\beta,\gamma}(S^{-1})_{\alpha\beta}(\gamma_\lambda)_{\beta\gamma}(S)_{\gamma\delta} = \sum_\nu a_{\lambda\nu}(\gamma_\nu)_{\alpha\delta}.$$

§See, for example, Dicke and Wittke (1960), p. 255. As an example, let us consider an electron whose spin is in the positive x_1-direction. Its wave function is represented by

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix},$$

which is evidently an eigenspinor of σ_1 with eigenvalue $+1$. In a primed system obtained by a rotation of 90° about the x_3-axis, the electron spin is in the negative x'_2-direction. Accordingly, the wave function in the primed system is

$$\left(\cos\frac{\pi}{4} + i\sigma_3 \sin\frac{\pi}{4}\right)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1+i\\1-i\end{pmatrix}.$$

This is indeed seen to be an eigenspinor of σ_2 with eigenvalue -1.

So the relation to be checked is (cf. Eq. 3.131)

$$\left(\cos\frac{\omega}{2} - \gamma_1\gamma_2 \sin\frac{\omega}{2}\right)\gamma_\lambda\left(\cos\frac{\omega}{2} + \gamma_1\gamma_2 \sin\frac{\omega}{2}\right) = \gamma_\nu a_{\lambda\nu}. \tag{3.135}$$

For $\lambda = 3, 4$ we have $a_{\lambda\nu} = \delta_{\lambda\nu}$; hence (3.135) is trivially satisfied by virtue of $\gamma_1\gamma_2\gamma_{3,4} = \gamma_{3,4}\gamma_1\gamma_2$ and $\gamma_1\gamma_2\gamma_{3,4}\gamma_1\gamma_2 = -\gamma_{3,4}$. For $\lambda = 1$,

$$\begin{aligned}&\left(\cos\frac{\omega}{2} - \gamma_1\gamma_2 \sin\frac{\omega}{2}\right)\gamma_1\left(\cos\frac{\omega}{2} + \gamma_1\gamma_2 \sin\frac{\omega}{2}\right)\\ &= \gamma_1\cos^2\frac{\omega}{2} + 2\gamma_2 \sin\frac{\omega}{2}\cos\frac{\omega}{2} - \gamma_1 \sin^2\frac{\omega}{2}\\ &= \gamma_1\cos\omega + \gamma_2\sin\omega.\end{aligned} \tag{3.136}$$

But according to (3.117) $\cos\omega = a_{11}$, $\sin\omega = a_{12}$. So (3.135) is indeed satisfied for $\lambda = 1$; similarly for $\lambda = 2$. Thus we have proved that the S_{rot} that satisfies (3.131) is given correctly by (3.133).

According to (3.119) a "pure" Lorentz transformation is nothing more than a rotation in the 1-4 plane by an imaginary angle $i\chi$. Hence for S corresponding to (3.119) we try a matrix analogous to (3.133) with $\omega \longrightarrow i\chi$, $\gamma_2 \longrightarrow \gamma_4$:

$$S_{\text{Lor}} = \cosh(\chi/2) + i\gamma_1\gamma_4 \sinh(\chi/2). \tag{3.137}$$

The relation we must check this time is

$$\left(\cosh\frac{\chi}{2} - i\gamma_1\gamma_4 \sinh\frac{\chi}{2}\right)\gamma_\lambda\left(\cosh\frac{\chi}{2} + i\gamma_1\gamma_4 \sinh\frac{\chi}{2}\right) = \gamma_\nu a_{\lambda\nu}. \tag{3.138}$$

Using similar techniques, we can verify this. For instance,

$$\begin{aligned}&\left(\cosh\frac{\chi}{2} - i\gamma_1\gamma_4 \sinh\frac{\chi}{2}\right)\gamma_4\left(\cosh\frac{\chi}{2} + i\gamma_1\gamma_4 \sinh\frac{\chi}{2}\right)\\ &= \gamma_4\cosh^2\frac{\chi}{2} - 2i\gamma_1\cosh\frac{\chi}{2}\sinh\frac{\chi}{2} + \gamma_4\sinh^2\frac{\chi}{2}\\ &= \gamma_4\cosh\chi + \gamma_1(-i\sinh\chi)\\ &= \gamma_4 a_{44} + \gamma_1 a_{41}.\end{aligned} \tag{3.139}$$

Since we have found S that satisfy (3.131) for the transformation (3.117) and (3.119), we have established the covariance of the Dirac equation under (3.117) and (3.119). In general, the S that relates ψ and ψ' via (3.127) is given by

$$S_{\text{rot}} = \cos\frac{\omega}{2} + i\sigma_{ij}\sin\frac{\omega}{2} \tag{3.140}$$

for a rotation about the kth-axis (ijk cyclic) by an angle ω, and

$$S_{\text{Lor}} = \cosh\frac{\chi}{2} - \sigma_{k4}\sinh\frac{\chi}{2} \tag{3.141}$$

for a Lorentz transformation along the kth-axis with $\beta = \tanh\chi$, where we have introduced new 4×4 matrices

$$\sigma_{\mu\nu} = (1/2i)[\gamma_\mu, \gamma_\nu] = -i\gamma_\mu\gamma_\nu, \qquad \mu \neq \nu. \tag{3.142}$$

More explicitly,

$$\begin{aligned}\sigma_{ij} &= -\sigma_{ji} = \Sigma_k = \begin{pmatrix}\sigma_k & 0\\ 0 & \sigma_k\end{pmatrix}, \qquad (ijk) \text{ cyclic},\\ \sigma_{k4} &= -\sigma_{4k} = \alpha_k = \begin{pmatrix}0 & \sigma_k\\ \sigma_k & 0\end{pmatrix}.\end{aligned} \tag{3.143}$$

It is easily verified that

$$S^\dagger_{\text{rot}} = \cos\frac{\omega}{2} - i\sigma_{ij}\sin\frac{\omega}{2} = S^{-1}_{\text{rot}}, \tag{3.144}$$

but

$$S^\dagger_{\text{Lor}} = \cosh\frac{\chi}{2} - \sigma_{k4}\sinh\frac{\chi}{2} = S_{\text{Lor}} \neq S^{-1}_{\text{Lor}}. \tag{3.145}$$

Thus, unlike S_{rot}, S_{Lor} is not unitary. This is not catastrophic, however; S_{Lor} should not be unitary if $\psi^\dagger\psi$ is to transform like the fourth component of a four-vector under Lorentz transformations. It is very important to note that for both pure rotations and pure Lorentz transformations we have

$$S^{-1} = \gamma_4 S^\dagger \gamma_4, \qquad S^\dagger = \gamma_4 S^{-1}\gamma_4, \tag{3.146}$$

since γ_4 commutes with σ_{ij} but anticommutes with σ_{k4}. We shall make extensive use of (3.146) in the next section.

Space inversion. So far we have considered only those transformations which can be obtained by compounding transformations that differ infinitesimally from the identity transformation. Under such transformations the right-handed coordinate system remains right-handed. In contrast, space inversion (often called the *parity operation*) represented by

$$\mathbf{x}' = -\mathbf{x}, \qquad t' = t \tag{3.147}$$

changes a right-handed coordinate system into a left-handed one; hence it is outside the class of transformations considered so far. We shall now demonstrate that the Dirac theory is covariant under space inversion even in the presence of A_μ.

According to the Maxwell theory the four-vector potential transforms under space inversion as‡

$$\mathbf{A}'(\mathbf{x}', t') = -\mathbf{A}(\mathbf{x}, t), \qquad A'_4(\mathbf{x}', t') = A_4(\mathbf{x}, t). \tag{3.148}$$

---

‡To prove this we first note that if the equation for the Lorentz force

$$m\frac{d}{dt}\left(\frac{\mathbf{v}}{\sqrt{1-(v/c)^2}}\right) = e[\mathbf{E} + (\mathbf{v}/c) \times \mathbf{B}]$$

is to take the same form in the space-inverted system, the electromagnetic fields must transform as $\mathbf{E}' = -\mathbf{E}$, $\mathbf{B}' = \mathbf{B}$ since $\mathbf{v}' = -\mathbf{v}$. Meanwhile, $\mathbf{E}$ and $\mathbf{B}$ are related to $\mathbf{A}$ and $A_4 = iA_0$ by

$$\mathbf{E} = -\nabla A_0 - \frac{1}{c}\frac{\partial A}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}.$$

Hence we must have (3.148).

If the Dirac equation is to look similar in the space-inverted system, we have

$$\left(\frac{\partial}{\partial x'_\mu} - \frac{ie}{\hbar c} A'_\mu\right)\gamma_\mu \psi' + \frac{mc}{\hbar}\psi' = 0, \tag{3.149}$$

or, equivalently,

$$\left[-\left(\frac{\partial}{\partial x_k} - \frac{ie}{\hbar c} A_k\right)\gamma_k + \left(\frac{\partial}{\partial x_4} - \frac{ie}{\hbar c} A_4\right)\gamma_4\right]\psi' + \frac{mc}{\hbar}\psi' = 0. \tag{3.150}$$

We try as before

$$\psi'(\mathbf{x}', t') = S_P \psi(\mathbf{x}, t), \tag{3.151}$$

where S_P is a 4×4 matrix independent of the space-time coordinates. Multiplying (3.150) by S_P^{-1} from the left, we see that (3.149) is equivalent to the Dirac equation if there exists S_P with the property

$$S_P^{-1}\gamma_k S_P = -\gamma_k, \qquad S_P^{-1}\gamma_4 S_P = \gamma_4. \tag{3.152}$$

Since γ_4 commutes with γ_4 but anticommutes with γ_k,

$$S_P = \eta\gamma_4, \qquad S_P^{-1} = \gamma_4/\eta \tag{3.153}$$

will do the job, where η is some multiplicative constant. The invariance of the probability density $\psi^\dagger\psi$ further requires $|\eta|^2 = 1$, or $\eta = e^{i\phi}$ with ϕ real. It is customary to set this phase factor η equal to 1 even though no experiment in the world can uniquely determine what this phase factor is.‡ We shall take

$$S_P = \gamma_4 \tag{3.154}$$

for an electron wave function.

Closely related to the parity operation is an operation known as *mirror reflection*, for example,

$$(x'_1, x'_2, x'_3) = (x_1, x_2, -x_3), \qquad x'_0 = x'_0. \tag{3.155}$$

Since (3.155) is nothing more than the parity operation followed by a 180° rotation about the third axis, the covariance of the Dirac equation under mirror reflection has already been demonstrated implicitly. Similar statements hold for more general transformations with

$$\det(a_{\mu\nu}) = -1, \qquad a_{44} > 0, \tag{3.156}$$

which are called *improper orthochronous Lorentz transformations* in contrast to proper orthochronous Lorentz transformations that satisfy

$$\det(a_{\mu\nu}) = 1, \qquad a_{44} > 0. \tag{3.157}$$

Simple examples. To appreciate the real physical significance of S_{rot}, S_{Lor}, S_P, it is instructive to work out some examples at this stage. As a first example, let us consider an infinitesimal form of (3.117) in which cos ω and sin ω are set, respectively, to 1 and $\delta\omega$. The wave functions in the two systems are related by

$$\psi'(x') = [1 + i\Sigma_3(\delta\omega/2)]\psi(x), \tag{3.158}$$

‡Some authors argue that one can narrow down the choice for η to ± 1, $\pm i$ by requiring that four successive inversion operations return the wave function to itself. However, there does not appear to be any deep physical significance attached to such a requirement.

where $\mathbf{x}' = \mathbf{x} + \delta\mathbf{x}$ with

$$\delta\mathbf{x} = (x_2\delta\omega, -x_1\delta\omega, 0). \tag{3.159}$$

But

$$\psi'(x') = \psi'(x) + \delta x_1 \frac{\partial\psi'}{\partial x_1} + \delta x_2 \frac{\partial\psi'}{\partial x_2}. \tag{3.160}$$

Consequently,

$$\begin{aligned}\psi'(x) &= \left[1 + i\Sigma_3\frac{\delta\omega}{2} - \left(x_2\delta\omega\frac{\partial}{\partial x_1} - x_1\delta\omega\frac{\partial}{\partial x_2}\right)\right]\psi(x) \\ &= \psi(x) + i\delta\omega\left[\frac{\Sigma_3}{2} + \frac{1}{\hbar}\left(-i\hbar x_1\frac{\partial}{\partial x_2} + i\hbar x_2\frac{\partial}{\partial x_1}\right)\right]\psi(x).\end{aligned} \tag{3.161}$$

We see that the change in the *functional form of* ψ induced by the infinitesimal rotation consists of two parts: the space-time independent operator $i\Sigma_3\delta\omega/2$ acting on the "internal" part of $\psi(x)$ and the familiar $iL_3\delta\omega/\hbar$ operator affecting just the spatial part of the wave function. The sum

$$(1/\hbar)[(\hbar\Sigma_3/2) + L_3] \tag{3.162}$$

is to be identified with the third component of the total angular-momentum operator in units of $\hbar$ since it generates an infinitesimal rotation around the third axis.‡

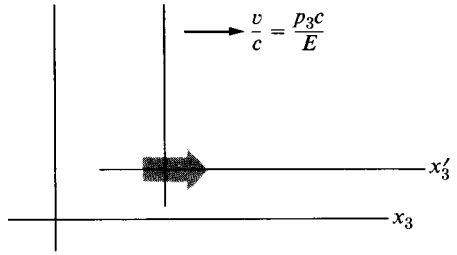

Fig. 3–2. A positive-helicity electron moving with momentum $\mathbf{p}$ along the x_3-axis. The electron is at rest in the primed system. The gray arrow indicates the spin direction.

As a second example, let us consider a free positive-energy electron of helicity $+1$ and momentum $\mathbf{p}$ along the positive x_3-direction. We choose a primed system in such a way that it will coincide with the rest system of the electron (Fig. 3–2). In the primed system the electron wave function can be written

$$\psi'(x') = \frac{1}{\sqrt{V'}}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} e^{-imc^2t/\hbar}. \tag{3.163}$$

‡Usually an operator that rotates the *physical system* around the third axis by an angle $\delta\omega$ is $1 - i\delta\omega(J_3/\hbar)$. But in our case we are rotating the coordinate system rather than the physical system. This explains why we have $1 + i\delta\omega J_3/\hbar$ instead of the above operator.

The question is: What is the wave function for the same physical situation in the unprimed system? According to (3.141) and (3.137)

$$\psi(x) = S_{\text{Lor}}^{-1}\psi'(x'), \tag{3.164}$$

where

$$S_{\text{Lor}}^{-1} = \cosh(\chi/2) - i\gamma_3\gamma_4 \sinh(\chi/2), \tag{3.165}$$

with χ given by

$$\cosh\chi = E/mc^2, \qquad \sinh\chi = p_3/mc. \tag{3.166}$$

Since

$$\begin{aligned}\cosh(\chi/2) &= \sqrt{(1+\cosh\chi)/2} = \sqrt{(E+mc^2)/2mc^2},\\ \sinh(\chi/2) &= \sqrt{\cosh^2(\chi/2) - 1} = p_3c/\sqrt{2mc^2(E+mc^2)},\end{aligned} \tag{3.167}$$

we obtain

$$\begin{aligned}S_{\text{Lor}}^{-1}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} &= \left(\begin{array}{c|c}\sqrt{\dfrac{E+mc^2}{2mc^2}} & \dfrac{p_3c\sigma_3}{\sqrt{2mc^2(E+mc^2)}}\\ \hline \dfrac{p_3c\sigma_3}{\sqrt{2mc^2(E+mc^2)}} & \sqrt{\dfrac{E+mc^2}{2mc^2}}\end{array}\right)\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\\ &= \sqrt{\frac{E+mc^2}{2mc^2}}\begin{pmatrix}1\\0\\\dfrac{p_3c}{E+mc^2}\\0\end{pmatrix}.\end{aligned} \tag{3.168}$$

This result is in complete agreement with $u^{(1)}(\mathbf{p})$, with $p_1 = p_2 = 0$ obtained earlier by solving directly the Dirac equation (cf. Eq. 3.114). It is amusing that the normalization constant which we pick up automatically is precisely the one that appears when $u(\mathbf{p})$ is normalized according to (3.110), which says that $u^\dagger u$ is the fourth component of a four-vector. As for the space-time dependence of the wave function, we merely note that

$$\begin{aligned}t' &= t\cosh\chi - (x_3/c)\sinh\chi\\ &= (E/mc^2)t - (p_3/mc^2)x_3.\end{aligned} \tag{3.169}$$

So we find that

$$\begin{aligned}\psi(x) &= S_{\text{Lor}}^{-1}\psi'(x')\\ &= \frac{1}{\sqrt{V'}}u^{(1)}(\mathbf{p})\exp\left[i\frac{p_3x_3}{\hbar} - \frac{iEt}{\hbar}\right]\\ &= \sqrt{\frac{mc^2}{EV}}u^{(1)}(\mathbf{p})\exp\left[i\frac{\mathbf{p}\cdot\mathbf{x}}{\hbar} - \frac{iEt}{\hbar}\right],\end{aligned} \tag{3.170}$$

where we have used $V = (mc^2/E)V'$ that follows from the Lorentz contraction of the normalization volume along the direction of motion. Thus we see that once we know the form of the wave function for a particle at rest, the correct wave function for a moving particle of definite momentum can be constructed just by applying S_{Lor}^{-1}. This operation is sometimes known as the *Lorentz boost*.

To work out an example that involves S_P, let us look at a Dirac wave function with a definite parity:

$$\Pi\psi(\mathbf{x}, t) = \pm\psi(\mathbf{x}, t). \tag{3.171}$$

According to (3.151) and (3.154) the functional form of the wave function in the space-inverted system is

$$\psi'(\mathbf{x}, t) = \gamma_4\psi(-\mathbf{x}, t), \tag{3.172}$$

which is to be identified with $\Pi\psi(x, t)$. Thus

$$\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}\begin{pmatrix}\psi_A(-\mathbf{x}, t)\\ \psi_B(-\mathbf{x}, t)\end{pmatrix} = \pm\begin{pmatrix}\psi_A(\mathbf{x}, t)\\ \psi_B(\mathbf{x}, t)\end{pmatrix}. \tag{3.173}$$

If, in addition, ψ_A and ψ_B can be assumed to be eigenstates of orbital angular momentum, then

$$\begin{aligned}\psi_A(-\mathbf{x}, t) &= (-1)^{l_A}\psi_A(\mathbf{x}, t) = \pm\psi_A(\mathbf{x}, t),\\ -\psi_B(-\mathbf{x}, t) &= -(-1)^{l_B}\psi_B(\mathbf{x}, t) = \pm\psi_B(\mathbf{x}, t),\end{aligned} \tag{3.174}$$

where l_A and l_B are the orbital angular momenta of the two-component wave functions. Thus

$$(-1)^{l_A} = -(-1)^{l_B}. \tag{3.175}$$

At first sight this appears to be a peculiar result, since it implies that if ψ_A is a two-component wave function with an even (odd) orbital angular momentum, then ψ_B is a two-component wave function with an odd (even) orbital angular momentum. Actually, this is not too surprising in view of the second part of (3.62), which, for a central force problem with $\mathbf{A} = 0$, takes the form

$$\psi_B = \frac{c}{E - V(r) + mc^2}(\sigma\cdot\mathbf{p})\psi_A. \tag{3.176}$$

Let us suppose that ψ_A is an $s_{1/2}$ state wave function with spin up so that

$$\psi_A = R(r)\begin{pmatrix}1\\0\end{pmatrix}e^{-i(Et/\hbar)}. \tag{3.177}$$

Then we find that

$$\begin{aligned}\psi_B &= -\frac{i\hbar c}{E - V + mc^2}\begin{pmatrix}\dfrac{\partial}{\partial x_3} & \dfrac{\partial}{\partial x_1} - i\dfrac{\partial}{\partial x_2}\\ \dfrac{\partial}{\partial x_1} + i\dfrac{\partial}{\partial x_2} & -\dfrac{\partial}{\partial x_3}\end{pmatrix}\begin{pmatrix}R(r)\\0\end{pmatrix}e^{-i(Et/\hbar)}\\ &= -\frac{i\hbar c}{E - V + mc^2}\frac{1}{r}\frac{dR}{dr}\begin{pmatrix}x_3 & x_1 - ix_2\\ x_1 + ix_2 & -x_3\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}e^{-i(Et/\hbar)}\\ &= \frac{i\hbar c}{E - V + mc^2}\frac{dR}{dr}\left[-\sqrt{\frac{4\pi}{3}}Y_1^0\begin{pmatrix}1\\0\end{pmatrix} + \sqrt{\frac{8\pi}{3}}Y_1^1\begin{pmatrix}0\\1\end{pmatrix}\right]e^{-i(Et/\hbar)},\end{aligned} \tag{3.178}$$

which is recognized as a wave function whose angular part is that of a $p_{1/2}$ wave function with $j_3 = \frac{1}{2}$. Thus ψ_A and ψ_B have opposite parities in agreement with (3.175). We could have guessed that this would be so, since the operator that multiplies ψ_A in (3.176) is a pseudoscalar operator which does not change j and

j_3 but does change the parity. We shall make use of this property in Section 3–8 when we discuss central-force problems in detail.

An even more striking consequence of $S_P = \gamma_4$ may now be discussed. Consider a positive-energy free-particle wave function and a negative-energy free-particle wave function both with $\mathbf{p} = 0$:

$$\begin{pmatrix} \chi^{(s)} \\ 0 \end{pmatrix} e^{-imc^2t/\hbar} \quad \text{and} \quad \begin{pmatrix} 0 \\ \chi^{(s)} \end{pmatrix} e^{imc^2t/\hbar}, \tag{3.179}$$

where $\chi^{(s)}$ may stand for

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since they are eigenstates of γ_4 with eigenvalues $+1$ and -1, respectively, we have the following far-reaching result: *a positive energy electron at rest and a negative energy electron at rest have opposite parities.* This will be shown to imply that an electron and a positron have opposite "intrinsic" parities when a negative energy state is properly interpreted in Section 3–9. For instance, the parity of an e^+e^--system in a relative s state must be *odd* despite its *even* orbital parity. This remarkable prediction of the Dirac theory has been checked experimentally in the decay of a positronium and will be discussed in Chapter 4.

3–5. BILINEAR COVARIANTS

Transformation properties of bilinear densities. We are in a position to discuss bilinear densities of the form $\bar{\psi}\Gamma\psi$, where Γ is a product of gamma matrices. Such densities are called bilinear covariants since they have definite transformation properties under Lorentz transformations, as will be shown in a moment. Let us first note that because of (3.146) the relation $\psi'(x') = S\,\psi(x)$ implies that

$$\bar{\psi}'(x') = \psi^\dagger(x)S^\dagger\gamma_4 = \psi^\dagger(x)\gamma_4\gamma_4 S^\dagger\gamma_4 = \bar{\psi}(x)S^{-1}, \tag{3.180}$$

whether S stands for S_{rot} or S_{Lor}. Clearly this relation holds also for S_P. Using (3.180) we immediately see that $\bar{\psi}\psi$ is invariant:

$$\bar{\psi}'(x')\,\psi'(x') = \bar{\psi}(x)\,\psi(x) \tag{3.181}$$

under pure rotations, pure Lorentz transformations, and space inversion; hence $\bar{\psi}\psi$ (not $\psi^\dagger\psi$) is a scalar density. To investigate the transformation properties of $\bar{\psi}\gamma_\mu\psi$, it is sufficient to recall (3.131). We have

$$\bar{\psi}'(x')\gamma_\mu\psi'(x') = \bar{\psi}(x)S^{-1}\gamma_\mu S\psi(x) = a_{\mu\nu}\bar{\psi}(x)\gamma_\nu\psi(x) \tag{3.182}$$

under pure rotations and pure Lorentz transformations. For the behavior under space inversion, we obtain

$$\bar{\psi}'\begin{Bmatrix}\gamma_k \\ \gamma_4\end{Bmatrix}\psi' = \bar{\psi}S_P^{-1}\begin{Bmatrix}\gamma_k \\ \gamma_4\end{Bmatrix}S_P\psi = \begin{Bmatrix}-\bar{\psi}\gamma_k\psi \\ \bar{\psi}\gamma_4\psi\end{Bmatrix}. \tag{3.183}$$

Hence $\bar{\psi}\gamma_\mu\psi$ is a four-vector density whose space components change under parity. Consequently the flux density and the probability density defined earlier (cf. Eq. 3.49) do indeed form a four-vector. Using similar techniques we find

Table 3–1

BEHAVIOR OF BILINEAR COVARIANTS UNDER LORENTZ TRANSFORMATIONS

		Proper orthochronous Lorentz transformations	Space inversion
Scalar	$\bar{\psi}\psi$	$\bar{\psi}\psi$	$\bar{\psi}\psi$
Vector	$\bar{\psi}\gamma_\mu\psi$	$a_{\mu\nu}\bar{\psi}\gamma_\nu\psi$	$\begin{Bmatrix}-\bar{\psi}\gamma_k\psi \\ \bar{\psi}\gamma_4\psi\end{Bmatrix}$
Tensor (antisymmetric, second rank)	$\bar{\psi}\sigma_{\mu\nu}\psi$	$a_{\mu\lambda}a_{\nu\sigma}\bar{\psi}\sigma_{\lambda\sigma}\psi$	$\begin{Bmatrix}\bar{\psi}\sigma_{kl}\psi \\ -\bar{\psi}\sigma_{k4}\psi\end{Bmatrix}$
Axial vector (pseudovector)	$i\bar{\psi}\gamma_5\gamma_\mu\psi$	$a_{\mu\nu}i\bar{\psi}\gamma_5\gamma_\nu\psi$	$\begin{Bmatrix}i\bar{\psi}\gamma_5\gamma_k\psi \\ -i\bar{\psi}\gamma_5\gamma_4\psi\end{Bmatrix}$
Pseudoscalar	$\bar{\psi}\gamma_5\psi$	$\bar{\psi}\gamma_5\psi$	$-\bar{\psi}\gamma_5\psi$

that $\bar{\psi}\sigma_{\mu\nu}\psi = -i\bar{\psi}\gamma_\mu\gamma_\nu\psi$ with $\mu \neq \nu$ (which is necessarily antisymmetric in μ and ν) is a second-rank *tensor* density. At this point it is advantageous to define a Hermitian 4×4 matrix

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4. \tag{3.184}$$

This γ_5 matrix has the remarkable property that it anticommutes with every one of γ_μ with $\mu = 1, \ldots, 4$,

$$\{\gamma_\mu, \gamma_5\} = 0, \qquad \mu \neq 5, \tag{3.185}$$

as seen, for instance, from $\gamma_2\gamma_1\gamma_2\gamma_3\gamma_4 = (-1)^3\gamma_1\gamma_2\gamma_3\gamma_4\gamma_2$, where we have used the fact that γ_2 commutes with γ_2 but anticommutes with $\gamma_1, \gamma_3, \gamma_4$. Note also that

$$\gamma_5^2 = 1 \tag{3.186}$$

just as γ_μ with $\mu = 1, \ldots, 4$. As for its explicit form it is easy to show that

$$\gamma_5 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} \tag{3.187}$$

in the standard (Dirac-Pauli) representation. Using (3.185), we see that

$$S_{\text{Lor}}^{-1}\gamma_5 S_{\text{Lor}} = \gamma_5, \qquad S_{\text{rot}}^{-1}\gamma_5 S_{\text{rot}} = \gamma_5, \tag{3.188}$$

since γ_5 commutes with $\sigma_{\mu\nu}$, but

$$S_P^{-1}\gamma_5 S_P = -\gamma_5, \tag{3.189}$$

since γ_5 anticommutes with γ_4. Hence $\bar{\psi}\gamma_5\psi$ transforms exactly like the scalar density $\bar{\psi}\psi$ under proper orthochronous Lorentz transformations but changes its sign under space inversion. This transformation is characteristic of a *pseudoscalar* density. Finally, using similar arguments, we can easily see that $i\bar{\psi}\gamma_5\gamma_\mu\psi$ transforms in the same way as $\bar{\psi}\gamma_\mu\psi$ under proper orthochronous Lorentz transformations but in exactly the opposite way under space inversion. This is expected of an *axial vector* (pseudovector) density. Table 3–1 summarizes the results.

The question naturally arises: Have we listed all possible bilinear covariants of the form $\bar{\psi}\Gamma\psi$? To answer this question let us start multiplying the γ_μ. If we

multiply any pair of gamma matrices, we get either $\gamma_\mu^2 = 1$ when the two matrices are the same or $\gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu = i\sigma_{\mu\nu}$ when the two matrices are different. When we multiply three gamma matrices, we get back only one of the γ_μ up to sign unless all three are different (for example, $\gamma_1\gamma_2\gamma_1 = -\gamma_1\gamma_1\gamma_2 = -\gamma_2$); when the three matrices are different, we do get a new matrix $\gamma_\mu\gamma_\nu\gamma_\lambda$. But $\gamma_\mu\gamma_\nu\gamma_\lambda$ with $\mu \neq \nu \neq \lambda$ can always be written in the form $\gamma_5\gamma_\sigma$ up to sign, where $\sigma \neq \mu, \nu, \lambda$ (for example, $\gamma_1\gamma_2\gamma_3\gamma_4\gamma_4 = \gamma_5\gamma_4$). Finally, when we multiply four gamma matrices, we get only one new matrix, $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ (which is, of course, equal to $-\gamma_2\gamma_3\gamma_4\gamma_1$, $\gamma_3\gamma_4\gamma_1\gamma_2$, etc.). Needless to say, when we multiply five or more gamma matrices, we obtain nothing new. So

$$\Gamma_A = 1, \quad \gamma_\mu, \quad \sigma_{\mu\nu} = -i\gamma_\mu\gamma_\nu \quad (\mu \neq \nu), \quad i\gamma_5\gamma_\mu, \quad \text{and} \quad \gamma_5 \tag{3.190}$$

represents all we can get. This means that there are in all sixteen independent 4×4 matrices (as we might have guessed): the identity matrix, the four γ_μ matrices, the six $\sigma_{\mu\nu}$ matrices (antisymmetric in μ and ν), the four $i\gamma_5\gamma_\mu$ matrices, and the γ_5 matrix. The factors $\pm i$ in (3.190) are inserted so that

$$\Gamma_A^2 = 1, \tag{3.191}$$

for $A = 1, \ldots, 16$. The Γ_A are all traceless with the obvious exception of the identity matrix, as the reader may easily verify by using the explicit forms of Γ_A in the standard (Dirac-Pauli) representation‡ (cf. Appendix B). Moreover, they are all linearly independent. Consequently any 4×4 matrix can be written as a unique linear combination of the sixteen Γ_A. We can find the coefficient λ_A in the expansion of an arbitrary 4×4 matrix Λ

$$\Lambda = \sum_A^{16} \lambda_A \Gamma_A, \tag{3.192}$$

by simply evaluating

$$Tr(\Lambda\Gamma_A) = Tr(\sum_B \lambda_B \Gamma_B \Gamma_A) = 4\lambda_A, \tag{3.193}$$

where we have used the fact that $\Gamma_B\Gamma_A$ is traceless when $B \neq A$ and is equal to the identity matrix when $A = B$. The algebra generated by Γ_A is called *Clifford algebra* after W. K. Clifford, who studied generalized quaternions half a century before the advent of the Dirac theory.

Let us return now to our discussion of the bilinear covariants. It is worth keeping in mind that Table 3–1 exhausts all possible bilinear covariants of the form $\bar{\psi}\Gamma\psi$, as first shown by J. von Neumann in 1928. For instance, note that we have no way to write a symmetric second-rank tensor of the form $\bar{\psi}\Gamma\psi$. This does not mean, however, that we cannot form a symmetric second-rank tensor in the Dirac

‡To prove this without appealing to any particular representation, first verify that for every Γ_A there exists at least one Γ_B (different from Γ_A) such that $\Gamma_A\Gamma_B = -\Gamma_B\Gamma_A$. Then

$$-Tr(\Gamma_A) = Tr(\Gamma_B\Gamma_A\Gamma_B) = Tr(\Gamma_B^2\Gamma_A) = Tr(\Gamma_A)$$

which would be impossible unless Γ_A were traceless.

theory. If we start introducing derivatives of ψ and $\bar{\psi}$, we can develop an expression like

$$T_{\mu\nu} = -\frac{i\hbar c}{2}\left(\bar{\psi}\gamma_\nu \frac{\partial\psi}{\partial x_\mu} - \frac{\partial\bar{\psi}}{\partial x_\mu}\gamma_\nu\psi\right) + \frac{i\hbar c}{4}\sum_{\substack{\sigma\neq\mu,\nu \\ \mu\neq\nu}} \frac{\partial}{\partial x_\sigma}(\bar{\psi}\gamma_\mu\gamma_\nu\gamma_\sigma\psi), \tag{3.194}$$

which can be shown to be the energy-momentum tensor of the Dirac wave function.

For not too relativistic electrons of positive energies some bilinear covariants are "large" while others are "small." To see this we first recall that if $E \sim mc^2$ and $V \ll mc^2$, then ψ_B is of order (v/c) compared to ψ_A. We can see then that $\bar{\psi}\psi$ and $\bar{\psi}\gamma_4\psi$ given by

$$\begin{aligned} \bar{\psi}\psi &= \psi^\dagger\gamma_4\psi = \psi_A^\dagger\psi_A - \psi_B^\dagger\psi_B, \\ \bar{\psi}\gamma_4\psi &= \psi^\dagger\psi = \psi_A^\dagger\psi_A + \psi_B^\dagger\psi_B \end{aligned} \tag{3.195}$$

are both "large" and in fact equal if terms of order $(v/c)^2$ or higher are ignored. Similarly, since

$$i\gamma_5\gamma_k = \begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix}, \qquad \sigma_{ij} = \sum_k = \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, \tag{3.196}$$

it follows that $i\bar{\psi}\gamma_5\gamma_k\psi$ and $\bar{\psi}\sigma_{ij}\psi$ (ijk cyclic) are "large" and indistinguishable up to order v/c:

$$\bar{\psi}\begin{Bmatrix} i\gamma_5\gamma_k \\ \sigma_{ij} \end{Bmatrix}\psi \approx \psi_A^\dagger\sigma_k\psi_A. \tag{3.197}$$

In contrast to 1, γ_4, $i\gamma_5\gamma_k$, and σ_{ij}, which connect $\psi_A^\dagger$ with ψ_A, the matrices γ_k, $i\gamma_5\gamma_4$, σ_{k4}, and γ_5 connect $\psi_A^\dagger(\psi_B^\dagger)$ with $\psi_B(\psi_A)$. Hence the corresponding bilinear covariants are "small" or, more precisely, of order v/c. For instance,

$$\begin{aligned} \bar{\psi}\gamma_5\psi &= (\psi_A^\dagger, -\psi_B^\dagger)\begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} \\ &= -\psi_A^\dagger\psi_B + \psi_B^\dagger\psi_A. \end{aligned} \tag{3.198}$$

Gordon decomposition of the vector current. The remaining part of this section is devoted to a detailed discussion of the vector covariant $\bar{\psi}\gamma_\mu\psi$, which occurs most frequently. We argued earlier that within the framework of the single-particle Dirac theory, $s_\mu = ic\bar{\psi}\gamma_\mu\psi$ is to be regarded as the four-vector probability current. We therefore define

$$j_\mu = es_\mu = iec\bar{\psi}\gamma_\mu\psi, \tag{3.199}$$

which is to be interpreted as the charge-current density. Using steps analogous to (3.45) through (3.48) we can show that j_μ satisfies the continuity equation even in the presence of the electromagnetic interaction. With (3.199) as the charge-current density, the Hamiltonian density for the electromagnetic interaction of the charged Dirac particle is given by

$$\begin{aligned} \mathscr{H}_{\text{int}} &= -j_\mu A_\mu/c = -ie\bar{\psi}\gamma_\mu\psi A_\mu \\ &= -e\psi^\dagger\boldsymbol{\alpha}\psi\cdot\mathbf{A} + e\psi^\dagger\psi A_0. \end{aligned} \tag{3.200}$$

This relationship can also be inferred from the Hamiltonian form of the Dirac equation (cf. Eqs. 3.38 and 3.39):

$$i\hbar(\partial/\partial t)\psi = [(-ic\hbar\boldsymbol{\nabla} - e\mathbf{A})\cdot\boldsymbol{\alpha} + \beta mc^2 + eA_0]\psi. \tag{3.201}$$

To appreciate the physical significance of j_μ we rewrite (3.199) as

$$\begin{aligned} j_\mu &= (iec/2)(\bar{\psi}\gamma_\mu\psi + \bar{\psi}\gamma_\mu\psi) \\ &= \frac{ie\hbar}{2m}\left[-\bar{\psi}\gamma_\mu\gamma_\nu\left(\frac{\partial}{\partial x_\nu} - \frac{ie}{\hbar c}A_\nu\right)\psi + \left\{\left(\frac{\partial}{\partial x_\nu} + \frac{ie}{\hbar c}A_\nu\right)\bar{\psi}\right\}\gamma_\nu\gamma_\mu\psi\right], \end{aligned} \tag{3.202}$$

where we have used (3.60) and its analog for the adjoint wave function $\bar{\psi}$. This encourages us to split j_μ into two parts:

$$j_\mu = j_\mu^{(1)} + j_\mu^{(2)}, \tag{3.203}$$

according to dependency on whether or not the summation index ν in (3.202) coincides or does not coincide with μ. We have

$$j_\mu^{(1)} = \frac{ie\hbar}{2m}\left(\frac{\partial\bar{\psi}}{\partial x_\mu}\psi - \bar{\psi}\frac{\partial\psi}{\partial x_\mu}\right) - \frac{e^2}{mc}A_\mu\bar{\psi}\psi \tag{3.204}$$

and

$$\begin{aligned} j_\mu^{(2)} &= \frac{ie\hbar}{2m}\left[-\bar{\psi}\gamma_\mu\gamma_\nu\frac{\partial}{\partial x_\nu}\psi + \left(\frac{\partial\bar{\psi}}{\partial x_\nu}\right)\gamma_\nu\gamma_\mu\psi\right. \\ &\qquad \left. + \frac{ie}{\hbar c}A_\nu\bar{\psi}\gamma_\mu\gamma_\nu\psi + \frac{ie}{\hbar c}A_\nu\bar{\psi}\gamma_\nu\gamma_\mu\psi\right]_{\mu\neq\nu} \\ &= -\frac{e\hbar}{2m}\frac{\partial}{\partial x_\nu}(\bar{\psi}\sigma_{\nu\mu}\psi). \end{aligned} \tag{3.205}$$

This decomposition is known as the *Gordon decomposition*, named for W. Gordon.

Let us look at each component of (3.204) and (3.205) a little more closely. The four-vector (3.204) does not contain any gamma matrix. In fact $j_k^{(1)}$ would be formally identical to the expression for the three-vector current density in the Schrödinger theory if we could replace the Dirac wave function by the Schrödinger wave function. This is rather gratifying since we know that in the nonrelativistic limit, $(\partial\bar{\psi}/\partial x_k)\psi$, etc., can be legitimately replaced by $(\partial\psi_A^\dagger/\partial x_k)\psi_A$, etc. As for $j_4^{(1)}$ we can easily show that

$$j_4^{(1)} = \frac{ieE}{mc}\bar{\psi}\psi - \frac{ie^2}{mc}\bar{\psi}\psi A_0 \tag{3.206}$$

when the time dependence $e^{-iEt/\hbar}$ is assumed. This reduces to ic times the charge current density $e\psi_A^\dagger\psi_A$ in the Schrödinger theory, provided $E \simeq mc^2$ and $|eA_0| \ll mc^2$. As for (3.205) we recall that in the nonrelativistic limit

$$\bar{\psi}\sigma_{k4}\psi = -\bar{\psi}\sigma_{4k}\psi$$

can be ignored while $\bar{\psi}\sigma_{jk}\psi$ can be interpreted as the spin density $\psi_A^\dagger\sigma_l\psi_A$ (jkl cyclic). In other words, the kth component of (3.205) is $-e\hbar/2m$ times

$$\frac{\partial}{\partial x_j}(\psi_A^\dagger\sigma_l A_A) - \frac{\partial}{\partial x_l}(\psi_A^\dagger\sigma_j\psi_A), \qquad (jkl) \text{ cyclic}, \tag{3.207}$$

which is just the kth component of the curl of the spin density.‡ Thus for slowly moving electrons the Gordon decomposition of j_k can be regarded as the separation of j_k into the convection current due to the moving charge and the current associated with the intrinsic magnetization (magnetic dipole density) of the electron.

When $j_\mu^{(2)}$ interacts with A_μ via (3.200), we have the interaction Hamiltonian density

$$\begin{aligned} -\frac{j_\mu^{(2)}A_\mu}{c} &= \frac{e\hbar}{2mc}\left[\frac{\partial}{\partial x_\nu}(\bar{\psi}\sigma_{\nu\mu}\psi)\right]A_\mu \\ &= -\frac{e\hbar}{2mc}\frac{\partial A_\mu}{\partial x_\nu}(\bar{\psi}\sigma_{\nu\mu}\psi) \\ &= -\frac{e\hbar}{2mc}\left[\frac{1}{2}\frac{\partial A_\mu}{\partial x_\nu}(\bar{\psi}\sigma_{\nu\mu}\psi) + \frac{1}{2}\frac{\partial A_\nu}{\partial x_\mu}(\bar{\psi}\sigma_{\mu\nu}\psi)\right] \\ &= -\frac{e\hbar}{2mc}[\tfrac{1}{2}F_{\nu\mu}\bar{\psi}\sigma_{\nu\mu}\psi], \end{aligned} \tag{3.208}$$

where we have dropped $(\partial/\partial x_\nu)(\bar{\psi}\sigma_{\nu\mu}\psi A_\mu)$ which is irrelevant when the interaction density is integrated. Noting that

$$\tfrac{1}{2}F_{\nu\mu}\bar{\psi}\sigma_{\nu\mu}\psi \approx \mathbf{B}\cdot(\psi_A^\dagger\boldsymbol{\sigma}\psi_A) \tag{3.209}$$

in the nonrelativistic limit, we see that (3.208) can indeed account for the spin magnetic moment interaction with the gyromagnetic ratio $g = 2$, in agreement with our earlier discussion based on (3.67).

Experimentally, as first shown by P. Kusch in 1947, the observed gyromagnetic ratio of the electron is not exactly 2 but rather

$$g = 2\left[1 + \left(\frac{e^2}{4\pi\hbar c}\right)\frac{1}{2\pi} + \cdots\right], \tag{3.210}$$

which holds also for the muon. The origin of the extra magnetic moment was satisfactorily explained in 1947 by J. Schwinger who took into account the fact that the physical electron can emit or absorb a virtual photon, as we noted in Section 2–8. (We shall come back to this point in Chapter 4.) When the magnetic moment is not correctly given by $g = 2$ we may add a phenomenological term to the interaction Hamiltonian of the form

$$\mathscr{H}_{\text{int}} = -\frac{e\hbar\kappa}{2mc}[\tfrac{1}{2}F_{\nu\mu}\bar{\psi}\sigma_{\nu\mu}\psi], \tag{3.211}$$

called an *anomalous moment* (Pauli moment) *interaction*.§ The total magnetic

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‡In this connection recall that according to classical electrodynamics a magnetic dipole density $\mathscr{M}$ gives rise to an effective current density

$$\mathbf{j}_{(\text{eff})} = c\boldsymbol{\nabla}\times\mathscr{M}.$$

See Panofsky and Phillips (1955), p. 120; Jackson (1962), p. 152.

§The interaction (3.211) is to be interpreted as an *effective* Hamiltonian density. To the extent that Schwinger's correction is computable on the basis of the interaction (3.200), there is no need to postulate an additional "fundamental" interaction of the type (3.211), at least for the electron and the muon.

moment computed from the sum of (3.208) and (3.211) is

$$\mu = (e\hbar/2mc)(1 + \kappa). \tag{3.212}$$

For the proton, replacing e and m in (3.208) and (3.211) by $|e|$ and the proton mass m_p, we have

$$\mu_p = (|e|\hbar/2m_p c)(1 + \kappa_p), \tag{3.213}$$

where experimentally $\kappa_p = 1.79$, as O. Stern first determined. We thus see that roughly 60% of the observed proton magnetic moment is "anomalous." According to (3.208), spin-½ particles should have zero magnetic moments as $e \to 0$, yet experimentally neutral particles, such as the neutron and the Λ-hyperon, are known to possess sizable magnetic moments. They can again be represented phenomenologically by interactions of the type (3.211). In contrast to the anomalous moments of the electron and the muon, the anomalous moments of the proton, the neutron, the Λ-hyperon, and so forth, cannot be accounted for by Schwinger's mechanism. Therefore their existence appears to indicate a failure of the simple prescription $p_\mu \to p_\mu - qA_\mu/c$. However, if we consider that these particles are complicated objects surrounded by virtual meson clouds, the failure of the simple prescription does not seem surprising. Spin-½ particles whose electromagnetic properties can be understood on the basis of (3.200) alone are sometimes referred to as *pure Dirac particles.*

Vector covariant for free particles. To investigate further the physical meaning of the gamma matrices, let us consider this time $\bar{\psi}_f \gamma_k \psi_i$, where ψ_i and ψ_f are $E > 0$ plane-wave solutions:

$$\begin{aligned}\psi_i &= \sqrt{\frac{mc^2}{EV}}\, u^{(r)}(\mathbf{p}) \exp\left(\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar} - \frac{iEt}{\hbar}\right),\\ \psi_f &= \sqrt{\frac{mc^2}{E'V}}\, u^{(r')}(\mathbf{p}') \exp\left(\frac{i\mathbf{p}'\cdot\mathbf{x}}{\hbar} - \frac{iE't}{\hbar}\right).\end{aligned} \tag{3.214}$$

This is of some practical interest in connection with the scattering of a Dirac particle by an external vector potential $\mathbf{A}$, since the transition matrix element which we compute in the Born approximation is essentially $-ie\int \bar{\psi}_f \gamma_k \psi_i A_k d^3x$. Assuming for simplicity that the vector potential is time independent so that $E = E'$ (elastic scattering), we can evaluate $i\gamma_k$ taken between $\bar{u}^{(r')}(\mathbf{p})$ and $u^{(r)}(\mathbf{p})$ as follows:

$$\begin{aligned} i\bar{u}^{(r')}(\mathbf{p}')\gamma_k u^{(r)}(\mathbf{p}) &= u^{(r')\dagger}(\mathbf{p}')\alpha_k u^{(r)}(\mathbf{p})\\ &= \left(\sqrt{\frac{E+mc^2}{2mc^2}}\right)^2 \left(\chi^{(s')\dagger}, \chi^{(s')\dagger}\frac{\boldsymbol{\sigma}\cdot\mathbf{p}'c}{E+mc^2}\right)\begin{pmatrix}0 & \sigma_k\\ \sigma_k & 0\end{pmatrix}\begin{pmatrix}\chi^{(s)}\\ \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}c}{E+mc^2}\chi^{(s)}\end{pmatrix}\\ &= \chi^{(s')\dagger}\left[\frac{p_k + p'_k}{2mc} + \frac{i\boldsymbol{\sigma}\cdot\{(\mathbf{p}'-\mathbf{p})\times\hat{\mathbf{n}}_k\}}{2mc}\right]\chi^{(s)}, \end{aligned} \tag{3.215}$$

where $\chi^{(s)}$ and $\chi^{(s')}$ are the initial and final Pauli two-component spinors. The first term, of course, corresponds to the convection current $j_k^{(1)}$ of (3.201). To see the meaning of the second term, we note that it appears in the transition matrix element as

$$\begin{aligned} &-e\left(\sqrt{\frac{mc^2}{E}}\right)^2 \frac{1}{V}\int_V d^3x \chi^{(s')\dagger}\left[\frac{i\boldsymbol{\sigma}\cdot\{(\mathbf{p}'-\mathbf{p})\times\mathbf{A}\}}{2mc}\right]\chi^{(s)} \exp\left[\frac{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}{\hbar}\right]\\ &= \frac{mc^2}{EV}\left(\frac{e\hbar}{2mc}\right)\int_V d^3x \chi^{(s')\dagger}\boldsymbol{\sigma}\cdot\left[\left(\nabla \exp\left[\frac{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}{\hbar}\right]\right)\times\mathbf{A}\right]\chi^{(s)}\\ &= -\frac{mc^2}{EV}\frac{e\hbar}{2mc}\chi^{(s')\dagger}\boldsymbol{\sigma}\chi^{(s)}\cdot\int_V d^3x(\nabla\times\mathbf{A})\exp\left[\frac{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}{\hbar}\right], \end{aligned} \tag{3.216}$$

which is recognized as the perturbation matrix element expected from the spin magnetic moment interaction.

Finally, let us consider the case $\psi_i = \psi_f$. Using (3.215) with $\mathbf{p} = \mathbf{p}'$, we have‡

$$i\int_V \bar{\psi}\gamma_k\psi d^3x = \int_V \psi^\dagger\alpha_k\psi d^3x = \left(\frac{mc^2}{EV}\right)\frac{2p_kV}{2mc} = \frac{p_kc}{E}. \tag{3.217}$$

But this is nothing more than the classical particle velocity divided by c. It is interesting to recall in this connection that the analog of $-ie\int\bar{\psi}\gamma_\mu\psi A_\mu d^3x$ in the *classical* electrodynamics of a point particle is

$$H_{\text{classical}} = (-e\mathbf{v}\cdot\mathbf{A}/c) + eA_0. \tag{3.218}$$

Since plane-wave solutions of the form (3.214) are orthonormal, the result (3.217) can be generalized to any wave function that can be expressed as a superposition of $E > 0$ free-particle plane-wave solutions:

$$\psi_{E>0} = \sum_{\mathbf{p}}\sum_{r=1,2}\sqrt{\frac{mc^2}{EV}}\, c_{\mathbf{p},r}u^{(r)}(\mathbf{p})\exp\left(i\frac{\mathbf{p}\cdot\mathbf{x}}{\hbar} - \frac{iEt}{\hbar}\right), \tag{3.219}$$

where $c_{\mathbf{p},r}$ is a Fourier coefficient whose modulus squared directly gives the probability for finding the electron in state $(\mathbf{p}, r)$. With the help of (3.215), we obtain

$$\begin{aligned}\langle\alpha_k\rangle_+ &\equiv \int_V \psi^\dagger_{E>0}\alpha_k\psi_{E>0}d^3x\\ &= \sum_{\mathbf{p}}\sum_{\mathbf{p}'}\sum_{r=1,2}\sum_{r'=1,2}\sqrt{(mc^2)^2/EE'V^2}\, c_{\mathbf{p},r}c^*_{\mathbf{p}',r'}u^{(r')\dagger}(\mathbf{p}')\alpha_k u^{(r)}(\mathbf{p})V\delta_{\mathbf{p},\mathbf{p}'}\\ &= \sum_{\mathbf{p}}\sum_{r=1,2}|c_{\mathbf{p},r}|^2(p_kc/E) = \langle p_kc/E\rangle, \end{aligned} \tag{3.220}$$

where + stands for positive energy. Using similar techniques with negative energy spinors, we can readily obtain

$$\langle\alpha_k\rangle_- = -\langle p_kc/|E|\rangle \tag{3.221}$$

for a wave function made up *exclusively* of negative-energy plane-wave solutions. We shall come back later to these very important relations.

Although we have treated only $i\bar{\psi}\gamma_k\psi = \psi^\dagger\alpha_k\psi$ in detail, the reader may work out the analogs of (3.216) for other matrices γ_4, $i\gamma_5\gamma_\mu$, σ_{k4}, γ_5, and so forth. For instance, the interpretation of $\bar{\psi}\sigma_{k4}\psi$ is of interest in connection with electron-neutron scattering (Problem 3–6).

‡Note that, although we have used the two-component language, we have not made a single nonrelativistic approximation in obtaining (3.215) and (3.217).

3–6. DIRAC OPERATORS IN THE HEISENBERG REPRESENTATION

Heisenberg equation of motion. Up to now we have regarded the Dirac matrices as short-hand devices that rearrange the various components of ψ. In discussing the time development of a matrix element such as $\int \psi'^{\dagger}(\mathbf{x}, t)\alpha_k\psi(\mathbf{x}, t)d^3x$, we shall find it sometimes more convenient to introduce a time-dependent operator $\alpha_k^{(H)}(t)$ (where H stands for Heisenberg) with the property

$$\int \psi'^{\dagger}(\mathbf{x}, t)\alpha_k\psi(\mathbf{x}, t)\, d^3x = \int \psi'^{\dagger}(\mathbf{x}, 0)\alpha_k^{(H)}(t)\psi(\mathbf{x}, 0)\, d^3x. \tag{3.222}$$

The time development of $\int \psi'^{\dagger}(\mathbf{x}, t)\alpha_k\psi(\mathbf{x}, t)d^3x$ can then be inferred directly from the differential equation that governs the behavior of the operator $\alpha_k^{(H)}$. All this amounts to working in the Heisenberg representation, where the state vector is time independent and the dynamical operator is time dependent. For this reason let us briefly review the connection between the Heisenberg representation and the Schrödinger representation.

By virtue of the Schrödinger equation [or the Dirac equation written in the Hamiltonian form (3.38) and (3.39)], a Schrödinger (or Dirac) wave function, which is not necessarily assumed to be an energy eigenfunction, can be written as

$$\psi(\mathbf{x}, t) = e^{-iHt/\hbar}\psi(\mathbf{x}, 0), \tag{3.223}$$

where H is the Hamiltonian *operator* that acts on the wave function. Associated with a time-independent operator $\Omega^{(S)}$ where S stands for Schrödinger, the corresponding operator in the Heisenberg representation can be defined as

$$\Omega^{(H)}(t) = e^{iHt/\hbar}\Omega^{(S)}\, e^{-iHt/\hbar}. \tag{3.224}$$

Clearly, at $t = 0$, $\Omega^{(H)}$ coincides with $\Omega^{(S)}$:

$$\Omega^{(H)}(0) = \Omega^{(S)}.$$

With (3.223) and (3.224), the matrix elements of a given operator in the two representations taken between any initial and final state are seen to be the same provided that in evaluating the matrix element in the Heisenberg representation we use the wave function at $t = 0$, that is,

$$\int \psi'^{\dagger}(\mathbf{x}, t)\Omega^{(S)}\psi(\mathbf{x}, t)\, d^3x = \int \psi'^{\dagger}(\mathbf{x}, 0)\, \Omega^{(H)}\psi(\mathbf{x}, 0)\, d^3x. \tag{3.225}$$

By considering an infinitesimal displacement in time, we are able to readily deduce the Heisenberg equation of motion:

$$\frac{d\Omega^{(H)}}{dt} = \frac{i}{\hbar}[H, \Omega^{(H)}] + \frac{\partial\Omega^{(H)}}{\partial t}, \tag{3.226}$$

where the last term is nonvanishing only when Ω depends explicitly on time. Note that this equation is equivalent to

$$\frac{d}{dt}\int \psi'^{\dagger}(\mathbf{x}, t)\Omega^{(S)}\psi(\mathbf{x}, t)\, d^3x$$
$$= \frac{i}{\hbar}\int \psi'^{\dagger}(\mathbf{x}, t)[H, \Omega^{(S)}]\psi(\mathbf{x}, t)\, d^3x + \int \psi'^{\dagger}(\mathbf{x}, t)(\partial\Omega^{(S)}/\partial t)\psi(\mathbf{x}, t)\, d^3x. \tag{3.227}$$

With a Dirac matrix, $\alpha_k(\beta)$ we may associate a Heisenberg dynamical operator $\alpha_k^{(H)}(\beta^{(H)})$ with the property that its explicit matrix representation at $t = 0$ is $\alpha_k(\beta)$. Since the operator $e^{-iHt/\hbar}$ that connects $\alpha_k^{(H)}$ with α_k, $\beta_k^{(H)}$ with β, and so forth, is unitary, the anticommutation relations among α_k and β hold also for the corresponding dynamical operators. In this section only, we shall omit the superscript (H) so that α_k will stand for the dynamical operator that corresponds to the matrix α_k discussed in the previous sections. Similarly, the symbols $\dot{\mathbf{x}}$, $\mathbf{p}$, and $\mathbf{L}$ will stand for the velocity operator, the momentum operator, and the orbital angular momentum operator in the Heisenberg representation.

Constants of the motion. With the aid of the Heisenberg equation of motion we can immediately determine whether or not a given observable is a constant of the motion. For instance, for a free particle whose Hamiltonian is‡

$$H = c\alpha_j p_j + \beta mc^2 \tag{3.228}$$

we have the Heisenberg equation for the momentum

$$\frac{dp_k}{dt} = \frac{i}{\hbar}[H, p_k] = 0, \tag{3.229}$$

since p_k commutes with both $c\alpha_j p_j$ and βmc^2. Equation (3.229) tells us that we can find a solution to the Dirac equation which is a simultaneous eigenfunction of the Hamiltonian and the momentum. But we know this already, since the plane wave solutions obtained in Section 3–3 are simultaneous eigenfunctions of H and $\mathbf{p}$.

As a less trivial example, let us consider $\mathbf{L}$ for a free particle. For the x-component of $\mathbf{L}$ we have

$$[H, L_1] = [c\alpha_k p_k, (x_2p_3 - x_3p_2)] = -i\hbar c(\alpha_2 p_3 - \alpha_3 p_2), \tag{3.230}$$

and similarly for L_2 and L_3. So

$$d\mathbf{L}/dt = c(\boldsymbol{\alpha} \times \mathbf{p}). \tag{3.231}$$

This means that for a free Dirac particle, $\mathbf{L}$ is *not* a constant of the motion, in sharp contrast with the corresponding operator in the Schrödinger theory.

Next let us consider $\boldsymbol{\Sigma}$. First it is useful to note that

$$\alpha_k = -\Sigma_k\gamma_5 = -\gamma_5\Sigma_k, \tag{3.232}$$

which is obvious from the explicit forms of α_k, γ_5, and Σ_k.§ Therefore

$$[H, \Sigma_1] = [c\alpha_k p_k, \Sigma_1] = -c[\gamma_5\Sigma_k p_k, \Sigma_1]$$
$$= 2ic(\alpha_2 p_3 - \alpha_3 p_2), \tag{3.233}$$

‡Since it is customary to use α_k and β to write H, we shall, in this section, make exclusive use of α_k and β rather than σ_{k4} and γ_4.

§To prove this without recourse to any particular representation, note, for instance, $\alpha_3 = -i\gamma_3\gamma_4 = i\gamma_1\gamma_2\gamma_1\gamma_2\gamma_3\gamma_4 = -\Sigma_3\gamma_5$, where we have used $(\gamma_1\gamma_2)^2 = -1$.

where we have used $\gamma_5\Sigma_2\Sigma_1 = -i\gamma_5\Sigma_3 = i\alpha_3$, and so forth. Consequently,

$$\frac{d\boldsymbol{\Sigma}}{dt} = -\frac{2c}{\hbar}(\boldsymbol{\alpha} \times \mathbf{p}). \tag{3.234}$$

Thus the spin angular momentum of a free electron is *not* a constant of the motion, either. But taking the dot product of (2.234) and $\mathbf{p}$ and remembering (3.229), we obtain

$$d(\boldsymbol{\Sigma}\cdot\mathbf{p})/dt = 0, \tag{3.235}$$

which means that the helicity $\boldsymbol{\Sigma}\cdot\mathbf{p}/|\mathbf{p}|$ is a constant of the motion, as shown in Section 3–3.

Let us now consider

$$\mathbf{J} = \mathbf{L} + \hbar\boldsymbol{\Sigma}/2. \tag{3.236}$$

Because of (3.231) and (3.234) we have

$$d\mathbf{J}/dt = 0. \tag{3.237}$$

Thus, although $\mathbf{L}$ and $\hbar\boldsymbol{\Sigma}/2$ taken separately are *not* constants of the motion, the sum (3.236) which, according to (3.161), should be identified with the total angular momentum *is* a constant of the motion. As is well known, the constancy of $\mathbf{J}$ is a consequence of invariance under rotation. Hence we may argue that $\mathbf{J}$ must be a constant of the motion even if we add to the free-particle Hamiltonian a central (spherically symmetric) potential $V(r)$. Indeed, the relation (3.237) still holds in the presence of $V(r)$ since both L_k and Σ_k commute with $V(r)$.

Next we shall consider the time derivative of the mechanical momentum

$$\boldsymbol{\pi} = \mathbf{p} - e\mathbf{A}/c \tag{3.238}$$

(as opposed to the *canonical* momentum $\mathbf{p}$) of an electron in the presence of A_μ. Using the Hamiltonian

$$H = c\boldsymbol{\alpha}\cdot\boldsymbol{\pi} + eA_0 + \beta mc^2, \tag{3.239}$$

we obtain

$$\dot{\pi}_k = \frac{i}{\hbar}[H, \pi_k] - \frac{e}{c}\frac{\partial A_k}{\partial t} = \frac{ic}{\hbar}\alpha_j[\pi_j, \pi_k] + \frac{ie}{\hbar}[A_0, \pi_k]. \tag{3.240}$$

But we know that

$$[A_0, \pi_k] = i\hbar\frac{\partial A_0}{\partial x_k},$$

and

$$[\pi_1, \pi_2] = \frac{ie\hbar}{c}\frac{\partial A_2}{\partial x_1} - \frac{ie\hbar}{c}\frac{\partial A_1}{\partial x_2} = \frac{ie\hbar}{c}B_3, \quad \text{etc.} \tag{3.241}$$

Hence

$$\dot{\boldsymbol{\pi}} = e(\mathbf{E} + \boldsymbol{\alpha}\times\mathbf{B}). \tag{3.242}$$

Since $\langle\alpha_k\rangle_+$ has been shown to correspond to the classical particle velocity in units of c (cf. Eq. 3.220), we may be tempted to identify (3.242) with the operator equation for the Lorentz force. However, as we shall see later, we have to be somewhat careful in regarding $c\boldsymbol{\alpha}$ as the operator that corresponds to the particle velocity in the usual sense.

It is also of interest to study the time dependence of $\boldsymbol{\Sigma}\cdot\boldsymbol{\pi}$ for an electron in the electromagnetic field. Assuming that A_μ is time-independent, we obtain

$$\frac{d(\boldsymbol{\Sigma}\cdot\boldsymbol{\pi})}{dt} = \frac{i}{\hbar}[(-c\gamma_5\boldsymbol{\Sigma}\cdot\boldsymbol{\pi} + \beta mc^2 + eA_0), \boldsymbol{\Sigma}\cdot\boldsymbol{\pi}] = \frac{ie}{\hbar}[A_0, \boldsymbol{\Sigma}\cdot\boldsymbol{\pi}] = e\boldsymbol{\Sigma}\cdot\mathbf{E}, \tag{3.243}$$

since both γ_5 and β commute with $\boldsymbol{\Sigma}\cdot\boldsymbol{\pi}$. Let us suppose that there is no electric field. We know that the magnitude of the mechanical momentum of a charged particle is unchanged in a time-independent magnetic field. The constancy of $\boldsymbol{\Sigma}\cdot\boldsymbol{\pi}$ then amounts to the constancy of the helicity. It follows that a longitudinally polarized electron (one whose helicity is $+1$ or -1) entering a region with a magnetic field will remain longitudinally polarized no matter how complicated $\mathbf{B}$ may be. When an electron is injected into a region with a uniform magnetic field $\mathbf{B}$ whose direction is perpendicular to the initial electron velocity, the electron follows a circular path with an angular frequency, known as the *cyclotron frequency*:

$$\omega_L = (|e\mathbf{B}|/mc)\sqrt{1-\beta^2}. \tag{3.244}$$

In this particularly simple case, the constancy of helicity implies that the electron spin precesses in such a way that its precession angular frequency ω_S is equal to ω_L, as pictorially represented in Fig. 3–3.

Fig. 3–3. Spin precession of a moving electron in a uniform magnetic field. The gray arrows indicate the spin direction.

In deriving the equality of ω_L and ω_S we implicitly assumed that the gyromagnetic ratio of the electron is strictly two. In reality, because of the anomalous moment of the electron, ω_L and ω_S are not quite equal. It can be shown that the correct relation is given by

$$\frac{\omega_S}{\omega_L} = 1 + \left(\frac{g-2}{2}\right)\frac{1}{\sqrt{1-\beta^2}}. \tag{3.245}$$

This means that as the initially longitudinally polarized electron makes one orbital turn, its spin direction departs from the direction of motion by a very small amount, 1/137 rad if the electron is nonrelativistic (cf. Eq. 3.210). This principle has been used experimentally to make precise measurements on the anomalous moments of the electron and the muon.

"Velocity" in the Dirac theory. Let us now return to the free-particle case. Consider

$$\dot{x}_k = (i/\hbar)[H, x_k] = (ic/\hbar)[\alpha_j p_j, x_k] = c\alpha_k, \tag{3.246}$$

which says that $\boldsymbol{\alpha}$ is the velocity in units of c. (Actually this relation holds even

in the presence of A_μ.) At first sight this appears quite reasonable in view of (3.220), which says that $\langle \alpha_k \rangle_+$ is the same as the expectation value of $p_k c H^{-1}$. Note, however, that the eigenvalue of α_k is $+1$ or -1. Hence the eigenvalue of the velocity operator is $\pm c$, as first pointed out by G. Breit in 1928. This is a truly remarkable result, since for a particle of finite mass the classical velocity cannot be equal to $\pm c$. We may also note that because α_k and α_l do not commute when $k \neq l$, a measurement of the x-component of the velocity is incompatible with a measurement of the y-component of the velocity; this may appear strange since we know that p_1 and p_2 commute.

In spite of these peculiarities, there is actually no contradiction with the results derived earlier. The plane-wave solutions (3.114) and (3.115) which are eigenfunctions of $\mathbf{p}$ are not eigenfunctions of α_k (unless the particle is massless), as the reader may verify by directly applying α_k. In fact, since α_k fails to commute with the Hamiltonian (see Eq. 3.248), no energy eigenfunctions are expected to be simultaneous eigenfunctions of α_k.

Let us now look at the time derivative of α_k:

$$\begin{aligned} \dot{\alpha}_k &= (i/\hbar)[H, \alpha_k] \\ &= (i/\hbar)(-2\alpha_k H + \{H, \alpha_k\}) \\ &= (i/\hbar)(-2\alpha_k H + 2c p_k), \end{aligned} \tag{3.247}$$

where we have used the fact that α_k anticommutes with every term in H except the term that involves α_k itself. This equation can also be written as

$$\dot{\alpha}_k = -\frac{2c}{\hbar}(\boldsymbol{\Sigma} \times \mathbf{p})_k - \frac{2i\alpha_k \beta mc^2}{\hbar}. \tag{3.248}$$

We see that the velocity operator $\dot{x}_k = c\alpha_k$ is *not* a constant of the motion despite the fact that the particle is free. Contrast this with Eq. (3.229), which says that the momentum of a free particle *is* a constant of the motion.

The relation (3.247) can be regarded as a differential equation for $\alpha_k(t)$. Keeping in mind that p_k and H are constants of the motion, we easily see by direct substitution that the solution of this differential equation is

$$\alpha_k(t) = c p_k H^{-1} + (\alpha_k(0) - c p_k H^{-1}) e^{-2iHt/\hbar}. \tag{3.249}$$

The first term of (3.249) is reasonable since, for an eigenstate of momentum and energy, it gives cp_k/E in agreement with (3.220) and (3.221). But what is the physical significance of the second term? Taken literally, it seems to say that the velocity of the electron has an additional term that fluctuates rapidly about its average value even in the absence of any potential.

As for the coordinate operator, the relation (3.249) can be easily integrated to yield

$$x_k(t) = x_k(0) + c^2 p_k H^{-1} t + (ic\hbar/2)(\alpha_k(0) - c p_k H^{-1}) H^{-1} e^{-2iHt/\hbar}. \tag{3.250}$$

The first and the second terms are understandable because their expectation values are seen to give the trajectory of the wave packet according to the *classical* law:

$$x_k^{(\text{class})}(t) = x_k^{(\text{class})}(0) + t(p_k c^2/E)^{(\text{class})}, \tag{3.251}$$

just as they do in nonrelativistic quantum mechanics. The presence of the third term in (3.250) (which, of course, is a consequence of the second term in Eq. 3.249) appears to imply that the free electron executes very rapid oscillations in addition to the uniform rectilinear motion (3.251). This oscillatory motion, first discussed by E. Schrödinger in 1930, is called *Zitterbewegung* (literally "quivering motion"). We shall say more about this in the next section.

3–7. *ZITTERBEWEGUNG* AND NEGATIVE-ENERGY SOLUTIONS

Expectation values of α and x. The algebraic techniques extensively employed in the previous section are quite powerful when we want to obtain the constants of the motion or establish a correspondence with the classical theory. However, in order to analyze the peculiarities we encountered at the end of the previous section, it is instructive to go back to the Schrödinger representation and reinterpret the operator relations (3.249) and (3.250), using specific wave functions.

Since the positive- and negative-energy plane-wave solutions (3.114) and (3.115) with all possible $\mathbf{p}$ form a complete orthonormal set, the most general free-particle wave function can be written as

$$\begin{aligned} \psi(\mathbf{x}, t) = &\sum_{\mathbf{p}} \sum_{r=1,2} \sqrt{\frac{mc^2}{|E|V}}\, c_{\mathbf{p},r} u^{(r)}(\mathbf{p}) \exp\left(\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar} - \frac{i|E|t}{\hbar}\right) \\ &+ \sum_{\mathbf{p}} \sum_{r=3,4} \sqrt{\frac{mc^2}{|E|V}}\, c_{\mathbf{p},r} u^{(r)}(\mathbf{p}) \exp\left(\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar} + \frac{i|E|t}{\hbar}\right), \end{aligned} \tag{3.252}$$

where $c_{\mathbf{p},r}$ is to be determined from the Fourier expansion of ψ at $t = 0$. By appropriately choosing $c_{\mathbf{p},r}$, we can write the wave function for an arbitrarily localized free-particle wave packet in the form (3.252). Let us now evaluate $\langle \alpha_k \rangle$. A straightforward calculation using (3.220) and (3.221) gives

$$\begin{aligned} \langle \alpha_k \rangle &= \int_V \psi^\dagger(\mathbf{x}, t) \alpha_k \psi(\mathbf{x}, t) d^3x \\ &= \sum_{\mathbf{p}} \sum_{r=1,2} |c_{\mathbf{p},r}|^2 \frac{p_k c}{|E|} - \sum_{\mathbf{p}} \sum_{r=3,4} |c_{\mathbf{p},r}|^2 \frac{p_k c}{|E|} \\ &\quad + \sum_{\mathbf{p}} \sum_{r=1,2} \sum_{r'=3,4} \frac{mc^2}{|E|} [c^*_{\mathbf{p},r'} c_{\mathbf{p},r} u^{(r')\dagger}(\mathbf{p}) \alpha_k u^{(r)}(\mathbf{p})\, e^{-2i|E|t/\hbar} \\ &\quad + c_{\mathbf{p},r'} c^*_{\mathbf{p},r} u^{(r)\dagger}(\mathbf{p}) \alpha_k u^{(r')}(\mathbf{p})\, e^{2i|E|t/\hbar}]. \end{aligned} \tag{3.253}$$

The first (second) term which is time independent represents the group velocity of the wave packet made up exclusively of positive- (negative-) energy plane-wave components. The last two terms which are time dependent are more interesting. First note that α_k taken between $u^{(3,4)\dagger}(\mathbf{p})$ and $u^{(1,2)}(\mathbf{p})$ is "large" when $|\mathbf{p}| \ll mc$, in sharp contrast with α_k taken between $u^{(1,2)\dagger}(\mathbf{p})$ and $u^{(1,2)}(\mathbf{p})$ which is of order v/c. Specifically,

$$u^{(3 \text{ or } 4)\dagger}(\mathbf{p}) \alpha_k u^{(1 \text{ or } 2)}(\mathbf{p}) = \chi^{(s_-)\dagger} \sigma_k \chi^{(s_+)} + 0([|\mathbf{p}|/mc]^2), \tag{3.254}$$

where $\chi^{(s_+)}$ and $\chi^{(s_-)}$ are the two-component Pauli spinors corresponding to $u^{(1 \text{ or } 2)}$ and $u^{(3 \text{ or } 4)}$. Therefore the last two terms of (3.253) represent a superposition of violent and rapid oscillations, each with an angular frequency $\sim 2mc^2/\hbar = 1.5 \times 10^{21}$ sec^{-1} and an amplitude $\sim |c^{(1,2)*} c^{(3,4)}|$.

As for $\langle x_k \rangle$ we first observe that the operator relation (3.246) implies

$$(d/dt) \int \psi^\dagger(\mathbf{x}, t) x_k \psi(\mathbf{x}, t)\, d^3x = c \int \psi^\dagger(\mathbf{x}, t) \alpha_k \psi(\mathbf{x}, t)\, d^3x. \tag{3.255}$$

We can therefore obtain $\langle x_k \rangle$ by integrating c times (3.253) with respect to time:

$$\begin{aligned}\langle x_k \rangle = \langle x_k \rangle_{t=0} + \Bigg[& \sum_{\mathbf{p}} \sum_{r=1,2} |c_{\mathbf{p},r}|^2 \frac{p_k c^2}{|E|} t - \sum_{\mathbf{p}} \sum_{r=3,4} |c_{\mathbf{p},r}|^2 \frac{p_k c^2}{|E|} t \\ & + \sum_{\mathbf{p}} \sum_{r=1,2} \sum_{r'=3,4} \frac{i\hbar}{2mc} \left(\frac{mc^2}{|E|}\right)^2 [c^*_{\mathbf{p},r'} c_{\mathbf{p},r} u^{(r')\dagger}(\mathbf{p}) \alpha_k u^{(r)}(\mathbf{p})\, e^{-2i|E|t/\hbar} \\ & - c_{\mathbf{p}r'} c^*_{\mathbf{p}r} u^{(r)\dagger}(\mathbf{p}) \alpha_k u^{(r')}(\mathbf{p})\, e^{2i|E|t/\hbar}] \Bigg].\end{aligned} \tag{3.256}$$

Thus on top of the rectilinear motion of the wave packet we again have a superposition of violent oscillations, each with an angular frequency $\sim 2mc^2/\hbar$. If both $c_{\mathbf{p},1 \text{ or } 2}$ and $c_{\mathbf{p},3 \text{ or } 4}$ are appreciable, the fluctuation of the electron coordinate due to these oscillations is of the order of $\hbar/mc = 3.9 \times 10^{-11}$ cm. It is very important to note that the peculiar oscillatory behavior, *Zitterbewegung*, of both $\langle \alpha_k \rangle$ and $\langle x_k \rangle$ is due solely to an interference between the positive- and negative-energy components in the wave packet. The *Zitterbewegung* is completely absent for a wave packet made up exclusively of positive- (negative-) energy plane-wave solutions.

Presence of negative-energy components. At this stage we may naturally ask why we cannot simply forget all about the trouble-making negative-energy components in constructing a wave packet. This can certainly be done for a free-particle wave packet since in a potential-free region a wave packet made up exclusively of positive-energy plane waves at a given time does not develop negative-energy components at later times. On the other hand, we can show that the wave function for a well-localized particle contains, in general, plane-wave components of negative energies. As an illustration, let us consider a harmless-looking four-component wave function at $t = 0$,

$$\psi(\mathbf{x}, 0) = \begin{pmatrix} \phi(\mathbf{x}) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{3.257}$$

where $|\phi(\mathbf{x})|^2$ is assumed to be appreciable only in a region whose linear dimension is Δx_k. From our earlier discussion in Section 3–3, we see that ψ corresponds to the four-component wave function for a nonrelativistic particle with spin-up localized to $\sim \Delta x_k$. We now propose to expand this wave function in various plane-wave components so that $\psi(\mathbf{x}, 0)$ takes the form (3.252), evaluated at $t = 0$. The appropriate Fourier coefficients can be readily found by multiplying (3.257) from the left by $u^{(r)\dagger}(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{x}/\hbar}$ and integrating over the space coordinates. Although the coefficients themselves depend on the detailed form of $\phi(\mathbf{x})$, we

easily see that

$$\frac{c_{\mathbf{p},3}}{c_{\mathbf{p},1}} = -\frac{p_3 c}{|E| + mc^2}, \qquad \frac{c_{\mathbf{p},4}}{c_{\mathbf{p},1}} = -\frac{(p_1 - ip_2)c}{|E| + mc^2}. \tag{3.258}$$

This means that a negative-energy component is comparable in importance to the corresponding positive-energy component whenever ϕ has a Fourier component with momentum comparable to mc. On the other hand, we know that the Fourier transform of ϕ is appreciable in a region in momentum-space whose linear dimension is

$$\Delta p_k \sim \hbar/\Delta x_k. \tag{3.259}$$

Suppose the state in question is localized to $\Delta x_k \lesssim \hbar/mc$. The uncertainty relation (3.259) tells us that we need plane-wave components of momenta $|\mathbf{p}| \gtrsim mc$. From (3.258), we then infer that there must be appreciable amounts of negative-energy components.

We have seen that a well-localized state contains, in general, plane-wave components of negative energies. Conversely, we may ask how well localized a state will be which we can form using only plane-wave components of positive energies. A rather careful analysis by T. D. Newton and E. P. Wigner indicates that the best-localized state we can construct in this way is one in which the characteristic linear dimension of the wave packet is $\sim \hbar/mc$ but no smaller. This statement can be shown to be valid also for a Klein-Gordon particle.

It is interesting to note that a *positive*-energy bound-state wave function when expanded in free-particle plane waves usually contains some *negative*-energy components. For instance, let us take the case of the ground-state wave function of the hydrogen atom whose energy is evidently positive, $E = mc^2 - (e^2/8\pi a_0) > 0$. When this bound-state wave function (whose explicit form will be given in the next section) is expanded in free-particle plane waves, we do obtain nonvanishing coefficients for negative-energy plane waves. An immediate consequence of this is that the electron in the hydrogen atom exhibits *Zitterbewegung*. As a result, the effective potential that the electron at $\mathbf{x}$ feels is no longer just $V(\mathbf{x})$ but rather $V(\mathbf{x} + \delta\mathbf{x})$, where $\delta\mathbf{x}$ characterizes the fluctuation of the electron coordinate. Note now that $V(\mathbf{x} + \delta\mathbf{x})$ can be expanded as follows:

$$V(\mathbf{x} + \delta\mathbf{x}) = V(\mathbf{x}) + \delta\mathbf{x}\cdot\nabla V + \frac{1}{2} \delta x_i \delta x_j \frac{\partial^2 V}{\partial x_i \partial x_j} + \cdots \tag{3.260}$$

Assuming that $\mathbf{x}$ fluctuates with magnitude $|\delta\mathbf{x}| \approx \hbar/mc$ without any preferred direction, we obtain for the time average of the difference between $V(\mathbf{x} + \delta\mathbf{x}) - V(\mathbf{x})$:

$$(\Delta V)_{\text{time average}} \approx \frac{1}{2} \frac{1}{3} \left(\frac{\hbar}{mc}\right)^2 \nabla^2 V = \frac{e^2}{6} \left(\frac{\hbar}{mc}\right)^2 \delta^{(3)}(\mathbf{x}). \tag{3.261}$$

Apart from a numerical factor ($\frac{1}{6}$ instead of $\frac{1}{8}$), this is just the effective potential needed to explain the Darwin term discussed earlier in connection with the approximate treatment of the hydrogen atom (cf. Eqs. 3.83 and 3.87).

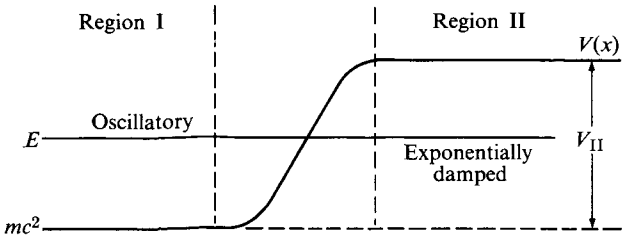


Fig. 3–4. One-dimensional potential with $mc^2 > E - V_{II} > -mc^2$.

Klein's paradox. As a final illustration of the peculiarities attending the negative-energy solutions, let us consider a simple one-dimensional potential (Fig. 3–4). In Region I the particle is free; the height of the potential in Region II is assumed to be V_{II}. When considering a region in which the potential is not varying rapidly, we can proceed directly to obtain the functional form of the wave function. For our purpose it is actually sufficient to look at ψ_A only:

$$(\boldsymbol{\sigma}\cdot\mathbf{p})c\frac{1}{E-V+mc^2}(\boldsymbol{\sigma}\cdot\mathbf{p})c\psi_A = (E-V-mc^2)\psi_A. \tag{3.262}$$

Wherever V is locally independent of x, we obtain

$$\psi_A \propto \exp\left(\frac{ipx}{\hbar} - \frac{iEt}{\hbar}\right)\chi, \quad \exp\left(-\frac{|p|x}{\hbar} - i\frac{Et}{\hbar}\right)\chi, \tag{3.263}$$

where

$$p^2c^2 = (E-V+mc^2)(E-V-mc^2). \tag{3.264}$$

With $p^2 > 0$ we have an *oscillatory* solution, while with $p^2 < 0$ we have an *exponentially damped* solution. Let us suppose that

$$mc^2 > E - V_{II} > -mc^2. \tag{3.265}$$

The Region II is a classically forbidden region ($p^2 < 0$), and the free-particle wave function in Region I dies out exponentially as it enters Region II.

So far everything has been straightforward. Let us now consider the potential given by Fig. 3–5. Our experience with nonrelativistic quantum mechanics tells us that the wave function in Region III is even *more strongly* damped. However, when the potential becomes so strongly repulsive that

$$V_{III} - E > mc^2, \tag{3.266}$$

(3.264) tells us that just the opposite is true. Since both $E-V+mc^2$ and $E-V-mc^2$ are now negative, we have $p^2 > 0$; hence the solution in Region III is *oscillatory* just as is the free-particle solution in Region I. This result is exactly the opposite of the one we set out to find. Semiclassically speaking, a particle initially confined in Region I can tunnel through Region II (just as the α-particle inside an α-emitting nucleus), and behaves in Region III as though it were in an attractive potential instead of the very strong repulsive potential implied by (3.266). This theory is named *Klein's paradox* for O. Klein, who worried about this interesting point in 1930.

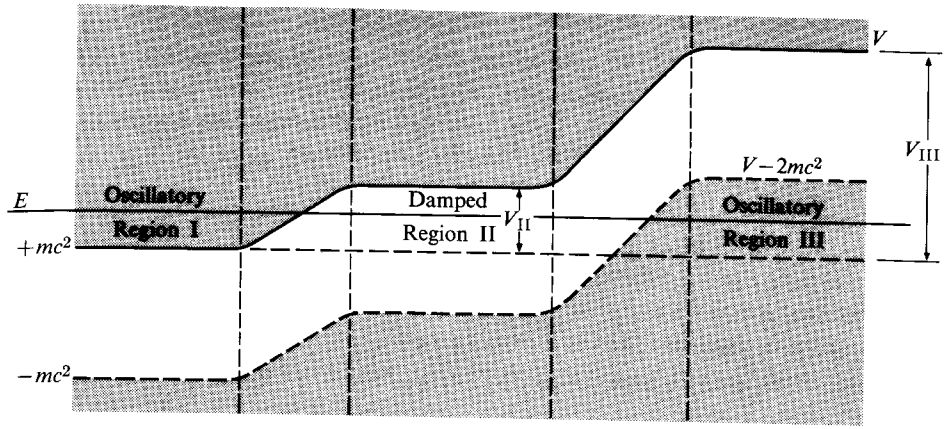


Fig. 3–5. Potential to illustrate Klein's paradox. Oscillatory solutions are expected in shaded regions.

What is the origin of this peculiar behavior? Let us recall that the free-particle solutions to the Dirac equation exhibit an energy spectrum ranging from $-mc^2$ to $-\infty$ as well as from $+mc^2$ to ∞. Now suppose we apply a small positive potential V. The condition that we have negative-energy oscillatory solutions now becomes

$$-\infty < E < -mc^2 + V. \tag{3.267}$$

As V is increased adiabatically, we see that eventually E need not even be negative for (3.267) to be satisfied. Coming back to Fig. 3–5, we see that the oscillatory solution in Region III is essentially a negative-energy solution despite $E > 0$, since it can be obtained from the wave function whose space-time dependence is $\exp[i(px/\hbar) + i(|E|t/\hbar)]$ by simply increasing V adiabatically. Klein's paradox arises because when the potential V is sufficiently positive, an oscillatory negative-energy solution in Region III can have the same positive energy as an oscillatory positive-energy solution in Region I. The tunneling of the electron from Region I into Region III must therefore be viewed as a transition from a positive-energy to a negative-energy state. We shall say more about such a transition in Section 3–9, pp. 131–143. In any case, we find that our intuitive notion that a strong positive potential can repulse the particle breaks down completely when V becomes comparable to $2mc^2$.

Similar peculiarities are present also for strongly attractive potentials. With a moderately attractive finite-ranged potential we can have bound-state solutions ($E < mc^2$) which fall off outside the range of the potential, just as they do in nonrelativistic quantum mechanics so long as the attraction does not exceed a certain critical strength. But when the potential becomes too strong, the Dirac theory starts accommodating solutions with E *less* than mc^2 which are *oscillatory and undamped outside* the range of the potential. The interested reader may verify this point in detail by studying the behavior of the Dirac particle in a deep spherical well (Problem 3–10c).

3–8. CENTRAL FORCE PROBLEMS; THE HYDROGEN ATOM

General considerations. In this section we shall first study some general properties of the wave function for an electron in a spherically symmetric potential. As we have already seen in Section 3–6, the total angular momentum operator $\mathbf{J}$ is a constant of the motion whenever the Hamiltonian is given by

$$H = c\boldsymbol{\alpha}\cdot\mathbf{p} + \beta mc^2 + V(r). \tag{3.268}$$

Let us now look for other constants of the motion. Intuitively, we expect that we must be able to specify whether the electron spin is parallel or antiparallel to the total angular momentum. In nonrelativistic quantum mechanics these two possibilities are distinguished by the eigenvalues of

$$\boldsymbol{\sigma}\cdot\mathbf{J} = \boldsymbol{\sigma}\cdot(\mathbf{L} + \hbar\boldsymbol{\sigma}/2) = (1/\hbar)(\mathbf{J}^2 - \mathbf{L}^2 + \tfrac{3}{4}\hbar^2). \tag{3.269}$$

Alternatively, we may specify l, which can be either $j + \frac{1}{2}$ or $j - \frac{1}{2}$. For a relativistic electron we might try the 4×4 generalization of (3.269), namely $\boldsymbol{\Sigma}\cdot\mathbf{J}$. However, the commutator of H with $\boldsymbol{\Sigma}\cdot\mathbf{J}$ turns out to be rather involved, as the reader may verify. Instead, then, let us try $\beta\boldsymbol{\Sigma}\cdot\mathbf{J}$ which has the same nonrelativistic limit as $\boldsymbol{\Sigma}\cdot\mathbf{J}$:

$$\begin{aligned}[H, \beta\,\boldsymbol{\Sigma}\cdot\mathbf{J}] &= [H, \beta]\,\boldsymbol{\Sigma}\cdot\mathbf{J} + \beta[H, \boldsymbol{\Sigma}]\cdot\mathbf{J} \\ &= -2c\beta(c\boldsymbol{\alpha}\cdot\mathbf{p})(\boldsymbol{\Sigma}\cdot\mathbf{J}) + 2ic\beta(\boldsymbol{\alpha}\times\mathbf{p})\cdot\mathbf{J},\end{aligned} \tag{3.270}$$

where we have used (3.233), and

$$\begin{aligned}[H, \beta] &= c\boldsymbol{\alpha}\cdot\mathbf{p}\beta - \beta c\boldsymbol{\alpha}\cdot\mathbf{p} \\ &= -2c\beta\boldsymbol{\alpha}\cdot\mathbf{p}.\end{aligned} \tag{3.271}$$

If we take advantage of

$$\begin{aligned}(\boldsymbol{\alpha}\cdot\mathbf{A})(\boldsymbol{\Sigma}\cdot\mathbf{B}) &= -\gamma_5(\boldsymbol{\Sigma}\cdot\mathbf{A})(\boldsymbol{\Sigma}\cdot\mathbf{B}) \\ &= -\gamma_5\mathbf{A}\cdot\mathbf{B} + i\boldsymbol{\alpha}\cdot(\mathbf{A}\times\mathbf{B}),\end{aligned} \tag{3.272}$$

we can simplify expression (3.270):

$$\begin{aligned}[H, \beta\,\boldsymbol{\Sigma}\cdot\mathbf{J}] &= 2c\beta\gamma_5(\mathbf{p}\cdot\mathbf{J}) \\ &= 2c\beta\gamma_5\mathbf{p}\cdot(\mathbf{L} + \hbar\boldsymbol{\Sigma}/2) \\ &= -c\hbar\beta\boldsymbol{\alpha}\cdot\mathbf{p} = (\hbar/2)[H, \beta],\end{aligned} \tag{3.273}$$

where we have used

$$\mathbf{p}\cdot\mathbf{L} = -i\hbar\nabla\cdot[\mathbf{x}\times(-i\hbar\nabla)] = 0 \tag{3.274}$$

and (3.271). Therefore an operator K defined by

$$K = \beta\,\boldsymbol{\Sigma}\cdot\mathbf{J} - \beta\hbar/2 = \beta(\boldsymbol{\Sigma}\cdot\mathbf{L} + \hbar) \tag{3.275}$$

does commute with H:

$$[H, K] = 0. \tag{3.276}$$

Furthermore, using the fact that $\mathbf{J}$ commutes with β and $\boldsymbol{\Sigma}\cdot\mathbf{L}$, we readily see that

$$[\mathbf{J}, K] = 0 \tag{3.277}$$

as well. Thus, for an electron in a central potential, we can construct a simultaneous eigenfunction of H, K, $\mathbf{J}^2$, and J_3. The corresponding eigenvalues are denoted by E, $-\kappa\hbar$, $j(j+1)\hbar^2$, and $j_3\hbar$.

We shall now derive an important relation between κ and j. First let us consider

$$\begin{aligned}K^2 &= \beta(\boldsymbol{\Sigma}\cdot\mathbf{L} + \hbar)\beta(\boldsymbol{\Sigma}\cdot\mathbf{L} + \hbar) \\ &= (\boldsymbol{\Sigma}\cdot\mathbf{L} + \hbar)^2 \\ &= \mathbf{L}^2 + i\,\boldsymbol{\Sigma}\cdot(\mathbf{L}\times\mathbf{L}) + 2\hbar\boldsymbol{\Sigma}\cdot\mathbf{L} + \hbar^2 \\ &= \mathbf{L}^2 + \hbar\boldsymbol{\Sigma}\cdot\mathbf{L} + \hbar^2.\end{aligned} \tag{3.278}$$

At the same time, since

$$\mathbf{J}^2 = \mathbf{L}^2 + \hbar\boldsymbol{\Sigma}\cdot\mathbf{L} + 3\hbar^2/4, \tag{3.279}$$

we obtain

$$K^2 = \mathbf{J}^2 + \tfrac{1}{4}\hbar^2, \tag{3.280}$$

which means that the eigenvalues of $\mathbf{J}^2$ and K^2 are related to each other by

$$\kappa^2\hbar^2 = j(j+1)\hbar^2 + \tfrac{1}{4}\hbar^2 = (j + \tfrac{1}{2})^2\hbar^2. \tag{3.281}$$

So we must have

$$\kappa = \pm(j + \tfrac{1}{2}). \tag{3.282}$$

Thus κ is a *nonzero integer* which can be positive or negative. Pictorially speaking, the sign of κ determines whether the spin is antiparallel ($\kappa > 0$) or parallel ($\kappa < 0$) to the total angular momentum in the nonrelativistic limit.

Explicitly, the operator K is given by

$$K = \begin{pmatrix} \boldsymbol{\sigma}\cdot\mathbf{L} + \hbar & 0 \\ 0 & -\boldsymbol{\sigma}\cdot\mathbf{L} - \hbar \end{pmatrix}. \tag{3.283}$$

Thus, if the four-component wave function ψ (assumed to be an energy eigenfunction) is a simultaneous eigenfunction of K, $\mathbf{J}^2$, and J_3, then

$$(\boldsymbol{\sigma}\cdot\mathbf{L} + \hbar)\psi_A = -\kappa\hbar\psi_A, \qquad (\boldsymbol{\sigma}\cdot\mathbf{L} + \hbar)\psi_B = \kappa\hbar\psi_B, \tag{3.284}$$

and

$$\begin{aligned}\mathbf{J}^2\psi_{A,B} &= (\mathbf{L} + \hbar\boldsymbol{\sigma}/2)^2\psi_{A,B} = j(j+1)\hbar^2\psi_{A,B}, \\ J_3\psi_{A,B} &= (L_3 + \hbar\sigma_3/2)\psi_{A,B} = j_3\hbar\psi_{A,B}.\end{aligned} \tag{3.285}$$

The operator $\mathbf{L}^2$ is equal to $\mathbf{J}^2 - \hbar\boldsymbol{\sigma}\cdot\mathbf{L} - \frac{3}{4}\hbar^2$ when it acts on the two-component wave functions ψ_A and ψ_B. This means that any two-component eigenfunction of $\boldsymbol{\sigma}\cdot\mathbf{L} + \hbar$ and $\mathbf{J}^2$ is automatically an eigenfunction of $\mathbf{L}^2$. Thus, although the four-component wave function ψ is not an eigenfunction of $\mathbf{L}^2$ (since H does not commute with $\mathbf{L}^2$), ψ_A and ψ_B *separately* are eigenfunctions of $\mathbf{L}^2$ whose eigenvalues are denoted by $l_A(l_A+1)\hbar^2$ and $l_B(l_B+1)\hbar^2$. From (3.284) and (3.285) we then obtain

$$-\kappa = j(j+1) - l_A(l_A+1) + \tfrac{1}{4}, \qquad \kappa = j(j+1) - l_B(l_B+1) + \tfrac{1}{4}. \tag{3.286}$$

Using (3.282) and (3.386), we can determine l_A and l_B for a given κ. The results are summarized in Table 3–2.

For a given j, l_A can assume two possible values corresponding to the two possible values of κ. This fact is already familiar from our study of nonrelativistic quantum mechanics. For example, for $j = \frac{1}{2}$, l_A can be either 0 or 1 ($s_{\frac{1}{2}}$ or $p_{\frac{1}{2}}$), depending on whether κ is negative or positive. What is new is that for a fixed κ the orbital

Table 3–2
RELATIONS AMONG κ, j, l_A, AND l_B

	l_A	l_B
$\kappa = j + \frac{1}{2}$	$j + \frac{1}{2}$	$j - \frac{1}{2}$
$\kappa = -(j + \frac{1}{2})$	$j - \frac{1}{2}$	$j + \frac{1}{2}$

parities of ψ_A and ψ_B are necessarily opposite. As we showed in Section 3–4, this result can also be derived from the requirement that the four-component wave function ψ have a definite parity (cf. Eqs. 3.172 through 3.175).

We can now write ψ as

$$\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \begin{pmatrix} g(r)\mathscr{Y}^{j_3}_{jl_A} \\ if(r)\mathscr{Y}^{j_3}_{jl_B} \end{pmatrix}, \tag{3.287}$$

where $\mathscr{Y}^{j_3}_{jl}$ stands for a normalized spin-angular function (an r-independent eigenfunction of $\mathbf{J}^2$, J_3, $\mathbf{L}^2$, and, of course, $\mathbf{S}^2$) formed by the combination of the Pauli spinor with the spherical harmonics of order l. Explicitly,

$$\mathscr{Y}^{j_3}_{jl} = \sqrt{\frac{l + j_3 + \frac{1}{2}}{2l+1}}\, Y_l^{j_3 - 1/2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{\frac{l - j_3 + \frac{1}{2}}{2l+1}}\, Y_l^{j_3 + 1/2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.288}$$

for $j = l + \frac{1}{2}$, and

$$\mathscr{Y}^{j_3}_{jl} = -\sqrt{\frac{l - j_3 + \frac{1}{2}}{2l+1}}\, Y_l^{j_3 - 1/2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{\frac{l + j_3 + \frac{1}{2}}{2l+1}}\, Y_l^{j_3 + 1/2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.289}$$

for $j = l - \frac{1}{2}$.‡ The radial functions f and g depend, of course, on κ. The factor i multiplying f has been inserted to make f and g real for bound-state (or standing-wave) solutions.

Before we substitute (3.287) in the Dirac equation written in the form

$$c(\boldsymbol{\sigma}\cdot\mathbf{p})\psi_B = (E - V(r) - mc^2)\psi_A, \quad c(\boldsymbol{\sigma}\cdot\mathbf{p})\psi_A = (E - V(r) + mc^2)\psi_B, \tag{3.290}$$

let us note that

$$\begin{aligned} \boldsymbol{\sigma}\cdot\mathbf{p} &= \frac{(\boldsymbol{\sigma}\cdot\mathbf{x})}{r^2}(\boldsymbol{\sigma}\cdot\mathbf{x})(\boldsymbol{\sigma}\cdot\mathbf{p}) \\ &= \frac{(\boldsymbol{\sigma}\cdot\mathbf{x})}{r^2}\left(-i\hbar r\frac{\partial}{\partial r} + i\boldsymbol{\sigma}\cdot\mathbf{L}\right). \end{aligned} \tag{3.291}$$

Moreover, the pseudoscalar operator $(\boldsymbol{\sigma}\cdot\mathbf{x})/r$ acting on $\mathscr{Y}^{j_3}_{jl_A}$ must give an eigenfunction of $\mathbf{J}^2$, J_3, and $\mathbf{L}^2$ with the same j and j_3 but of *opposite orbital parity*. Therefore $[(\boldsymbol{\sigma}\cdot\mathbf{x})/r]\,\mathscr{Y}^{j_3}_{jl_A}$ is equal to $\mathscr{Y}^{j_3}_{jl_B}$ itself up to a multiplicative phase factor. [*Note:* $(\boldsymbol{\sigma}\cdot\mathbf{x})^2/r^2 = 1$.] It is not difficult to show that this phase factor is *minus* one if we

‡See, for example, Merzbacher (1961), p. 402. Throughout this book we shall follow the phase convention used in Condon and Shortley (1951), Rose (1957, 1961), Merzbacher (1961), and Messiah (1962). The phase convention used by Bethe and Salpeter (1957) is slightly different due to an unconventional definition of Y_l^m.

conform to the phase convention used in writing (3.288) and (3.289). In fact, we have already verified this for the special case $j = j_3 = \frac{1}{2}$, $l_A = 0$, as seen from (3.178). Similarly $[(\boldsymbol{\sigma}\cdot\mathbf{x})/r]$ acting on $\mathscr{Y}^{j_3}_{jl_A}$ gives $\mathscr{Y}^{j_3}_{jl_B}$ apart from a minus sign. Thus

$$\begin{aligned} (\boldsymbol{\sigma}\cdot\mathbf{p})\psi_B &= i\frac{(\boldsymbol{\sigma}\cdot\mathbf{x})}{r^2}\left(-i\hbar r\frac{\partial}{\partial r} + i\boldsymbol{\sigma}\cdot\mathbf{L}\right) f\mathscr{Y}^{j_3}_{jl_A} \\ &= i\frac{(\boldsymbol{\sigma}\cdot\mathbf{x})}{r^2}\left(-i\hbar r\frac{df}{dr} + i(\kappa - 1)\hbar f\right)\mathscr{Y}^{j_3}_{jl_A} \\ &= -\hbar\frac{df}{dr}\mathscr{Y}^{j_3}_{jl_A} - \frac{(1-\kappa)\hbar}{r} f\mathscr{Y}^{j_3}_{jl_A}. \end{aligned} \tag{3.292}$$

Similarly

$$(\boldsymbol{\sigma}\cdot\mathbf{p})\psi_A = i\hbar\frac{dg}{dr}\mathscr{Y}^{j_3}_{jl_B} + i\frac{(1+\kappa)\hbar}{r} g\mathscr{Y}^{j_3}_{jl_B}. \tag{3.293}$$

We now observe that the spin-angular functions completely drop out when we rewrite (3.290), using (3.292) and (3.293):

$$\begin{aligned} -c\hbar\frac{df}{dr} - \frac{(1-\kappa)\hbar c}{r} f &= (E - V - mc^2)g, \\ c\hbar\frac{dg}{dr} + \frac{(1+\kappa)\hbar c}{r} g &= (E - V + mc^2)f. \end{aligned} \tag{3.294}$$

Introducing

$$F(r) = rf(r), \qquad G(r) = rg(r) \tag{3.295}$$

as in nonrelativistic quantum mechanics, we finally get radial equations:

$$\begin{aligned} \hbar c\left(\frac{dF}{dr} - \frac{\kappa}{r}F\right) &= -(E - V - mc^2)G, \\ \hbar c\left(\frac{dG}{dr} + \frac{\kappa}{r}G\right) &= (E - V + mc^2)F. \end{aligned} \tag{3.296}$$

Hydrogen atom. On the basis of the coupled equations (3.296) a variety of problems can be attacked. We shall consider only one problem along this line; the remaining part of this section will be devoted to a discussion of an electron bound to the atomic nucleus by a Coulomb potential. This classical problem (first treated by C. G. Darwin and W. Gordon in 1928) can be solved exactly. The reader who is interested in other central-force problems—the anomalous Zeeman effect, free spherical waves, exact solutions (as opposed to Born approximation solutions to be discussed in Chapter 4) to the Coulomb scattering problem, etc.—may consult Rose's book.‡

In order to simplify (3.296) when V is given by

$$V = -(Ze^2/4\pi r), \tag{3.297}$$

‡Rose (1961), Chapter 5.

we introduce

$$\alpha_1 = (mc^2 + E)/\hbar c, \qquad \alpha_2 = (mc^2 - E)/\hbar c,$$
$$\gamma = (Ze^2/4\pi\hbar c) = Z\alpha \simeq Z/137, \qquad \rho = \sqrt{\alpha_1\alpha_2}\, r. \tag{3.298}$$

Note that $\hbar\sqrt{\alpha_1\alpha_2} = \sqrt{m^2c^4 - E^2}/c$ is just the magnitude of the imaginary momentum of an electron of energy E. The coupled equations we must solve are

$$\left(\frac{d}{d\rho} - \frac{\kappa}{\rho}\right)F - \left(\sqrt{\frac{\alpha_2}{\alpha_1}} - \frac{\gamma}{\rho}\right)G = 0, \qquad \left(\frac{d}{d\rho} + \frac{\kappa}{\rho}\right)G - \left(\sqrt{\frac{\alpha_1}{\alpha_2}} + \frac{\gamma}{\rho}\right)F = 0. \tag{3.299}$$

As in the nonrelativistic treatment of the hydrogen atom, we seek solutions to (3.299) of the form

$$F = e^{-\rho}\rho^s \sum_{m=0} a_m \rho^m, \qquad G = e^{-\rho}\rho^s \sum_{m=0} b_m \rho^m. \tag{3.300}$$

Substituting (3.300) in (3.299), and equating the coefficients of $e^{-\rho}\rho^s\rho^{q-1}$, we obtain the recursion relations

$$(s + q - \kappa)a_q - a_{q-1} + \gamma b_q - \sqrt{\alpha_2/\alpha_1}\, b_{q-1} = 0,$$
$$(s + q + \kappa)b_q - b_{q-1} - \gamma a_q - \sqrt{\alpha_1/\alpha_2}\, a_{q-1} = 0. \tag{3.301}$$

For $q = 0$, we have

$$(s - \kappa)a_0 + \gamma b_0 = 0, \qquad (s + \kappa)b_0 - \gamma a_0 = 0. \tag{3.302}$$

Since a_0 and b_0 are not zero, the secular determinant of (3.302) must vanish; hence

$$s = \pm\sqrt{\kappa^2 - \gamma^2}. \tag{3.303}$$

We must require that $\int \psi^\dagger\psi d^3x$ be finite. This requirement amounts to

$$\int |F|^2 d\rho < \infty, \qquad \int |G|^2 d\rho < \infty. \tag{3.304}$$

Thus F and G must behave better than $\rho^{-1/2}$ at the origin, which means $s > -\frac{1}{2}$. Since

$$\kappa^2 - \gamma^2 \geq \min(\kappa^2) - \gamma^2 \simeq 1 - (Z/137)^2, \tag{3.305}$$

the above requirement cannot be satisfied if we take the negative root of (3.303). So we are led to take the positive root.‡

It is not difficult to show that F and G would increase exponentially as $\rho \longrightarrow \infty$ (that is, $F, G \sim e^{+\rho}$ at infinity) if the power series (3.300) did not terminate.§ Assuming that the two series terminate with the same power, there must exist n' with the property

$$a_{n'+1} = b_{n'+1} = 0, \qquad a_{n'} \neq 0, \qquad b_{n'} \neq 0. \tag{3.306}$$

‡For $|\kappa| = 1$, $f = F/r$ and $g = G/r$ diverge at the origin (since $s < 1$); yet (3.304) is satisfied.

§The e^{ρ} behavior for F and G at infinity is allowed if $E > mc^2$, which means a purely imaginary ρ. Indeed the oscillatory behavior of the radial functions at infinity is characteristic of scattering-state solutions which exhibit a continuous energy spectrum.

Setting $q = n' + 1$ in (3.301) we obtain

$$a_{n'} = -\sqrt{\alpha_2/\alpha_1}\, b_{n'} \tag{3.307}$$

from both parts of (3.301) (which incidentally justifies our assumption that the two series terminate with the same power). To write an equation that involves only $a_{n'}$ and $b_{n'}$ whose ratio is now known, we multiply the first of (3.301) by α_1, the second by $\sqrt{\alpha_1\alpha_2}$, set $q = n'$ this time, and subtract:

$$[\alpha_1(s + n' - \kappa) + \sqrt{\alpha_1\alpha_2}\,\gamma]a_{n'} - [\sqrt{\alpha_1\alpha_2}\,(s + n' + \kappa) - \alpha_1\gamma]b_{n'} = 0; \tag{3.308}$$

hence

$$2\sqrt{\alpha_1\alpha_2}\,(s + n') = \gamma(\alpha_1 - \alpha_2), \tag{3.309}$$

or

$$\sqrt{(mc^2)^2 - E^2}\,(s + n') = E\gamma. \tag{3.310}$$

Thus we obtain the energy eigenvalues‡

$$E = \frac{mc^2}{\sqrt{1 + \dfrac{\gamma^2}{(s + n')^2}}} = \frac{mc^2}{\sqrt{1 + \dfrac{Z^2\alpha^2}{(n' + \sqrt{(j + \frac{1}{2})^2 - Z^2\alpha^2})^2}}}. \tag{3.311}$$

Note that E depends only on n' and $j + \frac{1}{2} = |\kappa|$.

In order to compare (3.311) with the corresponding expression obtained in the Schrödinger theory, we define

$$n = n' + (j + \tfrac{1}{2}) = n' + |\kappa|. \tag{3.312}$$

Since the minimum value of n' is zero, we have

$$n > j + \tfrac{1}{2} = |\kappa|, \tag{3.313}$$

which is at least unity. Expanding (3.311) we get

$$E = mc^2\left[1 - \frac{1}{2}\frac{(Z\alpha)^2}{n^2} - \frac{1}{2}\frac{(Z\alpha)^4}{n^3}\left(\frac{1}{j + \frac{1}{2}} - \frac{3}{4n}\right) - \cdots\right]. \tag{3.314}$$

Since§

$$\tfrac{1}{2}\alpha^2 mc^2 = e^2/(8\pi a_{\text{Bohr}}), \tag{3.315}$$

we see that n is indeed identical with the familiar *principal quantum number* in nonrelativistic quantum mechanics. Note also that the leading correction to the Balmer formula is precisely the fine-structure splitting (3.88) which tells us that, for a given n, higher j-states are at higher levels.

In the Dirac theory each state of a hydrogen atom can be completely characterized by n', κ, and j_3. We can translate this classification scheme into the more familiar one based on spectroscopic notation. This is done in Table 3–3 which can be obtained with the help of Table 3–2 and Eqs. (3.312) and (3.313). Note that although $\mathbf{L}^2$ is not "good" in the relativistic theory, it is customary to use the

‡Formula (3.311) was first obtained by A. Sommerfeld, using a relativistic version of N. Bohr's old quantum theory.

§In this section we shall use a_{Bohr} rather than a_0 for the Bohr radius to avoid a possible confusion with the coefficient a_0 in (3.300).

Table 3–3

RELATIVISTIC QUANTUM NUMBERS AND SPECTROSCOPIC NOTATION

A pair of states which have the same energy according to (3.311) are denoted by "deg."

n	$n' = n - \lvert\kappa\rvert \geq 0$	$\kappa = \pm(j + \frac{1}{2})$	Notation
1	0	−1	$1s_{\frac{1}{2}}$
2	1	−1	$2s_{\frac{1}{2}}$ } deg
2	1	+1	$2p_{\frac{1}{2}}$ }
2	0	−2	$2p_{\frac{3}{2}}$
3	2	−1	$3s_{\frac{1}{2}}$ } deg
3	2	+1	$3p_{\frac{1}{2}}$ }
3	1	−2	$3p_{\frac{3}{2}}$ } deg
3	1	+2	$3d_{\frac{3}{2}}$ }
3	0	−3	$3d_{\frac{5}{2}}$

notation $p_{3/2}$, etc., which actually means $l_A = 1$ with $j = \frac{3}{2}$, etc. In other words the orbital angular momentum of the upper two-component wave function (which becomes the wave function of the Schrödinger-Pauli theory in the nonrelativistic limit) determines the orbital angular momentum in the spectroscopic language. The reader may wonder why we have omitted in Table 3–2 the $\kappa > 0$ states whenever $n' = 0$. The reason for this omission is evident if we go back to the second expression of (3.302) and (3.307) which together imply

$$(s + \kappa)/\gamma = -\sqrt{\alpha_2/\alpha_1}, \qquad n' = 0 \text{ only.} \tag{3.316}$$

This can be satisfied only if κ is negative because s is a positive number smaller than $|\kappa|$ (cf. Eq. 3.303). The absence of the $\kappa > 0$ states for $n' = 0$ corresponds to the familiar rule in nonrelativistic quantum mechanics: The maximum value of l is $n - 1$, not n.

For the ground state ($n' = 0$, $\kappa = -1$), the relation (3.311) simplifies to

$$E_{gd} = mc^2\sqrt{1 - (Z\alpha)^2}. \tag{3.317}$$

So

$$\sqrt{\alpha_1\alpha_2} = Z\alpha mc^2/\hbar c = Z/a_{\text{Bohr}}, \tag{3.318}$$

and

$$\frac{b_0}{a_0} = -\frac{Z\alpha}{1 + \sqrt{1 - (Z\alpha)^2}} = -\frac{(1 - \sqrt{1 - (Z\alpha)^2})}{Z\alpha}. \tag{3.319}$$

Up to an overall multiplicative constant denoted by N we can readily write the ground-state wave function

$$\psi_{gd} = \frac{N}{\sqrt{\pi}}\left(\frac{Z}{a_{\text{Bohr}}}\right)^{3/2} e^{-Zr/a_{\text{Bohr}}}\left(\frac{Zr}{a_{\text{Bohr}}}\right)^{\sqrt{1-(Z\alpha)^2}-1}\begin{pmatrix}\chi^{(s)} \\ \dfrac{i(1 - \sqrt{1 - (Z\alpha)^2})}{Z\alpha}\dfrac{(\boldsymbol{\sigma}\cdot\mathbf{x})}{r}\chi^{(s)}\end{pmatrix}, \tag{3.320}$$

where $\chi^{(s)}$ is the Pauli spinor

$$\begin{pmatrix}1\\0\end{pmatrix} \text{ or } \begin{pmatrix}0\\1\end{pmatrix},$$

depending on whether $j_3 = \frac{1}{2}$ or $-\frac{1}{2}$. A straightforward calculation shows that the normalization requirement for ψ gives

$$N = 2^{\sqrt{1-(Z\alpha)^2}-1}\sqrt{\frac{1 + \sqrt{1 - (Z\alpha)^2}}{\Gamma(1 + 2\sqrt{1 - (Z\alpha)^2})}}, \tag{3.321}$$

where

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt, \qquad \Gamma(m) = (m - 1)! \quad \text{for} \quad m = \text{positive integer}. \tag{3.322}$$

Note that N approaches 1 as $Z\alpha \to 0$. Furthermore $(Zr/a_{\text{Bohr}})^{\sqrt{1-(Z\alpha)^2}-1}$ is essentially unity except at distances of order

$$r \sim \frac{137\hbar}{2mcZ}e^{-2(137)^2/Z^2}. \tag{3.323}$$

As $r \to 0$, ψ exhibits a mild singularity. This, however, is of academic interest only, since the wave function at short distances must be modified because of the finite charge distribution of the nucleus. Thus we see that for the ground states of hydrogen-like atoms with low Z, the upper two-component wave function is essentially identical to the Schrödinger wave function multiplied by a Pauli spinor. As for the lower two-component wave function, we merely remark that, apart from $i(\boldsymbol{\sigma}\cdot\mathbf{x})/r$, the ratio of the lower to the upper components is given by (cf. Eq. 3.307)

$$-\frac{a_0}{b_0} = \sqrt{\frac{mc^2 - E_{gd}}{mc^2 + E_{gd}}} \simeq \sqrt{\left(\frac{mv^2}{2}\right)\left(\frac{1}{2mc^2}\right)} = \frac{v}{2c}, \tag{3.324}$$

where v is the "velocity" of the electron in Bohr's circular orbit theory. This result is in agreement with our earlier discussion in Section 3–3 on the ratio of ψ_B to ψ_A.

In 1947 W. E. Lamb and R. C. Retherford observed a splitting between the $2s_{\frac{1}{2}}$- and $2p_{\frac{1}{2}}$-states of the hydrogen atom not given by (3.311). As already discussed in Section 2–8, the main part of this "Lamb shift" can be satisfactorily accounted for when we consider the interaction of the electron with the quantized radiation field.

Another important effect not contained in (3.311) arises from the interaction between the magnetic moment of the nucleus and the magnetic moment of the electron. In the case of the hydrogen atom, for instance, when we compound the electron spin with the proton spin, the net result is $F = 1$ (triplet) or $F = 0$ (singlet), where F is the quantum number corresponding to the *total* spin. Since the magnetic interaction is dependent on the relative orientation of the two magnetic dipole moments, each level of the hydrogen atom characterized by n, j, $l(= l_A)$ is split further into two sublevels corresponding to the two possible values of F

even in the absence of any external magnetic field. This is known as a *hyperfine splitting*. Let us estimate it for the s-states using nonrelativistic quantum mechanics. Classically the magnetic field created by the proton magnetic moment $\mathbf{M}^{(\text{class})}$ is‡

$$\mathbf{B} = \nabla \times \left(\mathbf{M}^{(\text{class})} \times \nabla \frac{1}{4\pi r}\right). \tag{3.325}$$

Quantum-mechanically we replace $\mathbf{M}^{(\text{class})}$ by the magnetic moment operator

$$\mathbf{M} = \frac{|e|\hbar(1+\kappa)}{2m_p c}\boldsymbol{\sigma}_p, \tag{3.326}$$

where $\boldsymbol{\sigma}_p$ is the Pauli matrix for the proton spin and $2(1+\kappa)$ is the g-factor of the proton. (We assume that the magnetic moment distribution of the proton is point-like.) Within the framework of the Schrödinger-Pauli theory we obtain the interaction Hamiltonian operator

$$\begin{aligned} H^{(\text{hf})} &= -\boldsymbol{\mu}\cdot\nabla \times \left(\mathbf{M} \times \nabla \frac{1}{4\pi r}\right) \\ &= (\boldsymbol{\mu}\cdot\mathbf{M})\nabla^2\left(\frac{1}{4\pi r}\right) - [(\boldsymbol{\mu}\cdot\nabla)(\mathbf{M}\cdot\nabla)]\frac{1}{4\pi r} \\ &= \tfrac{2}{3}(\boldsymbol{\mu}\cdot\mathbf{M})\nabla^2\left(\frac{1}{4\pi r}\right) - \left[(\boldsymbol{\mu}\cdot\nabla)(\mathbf{M}\cdot\nabla) - \frac{1}{3}\boldsymbol{\mu}\cdot\mathbf{M}\nabla^2\right]\frac{1}{4\pi r}, \end{aligned} \tag{3.327}$$

where $\boldsymbol{\mu} = (e\hbar/2m_e c)\boldsymbol{\sigma}_e$. The quantity in the brackets transforms like a traceless tensor of rank two; so when it is integrated with a function of $\mathbf{x}$, $f(\mathbf{x})$, it gives a nonvanishing contribution only if the expansion of $f(\mathbf{x})$ in spherical harmonics contains Y_2^m. For the spherically symmetric s states, only the first term of (3.327) is relevant. Using the nonrelativistic wave function ψ_n, we obtain the energy shift

$$\begin{aligned} \Delta E_n &= -\tfrac{2}{3}\boldsymbol{\mu}\cdot\mathbf{M}\int \delta^{(3)}(\mathbf{x})|\psi_n(\mathbf{x})|^2 d^3x \\ &= \left[\frac{(1+\kappa)e^2\hbar^2}{6m_e m_p c^2}\right]\left[\frac{1}{\pi}\left(\frac{1}{a_{\text{Bohr}}n}\right)^3\right]\boldsymbol{\sigma}_e\cdot\boldsymbol{\sigma}_p \\ &= \tfrac{2}{3}\alpha^4(1+\kappa)\left(\frac{m_e}{m_p}\right)\frac{m_e c^2}{n^3}\begin{cases} 1, & F=1, \\ -3, & F=0, \end{cases} \end{aligned} \tag{3.328}$$

as first shown by E. Fermi in 1930. Note that the order of magnitude of this splitting is the fine-structure splitting multiplied by (m_e/m_p). For the $1s\frac{1}{2}$-state, the above energy difference corresponds to a radio microwave of 1420 Mc or 21 cm; the radiative transition between these two hyperfine levels is of fundamental importance in radio astronomy. We may parenthetically mention that this energy difference is one of the most accurately measured quantities in modern physics; S. B. Crampton, D. Kleppner, and N. F. Ramsey have determined that the corresponding radiofrequency is $(1420.405751800 \pm 0.000000028)$ Mc.

There are other corrections to formula (3.311). First, we must take into account the motion of the nucleus since the mass of the nucleus is not infinite; a major part of this correction can be taken care of if we use everywhere the reduced mass $m_e m_p/(m_e + m_p)$ in place of m_e. Second, there are other contributions to the Lamb shift not discussed in Chapter 2; especially important is the vacuum polarization effect to be discussed later. Third, the finite size of the nucleus also modifies formula (3.311) especially for the s states which are sensitive to small deviations from Coulomb's law at close distances; in the interesting case of the $2s$ state of the hydrogen atom, however, we can estimate the energy shift due to this effect to be only 0.1 Mc, using the observed proton charge radius $\sim 0.7 \times 10^{-13}$ cm.

‡Panofsky and Phillips (1955), p. 120; Jackson (1962) p. 146.

The utility of the Dirac theory in atomic physics is not limited to light hydrogen-like atoms. For heavy atoms where $(Z\alpha)^2$ is not very small compared with unity (0.45 for uranium), the relativistic effects must be taken into account even for understanding the qualitative features of the energy levels. Although we cannot, in practice, study one-electron ions of heavy atoms, it is actually possible to check the quantitative predictions of the Dirac theory by looking at the energy levels of the innermost (K-shell and L-shell) electrons of high Z atoms which can be inferred experimentally from X-ray spectra. Similar studies have been carried out with muonic atoms (atoms in which one of the electrons is replaced by a negative muon).

Although we shall not discuss the emission and absorption of radiation using the Dirac theory, the results of Section 2–4 are applicable *mutatis mutandis*. All we need to do is make the following replacements:‡

$$\begin{gathered} -\frac{e}{2mc}(\mathbf{p}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{p}) - \frac{e\hbar}{2mc}\boldsymbol{\sigma}\cdot\mathbf{B} \longrightarrow -e\boldsymbol{\alpha}\cdot\mathbf{A} \\ \psi^{(\text{Schrödinger-Pauli})} \longrightarrow \psi^{(\text{Dirac})} \\ \psi^{(\text{Schrödinger-Pauli})\dagger} \longrightarrow \psi^{(\text{Dirac})\dagger} \quad (\text{not } \bar{\psi}^{(\text{Dirac})}). \end{gathered} \tag{3.329}$$

3–9. HOLE THEORY AND CHARGE CONJUGATION

Holes and positrons. Although we have shown that the Dirac theory accommodates negative-energy solutions whose existence should not be ignored altogether, we have as yet not examined their physical significance. This section is devoted to the physical interpretation of the negative-energy states within the framework of a theory in which the electron field is not quantized.

As a simple example to illustrate some of the difficulties with the *original* Dirac theory of 1928, let us consider an atomic electron. According to the quantum theory of radiation developed in the previous chapter, an excited atomic state can lose its energy by spontaneously emitting a photon even in the absence of any external field. This is why all atomic states, with the exception of the ground states, have finite lifetimes. In the Dirac theory, however, the so-called ground state of an atom is not really the lowest state since there exists a continuum of negative-energy states from $-mc^2$ to $-\infty$ for any potential that vanishes at

‡We have to be a little more careful when we treat a process in which the quadratic $\mathbf{A}^2$ term in the nonrelativistic Hamiltonian is important. This point will be discussed in the next section with reference to Thomson scattering.

infinity. We know that an excited atomic state makes a radiative transition to the ground state; similarly, we expect that the atomic electron in the ground state with energy $mc^2 - |E_{BE}|$ can emit spontaneously a photon of energy $\gtrsim 2mc^2$ and fall into a negative-energy state. Furthermore, once it reaches a negative-energy state, it will keep on lowering its energy indefinitely by emitting photons since there is *no lower bound* to the negative-energy spectrum. Since we know that the ground state of an atom is stable, we must somehow prevent such catastrophic transitions.

Faced with this difficulty, Dirac proposed, in 1930, that *all the negative-energy states are completely filled under normal conditions.* The catastrophic transitions mentioned above are then prevented because of the Pauli exclusion principle. What we usually call the vacuum is actually an infinite sea of negative-energy electrons. Occasionally one of the negative-energy electrons in the Dirac sea can absorb a photon of energy $\hbar\omega > 2mc^2$ and become an $E > 0$ state. As a result, a "hole" is created in the Dirac sea. The *observable* energy of the Dirac sea is now the energy of the vacuum *minus* the *negative* energy of the vacated state, hence a *positive* quantity. In this way we expect that the absence of a negative-energy electron appears as the presence of a positive-energy particle. Similarly, when a hole is created in the Dirac sea, the total charge of the Dirac sea becomes

$$Q = Q_{\text{vacuum}} - e = Q_{\text{vacuum}} - (-|e|) = Q_{\text{vacuum}} + |e|; \tag{3.330}$$

hence the *observable* charge of the hole is

$$Q_{\text{obs}} = Q - Q_{\text{vacuum}} = |e|. \tag{3.331}$$

This means that a hole in the sea of negative-energy states looks like a positive-energy particle of charge $|e|$. Thus once we accept (a) that the negative-energy states are completely filled under normal conditions and (b) that a negative-energy electron can absorb a photon of energy $> 2mc^2$ (just as a positive-energy electron can) to become a positive-energy electron, we are unambiguously led to predict the existence of a particle of charge $|e|$ with a positive energy.

When Dirac proposed this "hole theory," there was no good candidate for the predicted positively charged particle. In the beginning Dirac even thought that the hole in the negative-energy state should be identified with the proton. However, it was quickly pointed out by J. R. Oppenheimer that if this interpretation were correct, the hydrogen atom would undergo a self-annihilation into two photons with a lifetime $\sim 10^{-10}$ sec.‡ Moreover, H. Weyl, who looked at the symmetry properties of the Dirac equation, proved that the mass of the particle associated with the hole must be the same as the electron mass. Prior to 1932, because of the experimental absence of the conjectured particle, Dirac's hole theory was not taken seriously. To recapture the prevailing atmosphere of the time, we quote from W. Pauli's *Handbuch* article.§

‡This number assumes that the energy released is $2m_ec^2$. If we take the energy released to be $m_ec^2 + m_pc^2$, the lifetime is even shorter.

§The translation from the original German text is the work of J. Alexander, G. F. Chew, W. Selove, and C. N. Yang.

> Recently Dirac attempted the explanation, already discussed by Oppenheimer, of identifying the holes with antielectrons, particles of charge $+|e|$ and the electron mass. Likewise, in addition to protons, there must be antiprotons. The experimental absence of such particles is then traced back to a special initial state in which only one of the two kinds of particles is present. We see that this already appears to be unsatisfactory because the laws of nature in this theory with respect to electrons and antielectrons are exactly symmetrical. Thus γ-ray photons (at least two in order to satisfy the laws of conservation of energy and momentum) must be able to transform, by themselves, into an electron and an antielectron. We do not believe, therefore, that this explanation can be seriously considered.

When the article appeared in print, however, C. D. Anderson had already demonstrated the existence of a positron. Many years later Pauli made the following famous remark on Dirac:

> . . . with his fine instinct for physical realities he started his argument without knowing the end of it.

We shall now examine a little more closely the absorption of a photon by one of the negative-energy electrons in the Dirac sea. As stated earlier, if the photon energy is sufficiently large, an electron in a negative-energy state may be "escalated" to a positive-energy state

$$e^-_{E<0} + \gamma \longrightarrow e^-_{E>0}. \tag{3.332}$$

According to the hole-theoretic interpretation, this appears as

$$\gamma \longrightarrow e^-_{E>0} + e^+_{E>0}, \tag{3.333}$$

since the vacated negative-energy state is observable as a positron state. Although a photon cannot produce an e^-e^+ pair in free space without violating energy and momentum conservation, the process (3.333) can take place in the Coulomb field of a nucleus. As is well known, the production of an electron-positron pair is a very frequent phenomenon when high-energy γ-rays go through matter. We may also consider a closely related process,

$$e^-_{E>0} \longrightarrow e^-_{E<0} + 2\gamma. \tag{3.334}$$

Since all the negative-energy states are supposed to be filled under normal conditions, (3.334) is forbidden except when there is a hole in the normally filled negative-energy states. This means that whenever (3.334) is allowed, we can interpret it as

$$e^-_{E>0} + e^+_{E>0} \longrightarrow 2\gamma. \tag{3.335}$$

This process has also been observed frequently as positrons slow down in solids. We shall present a quantitative treatment of this electron-positron annihilation process in Chapter 4.

At this stage we emphasize again that the electron must obey the Pauli exclusion principle if the hole theory is to make sense. Otherwise we cannot attach much meaning to the notion that the negative-energy states are completely filled. If it were not for the exclusion principle, we could keep on, for millions of years, piling up electrons in the *same* negative-energy state. Even though the energy spectrum

Table 3–4
DYNAMICAL QUANTITIES IN THE HOLE THEORY

	Charge	Energy	Momentum	Spin	Helicity	"Velocity"
$E<0$ Electron state	$-\lvert e\rvert$	$-\lvert E\rvert$	$\mathbf{p}$	$\frac{\hbar\langle\boldsymbol{\Sigma}\rangle}{2}$	$\boldsymbol{\Sigma}\cdot\hat{\mathbf{p}}$	$\mathbf{v}$
Positron state	$+\lvert e\rvert$	$+\lvert E\rvert$	$-\mathbf{p}$	$-\frac{\hbar\langle\boldsymbol{\Sigma}\rangle}{2}$	$\boldsymbol{\Sigma}\cdot\hat{\mathbf{p}}$	$\mathbf{v}$

of free Klein-Gordon particles is identical to that of free Dirac particles, it is not possible to construct a sensible hole theory out of Klein-Gordon particles which obey Bose-Einstein statistics.

Let us study the connection between the various dynamical quantities of the positron and those of the negative-energy electron whose absence appears as the presence of the positron in question. We have already seen that both the charge and the energy of the physical positron must be positive. What is the momentum of the positron? Just as in the case of energy, the absence of momentum $\mathbf{p}$ in the Dirac sea appears as the presence of momentum $-\mathbf{p}$. Hence the momentum of the physical ($E>0$) positron state is opposite to that of the corresponding negative-energy electron state. Similarly the absence of a spin-up $E<0$ electron is to be interpreted as the presence of a spin-down $E>0$ positron. Thus we can construct Table 3–4 for a free particle. (We have listed $\langle\boldsymbol{\Sigma}\rangle$ rather than the eigenvalue of Σ_3 since, in general, the plane-wave solutions are not eigenstates of Σ_3).

The entry "velocity" in Table 3–4 requires some explanation. Suppose we consider a wave packet made up of negative-energy solutions whose momenta center around a certain mean value. We can then associate a certain group-velocity with the wave packet. The absence of this $E<0$ wave packet must appear as a wave packet made up of $E>0$ positron states moving in the *same* direction, that is, the velocity of the positron wave packet must be the *same* as that of the corresponding $E<0$ electron wave packet. It is not hard to see that this is possible only if the "velocity" of the negative-energy electron is *opposite* in direction to its momentum. This appears somewhat strange but is completely consistent with (3.221), which says that the expectation value of the velocity operator $c\boldsymbol{\alpha}$ is the negative of the expectation value of $\mathbf{p}c^2/\lvert E\rvert$. The reader who is still not convinced may amuse himself by working out steps analogous to (3.163) through (3.170) for a negative-energy plane wave. If we apply S_{Lor}^{-1} to the wave function $\psi^{(3,4)}$ for an $E<0$ electron at rest in the primed system, we obtain the wave function $\psi^{(3,4)}$ which corresponds to the negative-energy electron whose momentum (defined as the eigenvalue of $-i\hbar\nabla$) in the unprimed system is *opposite* to the direction of motion of the primed system.

Thomson scattering in the Dirac theory. As a simple calculation that dramatically illustrates the importance of the negative-energy states in an unexpected domain, we shall now compute the cross section for Thomson scattering, that is, the scattering of a low-energy photon ($\hbar\omega \ll mc^2$) by a free electron. As in Section 2–5 (cf. Eqs. 2.158 and 2.168) we expect that the differential cross section is given by $r_0^2\lvert\boldsymbol{\epsilon}^{(\alpha)}\cdot\boldsymbol{\epsilon}^{(\alpha')}\rvert^2$, which is also the same as the classical result. In the Dirac theory there is no analog of the seagull graph Fig. 2–2 (c); we must compute the analogs of Fig. 2–2 (a) and (b) for the free electron. We characterize the initial, final, and intermediate states of the electron by $(\mathbf{p}, r)$, $(\mathbf{p}', r')$, and $(\mathbf{p}'', r'')$ respectively. We then obtain for the transition matrix element,

$$-\frac{e^2c^2\hbar}{2V\sqrt{\omega\omega'}}\sum_{\mathbf{p}''}\sum_{r''=1,2}\left(\frac{\langle\mathbf{p}'r'\rvert\boldsymbol{\alpha}\cdot\boldsymbol{\epsilon}^{(\alpha')}e^{-i\mathbf{k}'\cdot\mathbf{x}}\rvert\mathbf{p}''r''\rangle\langle\mathbf{p}''r''\rvert\boldsymbol{\alpha}\cdot\boldsymbol{\epsilon}^{(\alpha)}e^{i\mathbf{k}\cdot\mathbf{x}}\rvert\mathbf{p}r\rangle}{E''-E-\hbar\omega}\right.$$
$$\left.+\frac{\langle\mathbf{p}'r'\rvert\boldsymbol{\alpha}\cdot\boldsymbol{\epsilon}^{(\alpha)}e^{i\mathbf{k}\cdot\mathbf{x}}\rvert\mathbf{p}''r''\rangle\langle\mathbf{p}''r''\rvert\boldsymbol{\alpha}\cdot\boldsymbol{\epsilon}^{(\alpha')}e^{-i\mathbf{k}'\cdot\mathbf{x}}\rvert\mathbf{p}r\rangle}{E''-E+\hbar\omega'}\right), \tag{3.336}$$

where we have used the rule stated in (3.329). Since all the negative-energy states are supposed to be filled, the summation is over positive-energy states only ($r''=1,2$). The electron is initially at rest. So, as $\mathbf{p}\longrightarrow 0$, $\mathbf{k}\longrightarrow 0$, a typical matrix element in (3.336) becomes

$$\langle\mathbf{p}''r''\rvert\boldsymbol{\alpha}\cdot\boldsymbol{\epsilon}^{(\alpha)}\rvert 0r\rangle = (1/V)\sqrt{mc^2/E''}\int e^{-i\mathbf{p}''\cdot\mathbf{x}/\hbar}u^{(r'')\dagger}(\mathbf{p}'')(\boldsymbol{\alpha}\cdot\boldsymbol{\epsilon}^{(\alpha)})u^{(r)}(0)d^3x$$
$$=\delta_{\mathbf{p}''0}u^{(r'')\dagger}(\mathbf{p}'')(\boldsymbol{\alpha}\cdot\boldsymbol{\epsilon}^{(\alpha)})u^{(r)}(0)=0, \tag{3.337}$$

since the matrix element of α_k taken between two at-rest $E>0$ spinors vanishes. Because the final electron is also at rest for the scattering of a very soft photon, we easily see that (3.336) is identically zero. This means that the Thomson scattering cross section should vanish, in contradiction to both observation and nonrelativistic quantum mechanics.

What went wrong? In the hole theory we must take into account an additional process which has no analog in nonrelativistic quantum mechanics. Consider a negative-energy electron in the Dirac sea. It can absorb the incident photon (say, at $t=t_1$) and become a positive-energy electron. Even though this virtual transition does not conserve energy (unless $\hbar\omega>2mc^2$), there is a finite matrix element for it. At a subsequent time ($t=t_2$) the initial electron can fill up the vacated negative-energy state by emitting the outgoing photon. Meanwhile the escalated electron goes on as the positive-energy final-state electron. All this may be visualized physically as follows. The incident photon creates an electron-positron pair at $t=t_1$; subsequently at $t=t_2$ the positron is annihilated by the initial electron, emitting the outgoing photon, as shown in Fig. 3–6(a). Similarly it is possible for the outgoing photon to be emitted first as one of the $E<0$ electrons in the Dirac sea is escalated; subsequently the initial electron fills up the vacated negative-energy state by absorbing the incident photon. This is illustrated in Fig. 3–6(b), which physically represents the creation of an electron-positron pair plus the outgoing photon followed by the annihilation of the positron with the initial electron and the incident photon.

We shall now calculate the matrix elements for the two diagrams. According to the hole theory, for electrons, the initial state is made up of the incident electron and the Dirac sea. In Fig. 3–6(a), one of the negative-energy electrons denoted

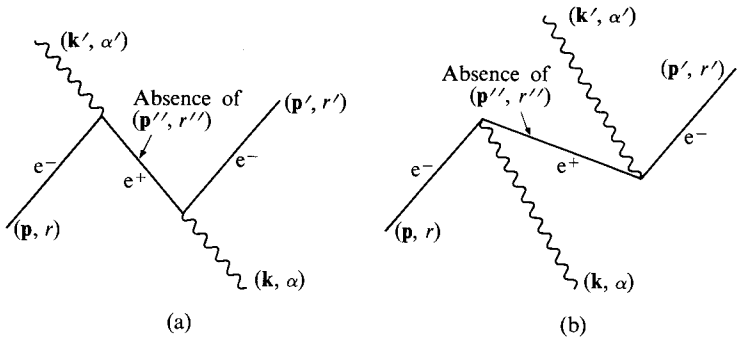


Fig. 3–6. Thomson scattering in the Dirac theory.

by $(\mathbf{p}'', r'')$ makes a transition to a positive-energy state $(\mathbf{p}', r')$ by absorbing the photon $(\mathbf{k}, \alpha)$. The relevant matrix element is $-ec\sqrt{\hbar/2V\omega}\langle \mathbf{p}'r' | \boldsymbol{\alpha}\cdot\boldsymbol{\epsilon}^{(\alpha)} e^{i\mathbf{k}\cdot\mathbf{x}} | \mathbf{p}''r''\rangle$. Since the absorption takes place first, the energy denominator according to the rules of Chapter 2 is $E_I - E_A - \hbar\omega$, where $E_I = E_{\text{vac}} - (-|E''|) + E + E'$ and $E_A = E_{\text{vac}} + E$. For the transition at t_2 we have the matrix element $-ec\sqrt{\hbar/2V\omega}\langle \mathbf{p}''r'' | \boldsymbol{\alpha}\cdot\boldsymbol{\epsilon}^{(\alpha')} e^{-i\mathbf{k}\cdot\mathbf{x}} | \mathbf{p}r\rangle$. Working out Fig. 3–6(b) in a similar way, we obtain for the transition matrix element for both diagrams combined:

$$-\frac{e^2c^2\hbar}{2V\sqrt{\omega\omega'}}\sum_{\mathbf{p}}\sum_{r''=3,4}\left(\frac{\langle \mathbf{p}''r'' | \boldsymbol{\alpha}\cdot\boldsymbol{\epsilon}^{(\alpha')} e^{-i\mathbf{k}'\cdot\mathbf{x}} | \mathbf{p}r\rangle\langle \mathbf{p}'r' | \boldsymbol{\alpha}\cdot\boldsymbol{\epsilon}^{(\alpha)} e^{i\mathbf{k}\cdot\mathbf{x}} | \mathbf{p}''r''\rangle}{E' + |E''| - \hbar\omega}\right.$$
$$\left. + \frac{\langle \mathbf{p}''r'' | \boldsymbol{\alpha}\cdot\boldsymbol{\epsilon}^{(\alpha)} e^{i\mathbf{k}\cdot\mathbf{x}} | \mathbf{p}r\rangle\langle \mathbf{p}'r' | \boldsymbol{\alpha}\cdot\boldsymbol{\epsilon}^{(\alpha')} e^{-i\mathbf{k}'\cdot\mathbf{x}} | \mathbf{p}''r''\rangle}{E' + |E''| + \hbar\omega'}\right). \quad (3.338)$$

As before, we set $E = E' = mc^2$, $\mathbf{p} = \mathbf{p}' = 0$ as $k \to 0$. Furthermore, because of the space integration that appears in the evaluation of each matrix element, $\mathbf{p}'' = 0$; hence $|E''| = mc^2$. It is now simple to evaluate each of the four matrix elements. For example,

$$\langle 0\, r'' | \boldsymbol{\alpha}\cdot\boldsymbol{\epsilon}^{(\alpha)} | 0\, r\rangle = (0, \chi^{(s'')\dagger})\begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha)} \\ \boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha)} & 0 \end{pmatrix}\begin{pmatrix} \chi^{(s)} \\ 0 \end{pmatrix}$$
$$= \chi^{(s'')}\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha)}\chi^{(s)}. \quad (3.339)$$

As $\hbar\omega, \hbar\omega' \ll mc^2$, the two terms in (3.338) combine. Taking into account the two spin states of the negative-energy electron at rest, we obtain

$$\sum_{r''=3,4}[\langle 0\, r'' | \boldsymbol{\alpha}\cdot\boldsymbol{\epsilon}^{(\alpha')} | 0\, r\rangle\langle 0\, r' | \boldsymbol{\alpha}\cdot\boldsymbol{\epsilon}^{(\alpha)} | 0\, r''\rangle + \langle 0\, r'' | \boldsymbol{\alpha}\cdot\boldsymbol{\epsilon}^{(\alpha)} | 0\, r\rangle\langle 0\, r' | \boldsymbol{\alpha}\cdot\boldsymbol{\epsilon}^{(\alpha')} | 0\, r''\rangle]$$
$$= \sum_{s''=1,2}[(\chi^{(s'')\dagger}\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha')}\chi^{(s)})(\chi^{(s')\dagger}\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha)}\chi^{(s'')}) + (\chi^{(s'')\dagger}\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha)}\chi^{(s)})(\chi^{(s')\dagger}\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha')}\chi^{(s'')})]$$
$$= \chi^{(s')\dagger}[(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha)})(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha')}) + (\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha')})(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^{(\alpha)})]\chi^{(s)}$$
$$= 2\boldsymbol{\epsilon}^{(\alpha)}\cdot\boldsymbol{\epsilon}^{(\alpha')}\delta_{ss'}, \quad (3.340)$$

where we have used (3.339) and the closure property of the Pauli spinors

$$\sum_{s''=1,2}\chi^{(s'')}\chi^{(s'')\dagger} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1 \quad 0) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(0 \quad 1)$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (3.341)$$

Thus (3.338) is

$$-\frac{e^2c^2\hbar}{2V\sqrt{\omega\omega'}}\frac{1}{mc^2}\delta_{ss'}\cos\Theta, \quad (3.342)$$

where Θ is the angle between $\boldsymbol{\epsilon}^{(\alpha)}$ and $\boldsymbol{\epsilon}^{(\alpha')}$. Apart from the minus sign in front this is exactly the time-independent part of (2.158), which has been shown to be responsible for Thomson scattering.‡ It is amusing to note that the seagull graph (Fig. 2–2c) which is the sole contributor to Thomson scattering in the nonrelativistic theory is replaced in the relativistic theory by the two diagrams of Fig. 3–6 in which the photons are emitted and absorbed one at a time.

The moral we can draw from this calculation is twofold. First, it illustrates (perhaps more vividly than any other example can) that it is absolutely necessary to take into account transitions involving *negative-energy* states if we are to obtain the correct *nonrelativistic* results. It is truly remarkable that only by invoking the concept of a negative-energy state (or a positron state), which is completely foreign to nonrelativistic quantum mechanics, can we arrive at the correct Thomson amplitude. Second, comparing (3.336) with (3.338) and noting that because of energy conservation, the energy denominators in (3.338) can be written as

$$E' + |E''| - \hbar\omega = -(E'' - E + \hbar\omega),$$
$$E' + |E''| + \hbar\omega' = -(E'' - E - \hbar\omega), \quad (3.343)$$

we observe that the negative of (3.338) is formally identical with (3.336) as we replace $r'' = 3,4$ with $r'' = 1, 2$ despite the reversal in the time orderings of the matrix elements.§ This kind of observation provides a natural justification of R.P. Feynman's point of view according to which a negative-energy electron going " *backward* in time " is to be regarded as a positron going "*forward* in time." We shall say more about this in the next chapter.

We might add that we can evaluate expressions (3.336) and (3.338) without making the approximation $\hbar\omega \ll mc^2$, $E' = mc^2$. We then obtain the famous

---

‡If (3.336) is finite (as in the case of the scattering of a high-energy photon), the relative sign of (3.336) and (3.338) is important. Actually the correct amplitude turns out to be the difference of (3.336) and (3.338). This is because for Fig. 3–6 we must take into account the minus sign arising from the fact that the initial $E > 0$ electron is "exchanged" with one of the $E < 0$ electrons in the Dirac sea. The reader need not worry about such subtle sign changes; we shall show in the next chapter that the covariant prescription based on the quantized Dirac theory automatically gives the correct signs for these matrix elements. (See Problem 4–12.)

§In writing (3.336) and (3.338) we have used the usual convention in which the matrix element standing to the right represents the perturbation acting earlier.

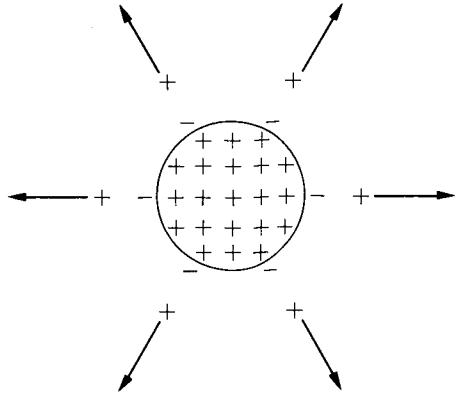


Fig. 3–7. Pictorial representation of vacuum polarization.

formula for Compton scattering

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{1}{4} r_0^2 \left(\frac{\omega'}{\omega}\right)^2 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - 2 + 4\cos^2\Theta\right) \tag{3.344}$$

derived by O. Klein and Y. Nishina in 1929.‡

Virtual electron-positron pairs. As another example illustrating the importance of electron-positron pairs we mention a phenomenon known as *vacuum polarization*. According to the hole theory, the completely filled sea of negative-energy electrons has no observable effects; in particular, what we normally call the vacuum is homogeneous and has no preferred direction. Let us, however, consider a nucleus of charge $Q = Z|e|$ placed in the Dirac sea. There is now a departure from complete homogeneity because the charge distribution of the negative-energy electrons is different from that of the free-field case. In terms of the electron-positron language this is due to the fact that a virtual electron-positron pair created in the Coulomb field behaves in such a way that the electron tends to be attracted to the nucleus while the positron tends to escape from the nucleus. This is pictorially represented in Fig. 3–7. Note that the "induced" negative charges in the vicinity of the nucleus are compensated for by positive charges that "escape to infinity." As a result, the net charge observed at large but finite distances is smaller than the bare charge of the nucleus. In fact what is usually called the observed charge of the nucleus is the original bare charge of the nucleus partially canceled by the charges of the virtual electrons surrounding the nucleus. This situation is rather analogous to that of a charge placed in a dielectric material; the effective charge in a polarized medium is the original charge divided by ϵ, where ϵ is the dielectric constant. In other words, because of virtual electron-positron pairs the vacuum behaves like a polarizable medium.

‡Formula (3.344) can be obtained more readily using covariant perturbation theory (cf. Section 4–4).

On the other hand, at very close distances to the nucleus the "bare charge" itself may be explored; the electron in a hydrogen-like atom should feel at very short distances a stronger attraction than the attraction determined by the Coulomb potential due to the usual observed charge. Since the s state electrons have a greater probability of penetrating the nucleus, we expect that the energy levels of the s states should be displaced to lower levels. Although the argument presented here is rather qualitative, the $2s_{\frac{1}{2}} - 2p_{\frac{1}{2}}$ splitting of the hydrogen atom due to this vacuum polarization effect turns out to be calculable. In 1935, E. A. Uehling predicted that the $2s_{\frac{1}{2}}$-state should lie *lower* by 27 Mc than the $2p_{\frac{1}{2}}$-state.‡ The experimentally observed Lamb shift, as we have seen in Section 2–8, has the opposite sign and is about 40 times larger in magnitude. Although the major part of the Lamb shift is not due to this Uehling effect, for a precise comparison of the experimental value with the theoretical value of the Lamb shift (measured to an accuracy of 0.2 Mc) it has been proved essential to take vacuum polarization seriously. The effect of vacuum polarization is also observable in π mesic and muonic atoms.

The notion that the negative-energy states are completely filled becomes rather treacherous when applied to a particle subject to an external potential. The results of Section 3–7 show that even the (positive-energy) wave function of the hydrogen atom when expanded in plane waves contains small negative-energy components. At first we may be tempted to simply drop the negative-energy components by saying that these states are completely filled. But this cannot be right because, if we do so, we do not even obtain the correct energy levels; in fact, we may recall that the Darwin term (needed for the $2s_{\frac{1}{2}}$–$2p_{\frac{1}{2}}$ degeneracy) can be qualitatively explained by invoking *Zitterbewegung*, which arises from interference of the positive- and negative-energy plane-wave components of the bound-state wave function.

A crude physical argument for the *Zitterbewegung* of an atomic electron within the framework of the hole theory goes as follows. We note that in the Coulomb field of the nucleus a negative-energy electron can make a virtual transition to a positive-energy state (which is equivalent to saying that the Coulomb field can create a virtual electron-positron pair). Now the fact that the wave function in the hydrogen atom contains negative-energy components implies that the atomic electron in the orbit can fill up the hole in the negative-energy state (which means that the atomic electron can annihilate with the positron of the virtual pair). The escalated electron which is left over can now go around the nucleus as the atomic electron. In short, the atomic electron and one of the $E < 0$ electrons in the Dirac sea are visualized as undergoing "exchange scattering." What is the order of magnitude of the distance over which this effect takes place? From the uncertainty principle we expect that the energy violation by an amount $2mc^2$ involved in the escalation of the negative energy electron is allowed only for a time interval $\Delta t \sim \hbar/2mc^2$. (Note incidentally that this is of the order of the reciprocal of the

‡Using covariant perturbation theory, we shall briefly outline the calculation of the Uehling effect in Chapter 5.

Zitterbewegung frequency.) At the time the original atomic electron fills up the vacated negative-energy state, the escalated electron is at most $c(\Delta t) \sim \hbar/2mc$ away from the original electron. This distance is precisely the order of magnitude of the fluctuation of the electron coordinate due to the *Zitterbewegung*.

Charge-conjugate wave function. It is not entirely obvious from the form of the Dirac equation that the space-time development of an electron state in a given potential A_μ is identical to that of the corresponding positron state in the potential $-A_\mu$. For this reason let us cast the Dirac theory into a form which makes the symmetry between the electron and the positron self-evident. This can be best done using a method originally exploited by H. A. Kramers, E. Majorana, and W. Pauli.

We first ask whether the theory based on the Dirac equation with the sign of eA_μ reversed,

$$\left(\frac{\partial}{\partial x_\mu} + \frac{ie}{\hbar c}A_\mu\right)\gamma_\mu\psi^C + \frac{mc}{\hbar}\psi^C = 0, \tag{3.345}$$

is equivalent to the one based on the original Dirac equation (3.60). We assume as usual that there is a definite prescription that relates ψ^C (called the charge-conjugate wave function) and ψ which are respectively solutions to (3.345) and (3.60). Motivated by our experience with the Klein-Gordon theory (cf. Problem 1–3), we try‡

$$\psi^C = S_C\psi^*, \tag{3.346}$$

where S_C is a 4×4 matrix. We must now show that

$$\left[\left(\frac{\partial}{\partial x_k} + \frac{ie}{\hbar c}A_k\right)\gamma_k + \left(\frac{\partial}{\partial x_4} + \frac{ie}{\hbar c}A_4\right)\gamma_4\right] S_C\psi^* + \frac{mc}{\hbar}S_C\psi^* = 0 \tag{3.347}$$

is as good as the original Dirac equation (3.60). Taking the complex conjugate of (3.347) we have

$$\left[\left(\frac{\partial}{\partial x_k} - \frac{ie}{\hbar}A_k\right)\gamma_k^* + \left(-\frac{\partial}{\partial x_4} + \frac{ie}{\hbar c}A_4\right)\gamma_4^*\right] S_C^*\psi + \frac{mc}{\hbar}S_C^*\psi = 0. \tag{3.348}$$

Multiplying (3.348) by $(S_C^*)^{-1}$ from the left and comparing the result with the original Dirac equation, we see that the equivalence of (3.346) and (3.60) can be established if there exists S_C such that

$$\begin{aligned}(S_C^*)^{-1}\gamma_k^* S_C^* &= \gamma_k,\\ (S_C^*)^{-1}\gamma_4^* S_C^* &= -\gamma_4.\end{aligned} \tag{3.349}$$

In the standard (Dirac-Pauli) representation, γ_2 and γ_4 are purely real while γ_1 and γ_3 are purely imaginary. It is easy to see that in this representation,

$$S_C = \gamma_2 = S_C^* = (S_C^*)^{-1} \tag{3.350}$$

‡Note that ψ^C is to be represented by a single *column* matrix, since ψ^* (complex conjugate) rather than $\psi^\dagger$ (Hermitian conjugate) enters.

will do:‡

$$\gamma_2\begin{Bmatrix}(-\gamma_1)\\ \gamma_2\\ (-\gamma_3)\end{Bmatrix}\gamma_2 = \begin{Bmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{Bmatrix}, \tag{3.351}$$

$$\gamma_2\gamma_4\gamma_2 = -\gamma_4.$$

Since we have demonstrated the existence of an S_C that satisfies (3.349), we have proved the equivalence of (3.345) and (3.60).

It is very important to note that $S_C = \gamma_2$ is true only in the standard (Dirac-Pauli) representation. In fact the particular forms of S_C depend on the particular representations we happen to use. For instance, in the Majorana representation in which γ_4 is purely imaginary and γ_k is purely real, ψ^C is simply ψ^* itself, as can readily be seen by complex-conjugating (3.60).§ This situation should be contrasted with the parity case where $S_P = \gamma_4$ (up to a phase factor) holds in any representation.

What does (3.346) mean for some of the familiar wave functions we have obtained in the previous sections? Take, for instance, the first of the $E > 0$ plane-wave solutions (3.114). We have

$$\begin{aligned}&\sqrt{\frac{mc^2}{EV}}\gamma_2\left[u^{(1)}(\mathbf{p})\exp\left(\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar} - \frac{iEt}{\hbar}\right)\right]^*\\ &= \sqrt{\frac{E+mc^2}{2EV}}\begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix}\begin{pmatrix}1\\0\\p_3c/(E+mc^2)\\(p_1+ip_2)c/(E+mc^2)\end{pmatrix}^*\exp\left(-\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar}+\frac{iEt}{\hbar}\right)\\ &= \sqrt{\frac{E+mc^2}{2EV}}\begin{pmatrix}-(p_1-ip_2)c/(|E|+mc^2)\\p_3c/(|E|+mc^2)\\0\\-1\end{pmatrix}\exp\left(-\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar}+\frac{i|E|t}{\hbar}\right)\\ &= -\sqrt{\frac{mc^2}{|E|V}}u^{(4)}(-\mathbf{p})\exp\left(-\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar}+\frac{i|E|t}{\hbar}\right).\end{aligned} \tag{3.352}$$

Note that the eigenvalues of $-i\hbar\nabla$ and $i\hbar(\partial/\partial t)$ are $-\mathbf{p}$ and $-|E|$ respectively. Similarly

$$\sqrt{\frac{mc^2}{EV}}\gamma_2\left[u^{(2)}(\mathbf{p})\exp\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar} - \frac{iEt}{\hbar}\right)\Big]^* = \sqrt{\frac{mc^2}{|E|V}}u^{(3)}(-\mathbf{p})\exp\left(-\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar}+\frac{i|E|t}{\hbar}\right). \tag{3.353}$$

‡More generally we have $S_C = \eta\gamma_2$, where η is an undetermined phase factor; however, this phase factor can be set to 1 by convention.

§In a more advanced treatment of the subject the relations (3.346) and (3.349) are often written as $\psi^C = C\bar{\psi}^T$, $C^{-1}\gamma_\mu C = -\gamma_\mu^T$, where T stands for "transpose." In the standard representation $C = \gamma_2\gamma_4$ up to a phase factor since $\psi^C = \gamma_2\gamma_4(\psi^\dagger\gamma_4)^T = \gamma_2\gamma_4\gamma_4^T\psi^* = \gamma_2\psi^*$.

Thus the charge-conjugate wave function ψ^c obtained from the positive-energy plane-wave solution ψ by means of (3.346) is the wave function for a negative-energy plane wave whose magnitude of the energy is the same and whose momentum is *opposite*. Moreover, the spin direction (or if $\mathbf{p}$ is not along the z-axis, the expectation value of $\boldsymbol{\Sigma}$) is also *reversed* since the index 4(3) goes with the index 1(2). If we now invoke the hole theory, we see that the charge-conjugate wave function describes the dynamical behavior of the negative-energy state whose absence appears as the $E > 0$ *positron* of the *same* $\mathbf{p}$ and *same* $\langle\boldsymbol{\Sigma}\rangle$ (cf. Table 3–4). Likewise, when ψ represents a negative-energy electron state whose absence appears as the positron state of $\mathbf{p}$ and $\langle\boldsymbol{\Sigma}\rangle$, then ψ^c represents the positive-energy electron state of $\mathbf{p}$ and $\langle\boldsymbol{\Sigma}\rangle$.

As another example, let us compare the probability distribution $\psi^\dagger\psi$ with the corresponding $\psi^{c\dagger}\psi^c$ where for the sake of definiteness, ψ may be taken to be the wave function for the ground state of the hydrogen atom. In general, we have

$$\psi^{c\dagger}\psi^c = (\gamma_2\psi^*)^\dagger(\gamma_2\psi^*) = \psi^{*\dagger}\psi^* = \psi^\dagger\psi. \tag{3.354}$$

For the electron in the hydrogen atom, the $A_0(= -iA_4)$ that appears in (3.60) is $|e|/4\pi r$; ψ^c, according to (3.345), is a solution to the Coulomb energy problem in the negative electrostatic potential $-|e|/4\pi r$. Evidently the energy eigenvalue of ψ^c is the negative of that of ψ because of the complex conjugation that appears in (3.346). Thus the relation (3.354) implies that the negative-energy electron going around the negative electrostatic potential has the same probability distribution as the corresponding positive-energy electron going around the positive electrostatic potential. This means that in an electrostatic potential that appears repulsive to the positive-energy electron (for example, in the Coulomb field of the antiproton), the negative-energy *electron* behaves dynamically as though it were in an *attractive* force field. Invoking now the hole-theoretic interpretation, we see that an antiatom in which a positron is bound to the center by $A_0 = -|e|/4\pi r$ looks like the usual atom in which an electron is bound to the center by $A_0 = |e|/4\pi r$.

We define the charge-conjugation operation such that its application on the electron (positron) state of momentum $\mathbf{p}$ and the spin-expectation value $\hbar\langle\boldsymbol{\Sigma}\rangle/2$ results in the positron (electron) state of momentum $\mathbf{p}$ and the spin-expectation value $\hbar\langle\boldsymbol{\Sigma}\rangle/2$. The equivalence of (3.345) and (3.60) implies that if $\psi(\mathbf{x}, t)$ characterizes the space-time behavior of an $E > 0$ electron state in a potential A_μ, then its charge-conjugate wave function $\psi^c(\mathbf{x}, t)$ characterizes the space-time behavior of the *negative-energy electron state whose absence appears as the charge-conjugate (positron) state* in the potential $-A_\mu$.

When we start computing the expectation values of the various dynamical variables using ψ^c, we obtain results which may appear somewhat confusing at first sight. For example, if we naively evaluate the expectation value of $\mathbf{p}$ with respect to ψ and ψ^c, we obtain the result: $\langle\mathbf{p}\rangle$ is opposite to $\langle\mathbf{p}\rangle_c$, as can be seen directly from (3.352) and (3.353) for the free-particle case, and similarly for $\langle\boldsymbol{\Sigma}\rangle$. But we know that the momentum and the spin direction are *unchanged* under charge conjugation which transforms the electron state of momentum $\mathbf{p}$ and

$\langle\boldsymbol{\Sigma}\rangle$ into the positron state of momentum $\mathbf{p}$ and $\langle\boldsymbol{\Sigma}\rangle$ (not $-\mathbf{p}$ and $-\langle\boldsymbol{\Sigma}\rangle$). This peculiarity is due to the fact that in the unquantized Dirac theory the so-called charge-conjugate wave function ψ^c is *not* the wave function of the charge-conjugate state but rather that of the state (subject to the potential whose sign is opposite to the original one) whose *absence* appears as the charge-conjugate state.

What is even more striking, the space integral of the charge density

$$Q = e\int \bar{\psi}\gamma_4\psi\, d^3x = e\int \psi^\dagger\psi\, d^3x, \tag{3.355}$$

which is the total charge, cannot possibly change its sign when we replace ψ by its charge-conjugate wave function ψ^c, in sharp contrast to the Klein-Gordon case where the substitution $\phi \rightleftarrows \phi^*$ results in the reversal of the charge-current density (cf. Eqs. 1.55 and 3.127).‡ Actually this is expected because when ψ is a positive-energy wave function, ψ^c is the wave function for an $E < 0$ *negatively* charged particle even though it "behaves dynamically" like a positively charged particle in an external electromagnetic field. For a more satisfactory formulation of charge conjugation, it is essential to quantize the electron field according to Fermi-Dirac statistics.

3–10. QUANTIZATION OF THE DIRAC FIELD

Difficulties of the unquantized Dirac theory. One of the great triumphs of relativistic quantum theory is that it has succeeded in providing a theoretical framework within which we can discuss quantitatively a variety of physical phenomena involving the creation and annihilation of various particles. We learned in the last chapter that the "natural language" used to describe the creation and annihilation of photons is that of quantum field theory. In the previous section we did discuss phenomena such as pair creation and pair annihilation. The language used there, however, is very different from that of quantum field theory; instead of saying that the number of electrons is not conserved in pair production, we have argued that the *number of electrons actually is conserved* and that all that happens is just the escalation of a negative-energy electron. In other words, we have tried to describe phenomena such as pair production without abandoning the single-particle interpretation of the Dirac wave function according to which the space integral of $\psi^\dagger\psi$ is a constant of the motion even in the presence of the electromagnetic interaction. In doing so, however, we were forced to depart very radically from the *single*-particle theory itself; in fact, we had to introduce a sea of an *infinite* number of negative-energy particles.

There are essentially two reasons why the hole-theoretic description works. First, as we have already mentioned, crucial to the success of the hole theory is the assumption that the electron obeys the Pauli exclusion principle. Second,

‡In fact $\psi^\dagger\psi$ cannot have its sign changed under any transformation that preserves its positive-definite form.

although electrons and positrons can be created or annihilated, the basic interaction in electrodynamics is such that the *difference* between the number of (positive-energy) electrons and the number of (positive-energy) positrons,

$$N = N(\mathrm{e}^-) - N(\mathrm{e}^+), \tag{3.356}$$

is conserved. In the hole-theoretic description what we do is just set

$$\begin{aligned} N(\mathrm{e}^-_{E>0}) &= N(\mathrm{e}^-), \\ N(\mathrm{e}^-_{E<0}) &= -N(\mathrm{e}^+) + \text{constant background}, \end{aligned} \tag{3.357}$$

so that the newly defined electron number, given by the *sum*

$$N' = N(\mathrm{e}^-_{E>0}) + N(\mathrm{e}^-_{E<0}), \tag{3.358}$$

is necessarily conserved whenever (3.356) is conserved.

In the "real world" there are nonelectromagnetic phenomena which do not conserve (3.356). Take, for instance, a beta (plus) decay

$$\mathrm{p} \longrightarrow \mathrm{n} + \mathrm{e}^+ + \nu. \tag{3.359}$$

Although the free proton cannot undergo this disintegration process because of energy conservation, a proton bound in a nucleus can emit a positron and neutrino and turns itself into a neutron. In the hole-theoretic interpretation we may try to attribute the presence of the e^+ in the final state to the absence of a negative-energy electron in the Dirac sea. But where is the electron which used to occupy the now vacated negative-energy state? It is apparent that the probability of finding the electron is no longer conserved.‡

Second quantization. Our beta-decay example reveals that it is actually much more sensible to construct a formalism in which we allow electrons and positrons to be destroyed or created more freely. Guided by the success of the quantum theory of radiation, we are tempted to follow, as much as possible, the quantization procedure we used in the photon case. We shall first construct a "classical" theory of the Dirac field using the standard Lagrangian formalism of Chapter 1 and then quantize the dynamical excitations of the Dirac field by replacing the Fourier coefficients by creation and annihilation operators. At this stage it is not completely clear whether this method is a legitimate one. As we emphasized in Section 2–3, the classical field theory is a limit of the quantum field theory where the occupation number goes to infinity, but we know that the occupation number of a particular electron state is *at most one*. However, let us go ahead with the Lagrangian formulation of the classical Dirac field.§

‡We might argue that in this β^+ process a negative-energy electron gives up its charge to the proton and gets escalated to a (positive-energy) neutrino state. But note that the electron and the neutrino are different particles which must be described by different wave equations.

§The reader who is unhappy with our procedure may study an alternative, more axiomatic, approach based on J. Schwinger's action principle, discussed, for instance, in Chapter 1 of Jauch and Rohrlich (1955). In Schwinger's formalism the field variables are treated as operators from the very beginning.

The basic free-field Lagrangian density from which the field equation may be derived is taken to be

$$\begin{aligned} \mathscr{L} &= -c\hbar\bar{\psi}\gamma_\mu(\partial/\partial x_\mu)\psi - mc^2\bar{\psi}\psi \\ &= -c\hbar\bar{\psi}_\alpha(\gamma_\mu)_{\alpha\beta}(\partial/\partial x_\mu)\psi_\beta - mc^2\,\delta_{\alpha\beta}\bar{\psi}_\alpha\psi_\beta, \end{aligned} \tag{3.360}$$

which is a Lorentz invariant scalar density. In the Lagrangian formulation each of the four components of ψ and $\bar{\psi}$ is to be regarded as an independent field variable. Varying $\bar{\psi}_\alpha$ [which actually stands for $(\psi^\dagger)_\gamma(\gamma_4)_{\gamma\alpha}$] we obtain four Euler-Lagrange equations of the form $\partial\mathscr{L}/\partial\bar{\psi}_\alpha = 0$ which can be summarized as the single Dirac equation (3.31). To obtain the field equation for $\bar{\psi}$, we first make the replacement

$$-c\hbar\bar{\psi}_\alpha(\gamma_\mu)_{\alpha\beta}\frac{\partial}{\partial x_\mu}\psi_\beta \longrightarrow c\hbar\left(\frac{\partial}{\partial x_\mu}\bar{\psi}\right)_\alpha(\gamma_\mu)_{\alpha\beta}\psi_\beta, \tag{3.361}$$

which is justified since the difference is just a four-divergence. Varying ψ_β we then get the adjoint equation (3.46). The "canonical momentum" π conjugate to ψ is‡

$$\pi_\beta = \frac{\partial\mathscr{L}}{\partial(\partial\psi_\beta/\partial t)} = i\hbar\bar{\psi}_\alpha(\gamma_4)_{\alpha\beta} = i\hbar\psi^\dagger_\beta. \tag{3.362}$$

The Hamiltonian density is then obtainable by the standard prescription (1.4):

$$\begin{aligned} \mathscr{H} &= c\pi_\beta\frac{\partial\psi_\beta}{\partial x_0} - \mathscr{L} \\ &= c\hbar\left(i\psi^\dagger\frac{\partial\psi}{\partial x_0} - i\bar{\psi}\gamma_4\frac{\partial\psi}{\partial x_0} + \bar{\psi}\gamma_k\frac{\partial}{\partial x_k}\psi\right) + mc^2\bar{\psi}\psi \\ &= \psi^\dagger(-i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta mc^2)\psi. \end{aligned} \tag{3.363}$$

Thus the total Hamiltonian of the free Dirac field is

$$H = \int\psi^\dagger(-i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta mc^2)\psi\,d^3x. \tag{3.364}$$

Since the plane-wave solutions (3.114) and (3.115) taken at $t = 0$ form a complete orthonormal set, an arbitrary four-component field at $t = 0$ can be expanded in free-particle plane waves. The Dirac field ψ becomes a quantized field if we replace the Fourier coefficients in the plane-wave expansion by operators of the type considered at the end of Section 2–2. We have

$$\psi(\mathbf{x}, t) = \frac{1}{\sqrt{V}}\sum_{\mathbf{p}}\sum_{r=1}^{4}\sqrt{\frac{mc^2}{|E|}}\,b^{(r)}_{\mathbf{p}}(t)u^{(r)}(\mathbf{p})\,e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}, \tag{3.365}$$

where ψ is now an operator assumed to act on state vectors in occupation number space. We interpret $b^{(r)}_{\mathbf{p}}$ and $b^{(r)\dagger}_{\mathbf{p}}$ as respectively the annihilation and the creation operators for state $(\mathbf{p}, r)$. A single electron state characterized by $(\mathbf{p}, r)$ is represented by $b^{(r)\dagger}_{\mathbf{p}}(0)|0\rangle$. As we have already seen in Chapter 2, the Pauli exclusion prin-

‡As we have written the Lagrangian density, the canonical momentum conjugate to $\bar{\psi}$ vanishes.

ciple is guaranteed if we use the Jordan-Wigner anticommutation relations (cf. Eq. 2.49)

$$\begin{aligned}\{b_{\mathbf{p}}^{(r)}, b_{\mathbf{p}'}^{(r')\dagger}\} &= \delta_{rr'}\delta_{\mathbf{pp}'},\\ \{b_{\mathbf{p}}^{(r)}, b_{\mathbf{p}'}^{(r')}\} &= 0,\\ \{b_{\mathbf{p}}^{(r)\dagger}, b_{\mathbf{p}'}^{(r')\dagger}\} &= 0,\end{aligned} \tag{3.366}$$

from which it follows that the eigenvalue of the number operator defined by

$$N_{\mathbf{p}}^{(r)} = b_{\mathbf{p}}^{(r)\dagger} b_{\mathbf{p}}^{(r)} \tag{3.367}$$

is zero or one.

We assume that the Hamiltonian operator of the quantized Dirac field has the same form as the classical expression (3.363). We then have‡

$$\begin{aligned}H &= \frac{1}{V}\int \sum_{\mathbf{p}}\sum_{\mathbf{p}'}\sum_{r=1}^{4}\sum_{r'=1}^{4}\left(\sqrt{\frac{mc^2}{|E|}}\, b_{\mathbf{p}}^{(r)\dagger} u^{(r)\dagger}(\mathbf{p})\, e^{-i\mathbf{p}\cdot\mathbf{x}/\hbar}\right)(-i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta mc^2)\\ &\quad\times\left(\sqrt{\frac{mc^2}{|E'|}}\, b_{\mathbf{p}'}^{(r')} u^{(r')}(\mathbf{p}')\, e^{i\mathbf{p}'\cdot\mathbf{x}/\hbar}\right) d^3x\\ &= \sum_{\mathbf{p}}\sum_{\mathbf{p}'}\sum_{r}\sum_{r'}\delta_{\mathbf{pp}'}\frac{mc^2E'}{\sqrt{|EE'|}}\, b_{\mathbf{p}}^{(r)\dagger} b_{\mathbf{p}'}^{(r)} u^{(r)\dagger}(\mathbf{p}) u^{(r')}(\mathbf{p}')\\ &= \sum_{\mathbf{p}}\sum_{r=1,2}|E|\, b_{\mathbf{p}}^{(r)\dagger} b_{\mathbf{p}}^{(r)} - \sum_{\mathbf{p}}\sum_{r=3,4}|E|\, b_{\mathbf{p}}^{(r)\dagger} b_{\mathbf{p}}^{(r)},\end{aligned} \tag{3.368}$$

because of the orthogonality and normalization relations (3.106) and (3.110), and the energy momentum relation

$$E' = \pm\sqrt{|\mathbf{p}'|^2c^2 + m^2c^4} \qquad \text{for} \quad r' = \begin{cases}1, 2,\\ 3, 4.\end{cases} \tag{3.369}$$

Recall that by our definition, the creation and annihilation operators $b_{\mathbf{p}}^{(r)\dagger}$ and $b_{\mathbf{p}}^{(r)}$ are *time-dependent* operators. Their time dependence can be inferred from the Heisenberg equation of motion,

$$\begin{aligned}\dot{b}_{\mathbf{p}}^{(r)} &= \frac{i}{\hbar}[H, b_{\mathbf{p}}^{(r)}] = \mp\frac{i}{\hbar}b_{\mathbf{p}}^{(r)}|E| \qquad \text{for} \quad r = \begin{cases}1, 2,\\ 3, 4,\end{cases}\\ \dot{b}_{\mathbf{p}}^{(r)\dagger} &= \frac{i}{\hbar}[H, b_{\mathbf{p}}^{(r)\dagger}] = \pm\frac{i}{\hbar}b_{\mathbf{p}}^{(r)}|E| \qquad \text{for} \quad r = \begin{cases}1, 2,\\ 3, 4,\end{cases}\end{aligned} \tag{3.370}$$

where we have taken advantage of the very useful relation

$$[AB, C] = A\{B, C\} - \{A, C\}B. \tag{3.371}$$

Thus we have

$$b_{\mathbf{p}}^{(r)}(t) = b_{\mathbf{p}}^{(r)}(0)\, e^{\mp i|E|t/\hbar} \qquad \text{for} \quad r = \begin{cases}1, 2,\\ 3, 4,\end{cases} \tag{3.372}$$

‡Previously we used the symbol H for the Hamiltonian operator $(-i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta mc^2)$ acting on the Dirac wave function. In this section, H stands for the total Hamiltonian operator of the free Dirac field acting on state vectors in occupation-number space.

and a similar relation for $b_{\mathbf{p}}^{(r)\dagger}(t)$. The expansion (3.365) now becomes

$$\begin{aligned}\psi(\mathbf{x}, t) = \frac{1}{V}\sum_{\mathbf{p}}\sqrt{\frac{mc^2}{|E|}}\Bigg(&\sum_{r=1,2} b_{\mathbf{p}}^{(r)}(0)\, u^{(r)}(\mathbf{p})\exp\left[\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar} - \frac{i|E|t}{\hbar}\right]\\ &+ \sum_{r=3,4} b_{\mathbf{p}}^{(r)}(0)\, u^{(r)}(\mathbf{p})\exp\left[\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar} + \frac{i|E|t}{\hbar}\right]\Bigg).\end{aligned} \tag{3.373}$$

The *quantized* free field is now seen to satisfy the same field equation (viz. the Dirac equation) as the one derived from the *classical* variational principle. Note that this is not *a priori* self-evident; recall, in particular, that b and $b^\dagger$ can no longer be written as linear combinations of P and Q satisfying $[Q, P] = i\hbar$. What is even more striking, the form of the field equation obtainable from the Heisenberg equation of motion is the same whether the creation and annihilation operators satisfy anticommutation relations or commutation relations, as the reader may readily verify.‡ Mathematically this remarkable feature is a consequence of the fact that the commutator $[AB, C]$ that appears in (3.371) can also be written as

$$[AB, C] = A[B, C] + [A, C]B. \tag{3.374}$$

From the fact that we can get the same field equation whether the electron satisfies Fermi-Dirac statistics or Bose-Einstein statistics, we may be tempted to infer that quantum field theory does not "know" which statistics the electron is supposed to satisfy. This inference, however, is *not* correct, as we shall see shortly.

At this point it turns out to be more convenient to redefine the b and $b^\dagger$ so that they now become *time-independent* operators. We do this because we would like to exhibit the time dependence of the field operator ψ more explicitly. We set

$$\begin{aligned}b_{\mathbf{p}}^{(r)(\text{new})} &= b_{\mathbf{p}}^{(r)(\text{old})}(0),\\ b_{\mathbf{p}}^{(r)(\text{old})}(t) &= b_{\mathbf{p}}^{(r)(\text{new})}\, e^{\mp i|E|t/\hbar} \qquad \text{for} \quad r = \begin{cases}1, 2,\\ 3, 4.\end{cases}\end{aligned} \tag{3.375}$$

The form of the anticommutation relations is unchanged under this replacement.

In the previous sections the symbol ψ stood for a single-particle wave function while in the present section the same symbol ψ is used to denote the quantized Dirac field which is an operator that can act on state vectors in occupation-number space. To distinguish between the two possibilities the wave function ψ is often called a c-number field; the field operator ψ, a q-number field. We naturally ask: what is the connection between the c-number ψ and the q-number ψ? For a single-particle plane-wave state the desired connection is easily established:

$$\psi^{(c\text{-number})} = \langle 0|\psi^{(q\text{-number})}|b_{\mathbf{p}}^{(r)\dagger}\Phi_0\rangle, \tag{3.376}$$

where $\psi^{(q\text{-number})}$ is given by (3.373) with $b_{\mathbf{p}}^{(r)}$ replacing $b_{\mathbf{p}}^{(r)}(0)$. To see this, just note

$$\langle 0|b_{\mathbf{p}'}^{(r')}b_{\mathbf{p}}^{(r)\dagger}|0\rangle = \delta_{rr'}\delta_{\mathbf{pp}'}. \tag{3.377}$$

‡Note that (3.368) would still be valid even if the electron satisfied Bose-Einstein statistics.

Therefore, for $r = 1, 2$, we have

$$\langle 0 | \psi^{(q\text{-number})} | b_{\mathbf{p}}^{(r)\dagger} \Phi_0 \rangle = \sqrt{\frac{mc^2}{EV}} u^{(r)}(\mathbf{p}) \exp\left(i\frac{\mathbf{p}\cdot\mathbf{x}}{\hbar} - \frac{iEt}{\hbar}\right), \tag{3.378}$$

which is indeed the wave function for a positive-energy plane wave characterized by $(\mathbf{p}, r)$. The transition from $\psi^{(c\text{-number})}$ to $\psi^{(q\text{-number})}$ is sometimes called *second quantization*.‡ Although in the above example we considered just a single-particle state, it is important to keep in mind that, in general, $\psi^{(q\text{-number})}$ can actually operate on the state vector for an *assembly* of electrons (and positrons). The Dirac equation in the quantized theory should be regarded as a differential equation that determines the dynamical behavior of the entire aggregate of electrons (and positrons).

Let us now go back to the expression for the Hamiltonian operator (3.368), which is unchanged under the replacement (3.375). This expression for the total Hamiltonian makes good sense since we showed in Chapter 2 that $b_{\mathbf{p}}^{(r)\dagger} b_{\mathbf{p}}^{(r)}$ (whose eigenvalue is zero or one) is to be interpreted as the number operator. The energy of a one-electron state characterized by $(\mathbf{p}, r)$ is just $|E|$ or $-|E|$ depending on whether $r = 1, 2$ or $r = 3, 4$. The total energy of an ensemble of $E > 0$ electrons is just the sum of the energies of the individual electrons.

We can also compute the total charge operator. Following steps analogous to (1.51) through (1.54), we can readily show that $i\bar{\psi}\gamma_\mu\psi$ satisfies the continuity equation. Assuming that the charge density operator is given by $e\psi^\dagger\psi$ even in the q-number theory, we get for the total charge operator,

$$\begin{aligned} Q &= e\int \psi^\dagger\psi\, d^3x \\ &= e\sum_{\mathbf{p}}\sum_{\mathbf{p}'}\sum_{r}\sum_{r'} (mc^2/\sqrt{|EE'|})\,\delta_{\mathbf{pp}'} b_{\mathbf{p}}^{(r)\dagger} b_{\mathbf{p}'}^{(r')} u^{(r)\dagger}(\mathbf{p}) u^{(r')}(\mathbf{p}') \\ &= e\sum_{\mathbf{p}}\sum_{r=1}^{4} b_{\mathbf{p}}^{(r)\dagger} b_{\mathbf{p}}^{(r)}. \end{aligned} \tag{3.379}$$

This is again expected.§ As for the total momentum of the Dirac field we may start with (cf. Problem 1–1)

$$P_k = -i\int \mathscr{T}_{4k}\, d^3x, \tag{3.380}$$

where

$$\mathscr{T}_{4k} = -\frac{\partial\mathscr{L}}{\partial(\partial\psi_\alpha/\partial x_4)}\frac{\partial\psi_\alpha}{\partial x_k} - \frac{\partial\bar{\psi}_\alpha}{\partial x_k}\frac{\partial\mathscr{L}}{\partial(\partial\bar{\psi}_\alpha/\partial x_4)}. \tag{3.381}$$

We then obtain

$$\mathbf{P} = -i\hbar\int \psi^\dagger\nabla\psi\, d^3x = \sum_{\mathbf{p}}\sum_{r} \mathbf{p}\, b_{\mathbf{p}}^{(r)\dagger} b_{\mathbf{p}}^{(r)}. \tag{3.382}$$

‡By *first quantization* one simply means $\mathbf{p}^{(\text{classical})} \longrightarrow -i\hbar\nabla$, etc., for the dynamical variables of a single particle.

§Recall that the negative-energy electron has electric charge $e = -|e|$ even though it "behaves dynamically" like a positively charged particle.

Positron operators and positron spinors. Although (3.368), (3.379), and (3.382) are satisfactory from the hole-theoretic point of view, the persistent appearance of negative energies seems somewhat distasteful. It is much better to have a formalism in which the free-particle energy is always positive while the total charge is positive or negative (depending on whether there are more positrons or electrons). With this aim in mind let us define $b_{\mathbf{p}}^{(s)}$, $d_{\mathbf{p}}^{(s)}$, $u^{(s)}(\mathbf{p})$, and $v^{(s)}(\mathbf{p})$ with $s = 1, 2$ such that

$$\begin{aligned} b_{\mathbf{p}}^{(s)} &= b_{\mathbf{p}}^{(r)}, \qquad (r = s) \quad \text{for } r = 1, 2, \\ d_{\mathbf{p}}^{(s)\dagger} &= \mp b_{-\mathbf{p}}^{(r)} \qquad \begin{cases} s = 1 & \text{for } r = 4, \\ s = 2 & \text{for } r = 3; \end{cases} \end{aligned} \tag{3.383}$$

and

$$\begin{aligned} u^{(s)}(\mathbf{p}) &= u^{(r)}(\mathbf{p}), \qquad (r = s) \quad \text{for } r = 1, 2, \\ v^{(s)}(\mathbf{p}) &= \mp u^{(r)}(-\mathbf{p}) \qquad \begin{cases} s = 1 & \text{for } r = 4, \\ s = 2 & \text{for } r = 3. \end{cases} \end{aligned} \tag{3.384}$$

The basic motivation for all this stems from the fact that the annihilation of a negative-energy electron of momentum $-\mathbf{p}$ and spin-*down* appears as the creation of a positron with momentum $+\mathbf{p}$ and spin-*up*. We later see that $d^\dagger$ (d) can indeed be interpreted as the creation (annihilation) operator of a positron. We have reshuffled the order of the r- and s-indices and inserted minus signs in such a way that

$$\begin{aligned} S_C u^{(s)*}(\mathbf{p}) &= \gamma_2 u^{(s)*}(\mathbf{p}) = v^{(s)}(\mathbf{p}), \\ S_C v^{(s)*}(\mathbf{p}) &= \gamma_2 v^{(s)*}(\mathbf{p}) = u^{(s)}(\mathbf{p}), \end{aligned} \tag{3.385}$$

with the *same* s ($= 1, 2$) (cf. 3.352). Note that the d and $d^\dagger$ satisfy the same anticommutation relations as the b and the $b^\dagger$:

$$\begin{aligned} \{d_{\mathbf{p}}^{(s)}, d_{\mathbf{p}'}^{(s')\dagger}\} &= \delta_{ss'}, \delta_{\mathbf{pp}'}, \\ \{d_{\mathbf{p}}^{(s)}, d_{\mathbf{p}'}^{(s')}\} &= \{d_{\mathbf{p}}^{(s)\dagger}, d_{\mathbf{p}'}^{(s')\dagger}\} = 0. \end{aligned} \tag{3.386}$$

We also have

$$\begin{aligned} \{b_{\mathbf{p}}^{(s)}, d_{\mathbf{p}'}^{(s')}\} = \{b_{\mathbf{p}}^{(s)}, d_{\mathbf{p}'}^{(s')\dagger}\} &= \{b_{\mathbf{p}}^{(s)\dagger}, d_{\mathbf{p}'}^{(s')\dagger}\} \\ &= \{b_{\mathbf{p}}^{(s)\dagger}, d_{\mathbf{p}'}^{(s')}\} = 0. \end{aligned} \tag{3.387}$$

For later purposes it turns out to be useful to collect formulas for u and v. First, (3.105) now becomes

$$\begin{aligned} (i\gamma\cdot p + mc)u^{(s)}(\mathbf{p}) &= 0, \\ (-i\gamma\cdot p + mc)v^{(s)}(\mathbf{p}) &= 0, \end{aligned} \tag{3.388}$$

where $p = (\mathbf{p}, iE/c)$ with E *positive*, even in the equation for $v^{(s)}(\mathbf{p})$. The orthogonality and normalization relations (3.106) and (3.110) become

$$\begin{aligned} u^{(s)\dagger}(\mathbf{p})u^{(s)}(\mathbf{p}) = \delta_{ss}(E/mc^2), \qquad v^{(s')\dagger}(\mathbf{p})v^{(s)}(\mathbf{p}) = \delta_{ss}(E/mc^2), \\ v^{(s')\dagger}(-\mathbf{p})u^{(s)}(\mathbf{p}) = u^{(s')\dagger}(-\mathbf{p})v^{(s)}(\mathbf{p}) = 0, \end{aligned} \tag{3.389}$$

where E is again understood to be positive. In terms of $\bar{u}^{(s)}(\mathbf{p})$ and $\bar{v}^{(s)}(\mathbf{p})$ we obtain from the Hermitian conjugate of (3.388),

$$\begin{aligned} \bar{u}^{(s)}(\mathbf{p})(i\gamma\cdot p + mc) &= 0, \\ \bar{v}^{(s)}(\mathbf{p})(-i\gamma\cdot p + mc) &= 0, \end{aligned} \tag{3.390}$$

where we have used $(\pm i\gamma\cdot p + mc)^\dagger\gamma_4 = \gamma_4(\pm i\gamma\cdot p + mc)$. It is also straightforward to prove with the aid of (3.389)

$$\begin{aligned} \bar{u}^{(s')}(\mathbf{p})u^{(s)}(\mathbf{p}) = \delta_{ss'}, \qquad \bar{v}^{(s')}(\mathbf{p})v^{(s)}(\mathbf{p}) &= -\delta_{ss'}, \\ \bar{u}^{(s')}(\mathbf{p})v^{(s)}(\mathbf{p}) = \bar{v}^{(s')}(\mathbf{p})u^{(s)}(\mathbf{p}) &= 0, \end{aligned} \tag{3.391}$$

for example, by multiplying the first expression of (3.388) from the left by $\bar{u}^{(s')}(\mathbf{p})\gamma_4$, multiplying the first expression of (3.390) (with s replaced by s') from the right by $\gamma_4 u^{(s)}(\mathbf{p})$, and adding the two. The expansions for ψ and $\bar{\psi}$ now take the form

$$\begin{aligned} \psi(\mathbf{x}, t) = \frac{1}{\sqrt{V}}\sum_{\mathbf{p}}\sum_{s=1,2}\sqrt{\frac{mc^2}{E}}\Big(b_{\mathbf{p}}^{(s)}u^{(s)}(\mathbf{p})\exp\left[\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar} - \frac{iEt}{\hbar}\right] \\ + d_{\mathbf{p}}^{(s)\dagger}v^{(s)}(\mathbf{p})\exp\left[\frac{-i\mathbf{p}\cdot\mathbf{x}}{\hbar} + \frac{iEt}{\hbar}\right]\Big), \\ \bar{\psi}(\mathbf{x}, t) = \frac{1}{\sqrt{V}}\sum_{\mathbf{p}}\sum_{s=1,2}\sqrt{\frac{mc^2}{E}}\Big(d_{\mathbf{p}}^{(s)}\bar{v}^{(s)}(\mathbf{p})\exp\left[\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar} - \frac{iEt}{\hbar}\right] \\ + b_{\mathbf{p}}^{(s)\dagger}\bar{u}^{(s)}(\mathbf{p})\exp\left[\frac{-i\mathbf{p}\cdot\mathbf{x}}{\hbar} + \frac{iEt}{\hbar}\right]\Big), \end{aligned} \tag{3.392}$$

where from now on it is understood that *E shall always stand for the positive square root* $\sqrt{|\mathbf{p}|^2c^2 + m^2c^4}$. Note that in obtaining (3.392) from (3.373) we used the fact that the sum over $\mathbf{p}$ runs over *all* directions ($-\mathbf{p}$ as well as $\mathbf{p}$).

Going back to the Hamiltonian operator and the total charge operator, we can now rewrite (3.368) and (3.379) as follows:

$$\begin{aligned} H &= \sum_{\mathbf{p}}\sum_{s} E(b_{\mathbf{p}}^{(s)\dagger}b_{\mathbf{p}}^{(s)} - d_{-\mathbf{p}}^{(s)}d_{-\mathbf{p}}^{(s)\dagger}) \\ &= \sum_{\mathbf{p}}\sum_{s} E(b_{\mathbf{p}}^{(s)\dagger}b_{\mathbf{p}}^{(s)} + d_{\mathbf{p}}^{(s)\dagger}d_{\mathbf{p}}^{(s)} - 1), \end{aligned} \tag{3.393}$$

and

$$\begin{aligned} Q &= e\sum_{\mathbf{p}}\sum_{s}(b_{\mathbf{p}}^{(s)\dagger}b_{\mathbf{p}}^{(s)} + d_{-\mathbf{p}}^{(s)}d_{-\mathbf{p}}^{(s)\dagger}) \\ &= e\sum_{\mathbf{p}}\sum_{s}(b_{\mathbf{p}}^{(s)\dagger}b_{\mathbf{p}}^{(s)} - d_{\mathbf{p}}^{(s)\dagger}d_{\mathbf{p}}^{(s)} + 1). \end{aligned} \tag{3.394}$$

We recall that the anticommutation relations for the d and $d^\dagger$ are completely identical in from with those of the b and $b^\dagger$. This means, among other things, that the eigenvalue of $d_{\mathbf{p}}^{(s)\dagger}d_{\mathbf{p}}^{(s)}$ is one or zero. From (3.393) and (3.394) we see that if we interpret $d_{\mathbf{p}}^{(s)\dagger}d_{\mathbf{p}}^{(s)}$ as the number operator for a positive-energy positron, then we have the following satisfactory result: *A state with an extra positron has the expected extra positive energy and positive* ($-e = |e|$) *charge*. Thus we take

$$N_{\mathbf{p}}^{(e^-,s)} = b_{\mathbf{p}}^{(s)\dagger}b_{\mathbf{p}}^{(s)}, \qquad N_{\mathbf{p}}^{(e^+,s)} = d_{\mathbf{p}}^{(s)\dagger}d_{\mathbf{p}}^{(s)} \tag{3.395}$$

to be the occupation-number operators for the electron and the positron state characterized by $(\mathbf{p}, s)$. In hole-theory language, where r runs from 1 to 4, the application of an $E < 0$ electron creation operator $b_{\mathbf{p}}^{(3,4)\dagger}$ to the "physical vacuum" must result in a null state since all the negative-energy states are already filled. In our new notation this means that the application of $d_{\mathbf{p}}^{(1,2)}$ to the vacuum must result in a null state. This is reasonable if $d_{\mathbf{p}}^{(1,2)}$ is to be understood as the annihilation operator for a positron with $(\mathbf{p}, s)$; in the vacuum there is no positron to be annihilated. So for the vacuum state we require

$$b_{\mathbf{p}}^{(s)}|0\rangle = 0, \qquad d_{\mathbf{p}}^{(s)}|0\rangle = 0. \tag{3.396}$$

From the anticommutation relation between $N_{\mathbf{p}}^{(e^+,s)}$, $d_{\mathbf{p}}^{(s)}$, and $d_{\mathbf{p}}^{(s)\dagger}$, it follows that the eigenvalue of $N_{\mathbf{p}}^{(e^+,s)}$ is zero for the vacuum state and one for a single positron state $d_{\mathbf{p}}^{(s)\dagger}|0\rangle$ (cf. Eqs. 2.55 through 2.57). In other words, $d_{\mathbf{p}}^{(s)\dagger}$ is the creation operator for a positron.

The expressions (3.393) and (3.394) are still not completely satisfactory. It is true that according to (3.393) the vacuum is the state with the lowest possible energy; however, if we apply H to the vacuum state, we get $-\Sigma_{\mathbf{p}}\Sigma_s E$ which is $-\infty$. Physically this means that the infinite negative energy of the Dirac sea has not yet been properly subtracted. We can redefine the energy scale so that H applied to $|0\rangle$ gives a zero eigenvalue. We then have

$$H = \sum_{\mathbf{p}}\sum_{s} E(N_{\mathbf{p}}^{(e^-,s)} + N_{\mathbf{p}}^{(e^+,s)}) \tag{3.397}$$

whose eigenvalue is necessarily positive semidefinite. Similarly subtracting the infinite negative charge of the Dirac sea, we obtain

$$\begin{aligned} Q &= e\sum_{\mathbf{p}}\sum_{s}(N_{\mathbf{p}}^{(e^-,s)} - N_{\mathbf{p}}^{(e^+,s)}) \\ &= -|e|\sum_{\mathbf{p}}\sum_{s}(N_{\mathbf{p}}^{(e^-,s)} - N_{\mathbf{p}}^{(e^+,s)}). \end{aligned} \tag{3.398}$$

This subtraction procedure amounts to starting with the charge density‡

$$\rho = e\psi^\dagger\psi - e\langle\psi^\dagger\psi\rangle_0. \tag{3.399}$$

Note that, unlike the total charge in the c-number theory (3.355) which is necessarily negative, the eigenvalue of (3.398) can be negative or positive. Now at last we can forget completely about negative energy electrons, the picturesque Dirac sea, the negatively charged particles with $E < 0$ that behave like positively charged particles, the absence of $\mathbf{p}$ appearing as the presence of $-\mathbf{p}$, and all that. From now on we can work with electrons and positrons of *positive energies only*.

We have seen that once we define both the energy and the charge of the vacuum state to be zero, then the total energy of the Dirac field is necessarily positive

‡The expression (3.399) can be shown to be equal to

$$(e/2)(\psi^\dagger\psi - \psi^T\psi^{\dagger T})$$

which is not zero in the q-number theory. This method of eliminating the undesirable vacuum expectation value is due to W. Heisenberg.

semidefinite, while the total charge can be negative or positive. We emphasize that in obtaining this satisfactory result the *anticommutation* relations for the creation and annihilation operators have played a crucial role. Had we used commutation relations instead, we would have ended up with an expression for the Hamiltonian operator whose eigenvalue has no lower bound [since $b_{\mathbf{p}}^{(r)\dagger} b_{\mathbf{p}}^{(r)}$ with $r = 3, 4$ in (3.368) can take on an arbitrarily large positive number]. Thus the Dirac field must be quantized according to Fermi-Dirac statistics if we demand that there be a state with the lowest energy. We have actually illustrated a special case of a very general theorem which states that half-integral spin fields must be quantized according to Fermi-Dirac statistics while integer spin fields must be quantized according to Bose-Einstein statistics. This spin-statistics theorem, first proved by W. Pauli in 1940, is one of the crowning achievements of relativistic quantum theory.

Let us now look at the total momentum operator (3.382). We get

$$\begin{aligned}\mathbf{P} &= \sum_{\mathbf{p}} \sum_{s=1,2} \mathbf{p}(b_{\mathbf{p}}^{(s)\dagger} b_{\mathbf{p}}^{(s)} + d_{-\mathbf{p}}^{(s)} d_{-\mathbf{p}}^{(s)\dagger}) \\ &= \sum_{\mathbf{p}} \sum_{s=1,2} \mathbf{p} b_{\mathbf{p}}^{(s)\dagger} b_{\mathbf{p}}^{(s)} + \sum_{\mathbf{p}} \sum_{s=1,2} (-\mathbf{p})(-d_{\mathbf{p}}^{(s)\dagger} d_{\mathbf{p}}^{(s)} + 1) \\ &= \sum_{\mathbf{p}} \sum_{s=1,2} \mathbf{p}(N_{\mathbf{p}}^{(e^-,s)} + N_{\mathbf{p}}^{(e^+,s)}) \qquad (3.400)\end{aligned}$$

(since $\sum_{\mathbf{p}} \mathbf{p} = 0$). This justifies our earlier assertion that the physical momentum of the positron state $d_{\mathbf{p}}^{(s)\dagger}|0\rangle$ is $\mathbf{p}$, not $-\mathbf{p}$ (cf. Eq. 3.384).

In order to convince ourselves that $b_{\mathbf{p}}^{(s)\dagger}|0\rangle$ and $d_{\mathbf{p}}^{(s)\dagger}|0\rangle$ with the same $(\mathbf{p}, s)$ really are an electron and a positron state with the *same* spin direction, it is instructive to work out the effect of applying the spin operator to these states. Taking the spin density to be $(\hbar/2)\psi^\dagger \boldsymbol{\Sigma} \psi$ we obtain

$$S_3 = (\hbar/2) \int \psi^\dagger \Sigma_3 \psi \, d^3x \qquad (3.401)$$

for the z-component of the spin operator.‡ For an electron,

$$\begin{aligned}S_3 b_{\mathbf{p}}^{(s)\dagger}|0\rangle &= [S_3, b_{\mathbf{p}}^{(s)\dagger}]|0\rangle \\ &= \frac{\hbar}{2} \int \psi^\dagger \Sigma_3 \{\psi, b_{\mathbf{p}}^{(s)\dagger}\} d^3x |0\rangle \\ &= \frac{\hbar}{2} \frac{mc^2}{E} u^{(s)\dagger}(\mathbf{p}) \Sigma_3 u^{(s)}(\mathbf{p}) b_{\mathbf{p}}^{(s)\dagger}|0\rangle. \qquad (3.402)\end{aligned}$$

‡Within the framework of the Lagrangian formalism the ultimate justification for interpreting $(\hbar/2)\psi^\dagger \boldsymbol{\Sigma} \psi$ as the spin density rests on the fact that the constancy of

$$\int \psi^\dagger [-i\hbar(\mathbf{x} \times \nabla)_3 + (\hbar/2)\Sigma_3] \psi \, d^3x$$

is guaranteed by the invariance of the Lagrangian density under an infinitesimal rotation around the z-axis [see, for example, Bjorken and Drell (1965), pp. 17–19, 55]. Note that no additive constant is needed because, with S_3 given by (3.401), $S_3|0\rangle = 0$ is automatically satisfied by rotational invariance.

In contrast,

$$\begin{aligned}S_3 d_{\mathbf{p}}^{(s)\dagger}|0\rangle &= [S_3, d_{\mathbf{p}}^{(s)\dagger}]|0\rangle \\ &= -\frac{\hbar}{2} \int \{\psi^\dagger, d_{\mathbf{p}}^{(s)\dagger}\} \Sigma_3 \psi \, d^3x |0\rangle \\ &= -\frac{\hbar}{2} \frac{mc^2}{E} v^{(s)\dagger}(\mathbf{p}) \Sigma_3 v^{(s)}(\mathbf{p}) d_{\mathbf{p}}^{(s)\dagger}|0\rangle. \qquad (3.403)\end{aligned}$$

If the free-particle spinor corresponding to an electron state has an eigenvalue $\Sigma_3 = +1$, then (3.402) tells us that $S_3 = \hbar/2$, as expected. On the other hand, if we want to describe a positron with spin-*up*, (3.403) demands that we use a free-particle spinor v for which the eigenvalue of Σ_3 is *minus* one. As an example, take the electron at rest with spin-up. The corresponding free particle spinor is $u^{(1)}(0)$ with $\Sigma_3 = +1$. For the positron state at rest with the same spin direction we are supposed to use $v^{(1)}(0) = -u^{(4)}(0)$ which indeed has $\Sigma_3 = -1$. This, of course, is expected on the basis of the hole theory.

Under the charge conjugation operation defined in the previous section, a single electron state $b_{\mathbf{p}}^{(s)\dagger}|0\rangle$ goes into the corresponding positron state $d_{\mathbf{p}}^{(s)\dagger}|0\rangle$ with the same $(\mathbf{p}, s)$. Thus

$$b_{\mathbf{p}}^{(s)} \rightleftarrows d_{\mathbf{p}}^{(s)}, \qquad b_{\mathbf{p}}^{(s)\dagger} \rightleftarrows d_{\mathbf{p}}^{(s)\dagger} \qquad (3.404)$$

under charge conjugation. We see that the total charge operator (3.398) indeed changes its sign under charge conjugation in contrast to its c-number analog (3.355), which always has the same sign. The Hamiltonian operator (3.397) does remain invariant under charge conjugation, as it should.

In the previous section we also examined the invariance property of the Dirac equation under $eA_\mu \longrightarrow -eA_\mu$. In the quantized theory the steps (3.345) through (3.351) go through just as before if we replace ψ^* by $\psi^{\dagger T}$ (where the transpose operation brings the free-particle spinor back to the column matrix form without affecting the creation and annihilation operators); ψ^C, given by

$$\psi^C = \gamma_2 \psi^{\dagger T}, \qquad (3.405)$$

is now called the charge-conjugate field (rather than the charge-conjugate wave function). Using (3.385), we find that ψ^C in the free-field case is given by

$$\psi^C = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} \sum_s \sqrt{\frac{mc^2}{E}} \left(b_{\mathbf{p}}^{(s)\dagger} v_{\mathbf{p}}^{(s)} \exp\left[\frac{-i\mathbf{p}\cdot\mathbf{x}}{\hbar} + \frac{iEt}{\hbar}\right] + d_{\mathbf{p}}^{(s)} u^{(s)}(\mathbf{p}) \exp\left[\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar} - \frac{iEt}{\hbar}\right] \right). \qquad (3.406)$$

Comparing this with (3.392), we see that the transformation

$$\psi \longrightarrow \psi^C \qquad (3.407)$$

is achieved if we make the substitution (3.404) which is precisely the charge conjugation operation; ψ^C annihilates positrons and creates electrons just as ψ annihilates electrons and creates positrons.

We shall now briefly mention the anticommutation relations among ψ, $\psi^\dagger$, and $\bar{\psi}$. First, it is evident that

$$\{\psi_\alpha(x), \psi_\beta(x')\} = \{\psi_\alpha^\dagger(x), \psi_\beta^\dagger(x')\} = \{\bar{\psi}_\alpha(x), \bar{\psi}_\beta(x')\} = 0, \qquad (3.408)$$

where x now stands for the four-vector $(\mathbf{x}, ict)$. For $\psi(x)$ and $\psi^\dagger(x')$, we have the equal-time anticommutation relation first written by W. Heisenberg and W. Pauli:

$$\{\psi_\alpha(\mathbf{x}, t), \psi^\dagger_\beta(\mathbf{x}', t')\}_{t=t'} = \delta_{\alpha\beta}\delta^{(3)}(\mathbf{x} - \mathbf{x}'), \tag{3.409}$$

the proof of which is left as an exercise (Problem 3–13). This also implies

$$\{\psi_\alpha(\mathbf{x}, t), \bar{\psi}_\beta(\mathbf{x}', t')\}_{t=t'} = (\gamma_4)_{\alpha\beta}\delta^{(3)}(\mathbf{x} - \mathbf{x}'). \tag{3.410}$$

As for the anticommutation relation between ψ and $\bar{\psi}$ at different times, for our purpose it suffices to remark that $\{\psi_\alpha(x), \bar{\psi}_\beta(x')\}$ is a function of the four-vector $x - x'$ such that it vanishes when x and x' are separated by a space-like distance:‡

$$\{\psi_\alpha(x), \bar{\psi}_\beta(x')\} = 0 \quad \text{if} \quad (x - x')^2 = (\mathbf{x} - \mathbf{x}')^2 - c^2(t - t')^2 > 0. \tag{3.411}$$

Because of the *anticommutation* relation, it is clear that $\psi(x)$ and $\bar{\psi}(x')$ *do not commute* when $x - x'$ is spacelike. This is not disturbing since ψ and $\bar{\psi}$, having no classical analog, are not "measurable" in the same sense as $\mathbf{E}$ and $\mathbf{B}$ are measurable. On the other hand, for the charge-current density

$$j_\mu(x) = ie\bar{\psi}\gamma_\mu\psi - ie\langle 0|\bar{\psi}\gamma_\mu\psi|0\rangle, \tag{3.412}$$

which *is* "measurable," we obtain from (3.408) and (3.410),

$$[j_\mu(x), j_\nu(x')] = 0 \quad \text{if} \quad (x - x')^2 > 0, \tag{3.413}$$

where we have used

$$[AB, CD] = -AC\{D, B\} + A\{C, B\}D - C\{D, A\}B + \{C, A\}DB. \tag{3.414}$$

Thus measurements of charge-current densities performed at two different points separated by a spacelike distance cannot influence each other, in agreement with the causality principle.

Electromagnetic and Yukawa couplings. Let us now talk about the interaction of electrons and positrons with the electromagnetic field. The Hamiltonian density for the basic interaction is taken to be

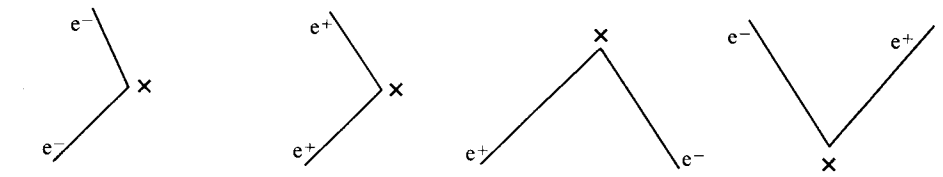
$$\mathscr{H}_{\text{int}} = -ie\bar{\psi}\gamma_\mu\psi A_\mu, \tag{3.415}$$

where ψ is now the quantized electron field; A_μ can be either classical or quantized. This interaction can be derived from the Lagrangian density

$$\mathscr{L}_{\text{int}} = ie\bar{\psi}\gamma_\mu\psi A_\mu, \tag{3.416}$$

since $\mathscr{H}_{\text{int}}$ is just the negative of $\mathscr{L}_{\text{int}}$ whenever $\mathscr{L}_{\text{int}}$ does not contain time derivatives of field operators. Strictly speaking, $ie\bar{\psi}\gamma_\mu\psi$ should be replaced by (3.412), but in practice the form (3.415) is sufficient since a constant (c-number) interaction cannot cause a transition between different states.

‡The explicit form of $\{\psi_\alpha(x), \bar{\psi}_\beta(x')\}$ known as $-iS_{\alpha\beta}(x - x')$ may be found in more advanced textbooks, for example, Mandl (1959), pp. 30–35, pp. 54–55; Schweber (1961), pp. 180–182, pp. 225–227.



(a) $ie\bar{\psi}^{(-)}\gamma_\mu\psi^{(+)}A_\mu$ (b) $ie\bar{\psi}^{(+)}\gamma_\mu\psi^{(-)}A_\mu$ (c) $ie\bar{\psi}^{(+)}\gamma_\mu\psi^{(+)}A_\mu$ (d) $ie\bar{\psi}^{(-)}\gamma_\mu\psi^{(-)}A_\mu$

Fig. 3–8. Four possibilities realized by the interaction (3.415).

In lowest-order perturbation theory (and also in covariant perturbation based on the interaction representation to be discussed in the next chapter), we may regard ψ in (3.415) as the free field ψ given by (3.392) and look at the effect of $\mathscr{H}_{\text{int}}$ on a given initial state. It is convenient to decompose each of ψ and $\bar{\psi}$ into two parts as follows:

$$\psi = \psi^{(+)} + \psi^{(-)}, \qquad \bar{\psi} = \bar{\psi}^{(+)} + \bar{\psi}^{(-)}, \tag{3.417}$$

where

$$\begin{aligned}
\psi^{(+)} &= \frac{1}{\sqrt{V}}\sum_{\mathbf{p}}\sum_s \sqrt{\frac{mc^2}{E}}\, b_{\mathbf{p}}^{(s)} u^{(s)}(\mathbf{p}) \exp\left(\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar} - \frac{iEt}{\hbar}\right),\\
\psi^{(-)} &= \frac{1}{\sqrt{V}}\sum_{\mathbf{p}}\sum_s \sqrt{\frac{mc^2}{E}}\, d_{\mathbf{p}}^{(s)\dagger} v^{(s)}(\mathbf{p}) \exp\left(\frac{-i\mathbf{p}\cdot\mathbf{x}}{\hbar} + \frac{iEt}{\hbar}\right),\\
\bar{\psi}^{(+)} &= \frac{1}{\sqrt{V}}\sum_{\mathbf{p}}\sum_s \sqrt{\frac{mc^2}{E}}\, d_{\mathbf{p}}^{(s)} \bar{v}^{(s)}(\mathbf{p}) \exp\left(\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar} - \frac{iEt}{\hbar}\right),\\
\bar{\psi}^{(-)} &= \frac{1}{\sqrt{V}}\sum_{\mathbf{p}}\sum_s \sqrt{\frac{mc^2}{E}}\, b_{\mathbf{p}}^{(s)\dagger} \bar{u}^{(s)}(\mathbf{p}) \exp\left(\frac{-i\mathbf{p}\cdot\mathbf{x}}{\hbar} + \frac{iEt}{\hbar}\right),
\end{aligned} \tag{3.418}$$

Note that $\psi^{(+)}$ (which is called the positive *frequency* part of ψ) is linear in the electron annihilation operators; it can therefore annihilate electrons but cannot do anything else. Likewise $\psi^{(-)}$, $\bar{\psi}^{(+)}$, and $\bar{\psi}^{(-)}$ respectively create positrons, annihilate positrons, and create electrons. So we see that the interaction (3.415) can do four different types of things, as shown in Fig. 3–8 where the symbol $\times$ may stand for an interaction with an external classical potential, the emission of a photon, or the absorption of a photon. The question of which of the four possibilities given in Fig. 3–8 is actually realized in a particular physical process depends entirely on what kind of initial and final state are present. For instance, if initially there is only a positron, the action of (3.415) may result in the scattering of the positron, represented by Fig. 3–8(b). On the other hand, if initially there is an electron-positron pair, and if we also know that there is neither an electron nor a positron in the final state, the matrix element of $\mathscr{H}_{\text{int}}$ is that of a pair annihilation process represented by Fig. 3–8(c). Note that in all cases the difference between the number of electrons and the number of positrons (3.356) is conserved. In Chapter 4 we shall work out a number of problems in quantum electrodynamics based on (3.415).

The meson-nucleon interaction proposed by H. Yukawa in 1935 is patterned after (3.415).‡ We have

$$\mathscr{H}_{\text{int}} = G\bar{\psi}\psi\phi, \tag{3.419}$$

where ψ and ϕ are now the nucleon and a neutral spin-zero meson field. Note that ψ can annihilate nucleons and create antinucleons while $\bar{\psi}$ can create nucleons and annihilate antinucleons. The constant, G, characterizes the strength of the coupling of the meson field to the nucleon. Alternatively we may have§

$$\mathscr{H}_{\text{int}} = iG\bar{\psi}\gamma_5\psi\phi. \tag{3.420}$$

Both (3.419) and (3.420) are invariant under proper orthochronous Lorentz transformations, as is evident from Table 3–1. The form of the interaction (3.419) can be made invariant under space inversion if ϕ in the space-inverted system is given by

$$\phi'(x') = \phi(x), \qquad x' = (-\mathbf{x}, ict). \tag{3.421}$$

On the other hand (3.420) can also be made invariant under space inversion if ϕ in the space-inverted system is given by

$$\phi'(x') = -\phi(x), \qquad x' = (-\mathbf{x}, ict), \tag{3.422}$$

since $\bar{\psi}\gamma_5\psi$ changes its sign. If we require that parity be conserved, we may have *either* (3.419) *only or* (3.420) *only* but not both simultaneously. In the next section we shall point out that a Yukawa interaction of the type (3.419) leads to an *s*-wave emission (absorption) of a meson by the nucleon while an interaction of the type (3.420) leads to a *p*-wave emission (absorption). The meson fields that transform like (3.421) and (3.422) under space inversion are called respectively a *scalar* and a *pseudoscalar* field; the couplings (3.419) and (3.420) are respectively called a *scalar* coupling and a *pseudoscalar* (*pseudoscalar*) coupling.‖ The meson fields corresponding to the observed neutral spin-zero mesons, π^0 and η, turn out to be both pseudoscalar. We shall say more about the meson-nucleon interaction when we discuss the one-pion exchange potential in the next chapter.

3–11. WEAK INTERACTIONS AND PARITY NONCONSERVATION; THE TWO-COMPONENT NEUTRINO

Classification of interactions. To illustrate the power of the field-theoretic formalism we have developed in the last section, we shall discuss in this section some simple examples taken from the physics of so-called "weak interactions." As is well

‡We emphasize that the field-theoretic description of the meson-nucleon interaction is not as firmly founded as that of the electromagnetic interaction of electrons and positrons. Quantum field theory does, however, provide a convenient language to describe some phenomena involving mesons and nucleons.

§The factor i is necessary to make the interaction density Hermitian. Note $(\bar{\psi}\gamma_5\psi)^\dagger = \psi^\dagger\gamma_5\gamma_4\psi = -\bar{\psi}\gamma_5\psi$.

‖So called to distinguish it from a pseudoscalar (pseudovector) coupling of the type $(iF\hbar/m_\pi c)(\partial\phi/\partial x_\mu)\,\bar{\psi}\gamma_5\gamma_\mu\psi$.

known, the basic interactions of elementary particles can be classified into three categories:

a) *strong* interactions, for example, $n + p \longrightarrow n + p$, $\pi^- + p \longrightarrow \Lambda + K^0$, $\rho^0 \longrightarrow \pi^+ + \pi^-$;

b) *electromagnetic* interactions, for example, $e^+ + e^- \longrightarrow 2\gamma$, $\gamma + p \longrightarrow p + \pi^0$, $\Sigma^0 \longrightarrow \Lambda + \gamma$;

c) *weak* interactions, for example, $n \longrightarrow p + e^- + \bar{\nu}$, $\bar{\nu}' + p \longrightarrow \mu^+ + \Lambda$, $K^+ \longrightarrow \pi^+ + \pi^0$.

To this list we should add a fourth (and the oldest) class of interactions, the *gravitational* interactions. But gravity turns out to be of little interest in our present-day understanding of elementary particles. What is remarkable is that all four classes of interactions are characterized by *dimensionless coupling constants* that differ by many orders of magnitude; the dimensionless coupling constant that characterizes electromagnetic interaction is $(e^2/4\pi\hbar c) = \alpha \simeq \frac{1}{137}$. As we shall show in Section 4–6, for the analogous coupling constant in pion-nucleon physics, defined as in (3.420), we have $(G^2/4\pi\hbar c) \simeq 14$, which is a typical strong-interaction coupling constant. In contrast, the constant that characterizes the strength of a typical weak-interaction process, when defined in a similar manner, turns out to be as small as 10^{-12} to 10^{-14}; this will be illustrated in a moment when we discuss Λ decay and pion decay. Gravitational interactions are even weaker; at the same separation distance the gravitational attraction between two protons is about 10^{-37} times weaker than the electrostatic repulsion between them.

Another striking feature of the elementary particle interactions is that some conservation laws which are obeyed to high degrees of accuracy by the strong and electromagnetic interactions are known to be violated by the weak interactions. In particular, weak-interaction processes *in general* do not conserve parity. In this section we shall illustrate this point using the language of quantum field theory.

Parity. One learns in nonrelativistic quantum mechanics that a state with momentum $\mathbf{p}$ goes into a state with momentum $-\mathbf{p}$ under parity, as is evident from the operator form of momentum, $\mathbf{p} = -i\hbar\nabla$. For instance, in the Schrödinger theory the plane-wave solution $\psi(\mathbf{x}, t) = \exp(i\mathbf{p}\cdot\mathbf{x}/\hbar - iEt/\hbar)$ goes into $\psi(-\mathbf{x}, t) = \exp[i(-\mathbf{p})\cdot\mathbf{x}/\hbar - iEt/\hbar]$, which we recognize as the plane-wave solution with momentum *opposite* to the original one. The orbital angular momentum $\mathbf{L}$ is unchanged under parity since it is given by $\mathbf{x} \times \mathbf{p}$. Furthermore, we argue that the spin angular momentum is also even under parity since space inversion commutes with an infinitesimal rotation. As a result the magnetic moment interaction $-(e\hbar/2mc)\boldsymbol{\sigma}\cdot\mathbf{B}$, for instance, is invariant under parity since $\mathbf{B}$ does not change under parity either. From these considerations we expect that in field theory an electron state $b_{\mathbf{p}}^{(s)\dagger}|0\rangle$ goes into $b_{-\mathbf{p}}^{(s)\dagger}|0\rangle$ (with the same s), etc. We shall see, in a moment, that this is essentially the case except that we have to be careful with the relative transformation properties of the electron and positron states.

In Section 3–4 we proved that the Dirac wave function in the space-inverted system $\mathbf{x}' = -\mathbf{x}$, $t' = t$ is related to the wave function in the original system by $\psi'(\mathbf{x}', t) = \gamma_4\psi(\mathbf{x}, t)$ (up to a phase factor). The proof given there goes through just as well even if ψ stands for the quantized Dirac field. If we now consider the space inversion operation (commonly called the parity operation) that transforms $\mathbf{x}$ into $-\mathbf{x}$, we see that the functional form of the field operator changes as

$$\psi(\mathbf{x}, t) \longrightarrow \psi'(\mathbf{x}, t) = \gamma_4\psi(-\mathbf{x}, t), \tag{3.423}$$

since what used to be called $\mathbf{x}' = -\mathbf{x}$ is now $\mathbf{x}$. What does this mean? Going back to the plane-wave expansion of the filed operator, we have

$$\gamma_4\psi(-\mathbf{x}, t) = \sum_{\mathbf{p}}\sum_{s}\sqrt{\frac{mc^2}{E}}\left[b_{\mathbf{p}}^{(s)}\gamma_4 u^{(s)}(\mathbf{p})\exp\left(-\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar} - \frac{iEt}{\hbar}\right)\right.$$
$$\left. + d_{\mathbf{p}}^{(s)\dagger}\gamma_4 v^{(s)}(\mathbf{p})\exp\left(\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar} + \frac{iEt}{\hbar}\right)\right]. \tag{3.424}$$

But it is easy to show (cf. Problem 3–4) that

$$\gamma_4 u^{(s)}(\mathbf{p}) = u^{(s)}(-\mathbf{p}), \qquad \gamma_4 v^{(s)}(\mathbf{p}) = -v^{(s)}(-\mathbf{p}). \tag{3.425}$$

Note the minus sign preceding the positron spinor, $v^{(s)}(-\mathbf{p})$; a special case of this has already been worked out in Section 3–4, where we learned that a positive-energy at-rest spinor and a negative-energy at-rest spinor behave oppositely under parity. Thus

$$\gamma_4\psi(-\mathbf{x}, t) = \sum_{\mathbf{p}}\sum_{s}\sqrt{\frac{mc^2}{E}}\left[b_{\mathbf{p}}^{(s)}u^{(s)}(\mathbf{p})\exp\left(\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar} - \frac{iEt}{\hbar}\right)\right.$$
$$\left. - d_{\mathbf{p}}^{(s)\dagger}v^{(s)}(\mathbf{p})\exp\left(\frac{-i\mathbf{p}\cdot\mathbf{x}}{\hbar} + \frac{iEt}{\hbar}\right)\right]. \tag{3.426}$$

Comparing this with (3.392) we deduce that the transformation (3.423) is accomplished if the creation and annihilation operators change in the following manner:

$$\begin{aligned} b_{\mathbf{p}}^{(s)} &\longrightarrow b_{-\mathbf{p}}^{(s)}, \qquad d_{\mathbf{p}}^{(s)} \longrightarrow -d_{-\mathbf{p}}^{(s)}, \\ b_{\mathbf{p}}^{(s)\dagger} &\longrightarrow b_{-\mathbf{p}}^{(s)\dagger}, \qquad d_{-\mathbf{p}}^{(s)\dagger} \longrightarrow -d_{-\mathbf{p}}^{(s)\dagger}. \end{aligned} \tag{3.427}$$

An immediate consequence of this is that a single-particle state with $(\mathbf{p}, s)$ goes into a state with $(-\mathbf{p}, s)$, as expected from nonrelativistic quantum mechanics, except that a positron state acquires a minus sign,

$$\begin{aligned} b_{\mathbf{p}}^{(s)\dagger}|0\rangle &\longrightarrow \Pi\, b_{\mathbf{p}}^{(s)\dagger}|0\rangle = b_{-\mathbf{p}}^{(s)\dagger}|0\rangle, \\ d_{\mathbf{p}}^{(s)\dagger}|0\rangle &\longrightarrow \Pi\, d_{\mathbf{p}}^{(s)\dagger}|0\rangle = -d_{-\mathbf{p}}^{(s)\dagger}|0\rangle, \end{aligned} \tag{3.428}$$

where Π is the parity operator that acts on state vectors.‡

We shall now discuss the physical significance of the minus sign in the second part of (3.428). Consider, for simplicity, an electron-positron system in which the electron and the positron are both at rest, hence are in a relative s-state. According

‡Using the operator Π, we can write (3.427) as $\Pi b_{\mathbf{p}}^{(s)}\Pi^{-1} = b_{-\mathbf{p}}^{(s)}$, etc. We assume that the vacuum state is even by convention: $\Pi|0\rangle = |0\rangle$.

to (3.428), such a system transforms as

$$b_{\mathbf{p}=0}^{(s)\dagger}d_{\mathbf{p}=0}^{(s')\dagger}|0\rangle \longrightarrow -b_{\mathbf{p}=0}^{(s)\dagger}d_{\mathbf{p}=0}^{(s')\dagger}|0\rangle \tag{3.429}$$

under parity. This means that the parity of an s-state e^-e^+ system is *odd*.‡ Actually we could have anticipated this from the hole theory. According to the hole-theoretic interpretation, what is physically observable is the relative parity between the completely filled Dirac sea and the Dirac sea with one negative-energy electron missing. Remembering that parity is a multiplicative concept in the sense that the parity of a composite system is the product of the parities of the constituent systems, we infer that the observable parity of the positron is the *same* as that of the missing negative-energy electron. But, according to our earlier discussion following (3.179), the parity of a negative-energy electron at rest is odd (when the phase factor η is so chosen that the parity of a positive-energy electron at rest is even). Hence the parity of a positron at rest is odd (relative to that of an electron at rest).

If we follow steps similar to (3.423) through (3.429) with a non-Hermitian (charged) field $\phi_{\text{ch}}(\mathbf{x}, t)$ which transforms as

$$\phi_{\text{ch}}(\mathbf{x}, t) \longrightarrow \pm\phi_{\text{ch}}(-\mathbf{x}, t) \tag{3.430}$$

under parity, it is not difficult to show that a π^- state and a π^+ state transform in the *same* way under parity and that a $\pi^+\pi^-$ system in a relative s-state is *even* (in sharp contrast to the e^+e^- case). Quite generally, the "intrinsic parity" of the "antiparticle" is opposite to that of the corresponding "particle" in the fermion case and is the same in the boson case. This is one of the most important results of relativistic quantum theory. In Section 4–4, we shall present experimental evidence in favor of the odd parity of an s-state e^+e^- system.

Hyperon decay. So much for the transformation properties of the free-particle states. Let us now examine how we may describe parity-nonconserving decay processes using the language of quantum field theory. As a particularly simple example, we shall consider the decay of a free Λ hyperon (known to be a spin-½ particle):

$$\Lambda \longrightarrow \text{p} + \pi^-. \tag{3.431}$$

A simple interaction density (operator) that can account for this process is

$$\mathscr{H}_{\text{int}} = \phi_\pi^\dagger\bar{\psi}_{\text{p}}(g + g'\gamma_5)\psi_\Lambda + \text{Hc}, \tag{3.432}$$

where ψ_Λ can annihilate Λ hyperons and create anti-Λ hyperons,§ $\bar{\psi}_{\text{p}}$ can annihilate antiprotons and create protons, and $\phi_\pi^\dagger$ can create π^- and annihilate π^+. We have added Hc which stands for the "Hermitian conjugate" because, without

‡The skeptical reader may demonstrate that this conclusion (which can be verified experimentally) is independent of our choice $\eta = 1$ in $\psi'(x') = \eta\gamma_4\psi(x)$.

§The anti-Λ hyperon is to be distinguished from the Λ hyperon. First, even though they have the same charge (namely, zero), the same mass, and the same lifetime, their magnetic moments are opposite. Second, annihilations of a $\bar{\Lambda}$p system into mesons, for example, $\bar{\Lambda} + \text{p} \longrightarrow \text{K}^+ + \pi^+ + \pi^-$, have been observed, whereas reactions of the type $\Lambda + \text{p} \longrightarrow$ mesons are strictly forbidden.

it, the interaction density would not be Hermitian.‡ Explicitly stated,

$$\begin{aligned} \mathrm{Hc} &= \psi_\Lambda^\dagger(g^* + g'^*\gamma_5)\gamma_4\psi_\mathrm{p}\phi_\pi \\ &= \bar{\psi}_\Lambda(g^* - g'^*\gamma_5)\psi_\mathrm{p}\phi_\pi. \end{aligned} \tag{3.433}$$

We see that the Hc can give rise to processes like

$$\bar{\Lambda} \longrightarrow \bar{\mathrm{p}} + \pi^+, \tag{3.434}$$

where $\bar{\Lambda}$ and $\bar{\mathrm{p}}$ stand for an anti-Λ hyperon and an antiproton. It is a characteristic feature of the quantum-field-theoretic description that if (3.431) takes place, then (3.434) must also take place. At the end of this section, we shall show that this feature is a consequence of what is known as CPT invariance. Experimentally the decay process (3.434) has indeed been observed to be as follows:

$$\begin{array}{l} \mathrm{p} + \bar{\mathrm{p}} \longrightarrow \Lambda + \bar{\Lambda} \\ \qquad\qquad\quad \big\lfloor\!\!\longrightarrow \bar{\mathrm{p}} + \pi^+ \\ \qquad\quad \big\lfloor\!\!\longrightarrow \mathrm{p} + \pi^-. \end{array} \tag{3.435}$$

We do not actually believe that the Hamiltonian density (3.432) is a "fundamental" interaction in nature in the same way as we believe that $-ie\bar{\psi}_\mathrm{e}\gamma_\mu\psi_\mathrm{e}A_\mu$ is "fundamental." So many new particles and new decay processes are observed nowadays in high-energy nuclear physics that if we were to introduce a new fundamental interaction every time a new decay process was discovered, a complete list of the fundamental interactions would become ridiculously long. Anybody in his right mind would then say that most (or perhaps all) of the interactions in the list could not possibly be "fundamental." Unfortunately, as yet we do not know what the basic interaction mechanisms are which give rise to a phenomenological interaction of the kind described by (3.432). In any case, for computational purposes, let us go along with (3.432).

We shall now investigate the transformation properties of (3.432). First, we can easily see that (3.432) is invariant under a proper orthochronous Lorentz transformation,

$$\begin{aligned} \bar{\psi}'_\mathrm{p}\begin{Bmatrix}1\\ \gamma_5\end{Bmatrix}\psi'_\Lambda\phi'_\pi &= \bar{\psi}_\mathrm{p}S_\mathrm{Lor}^{-1}\begin{Bmatrix}1\\ \gamma_5\end{Bmatrix}S_\mathrm{Lor}\psi_\Lambda\phi_\pi \\ &= \bar{\psi}_\mathrm{p}\begin{Bmatrix}1\\ \gamma_5\end{Bmatrix}\psi_\Lambda\phi_\pi, \end{aligned} \tag{3.436}$$

since the form of $S_\mathrm{Lor}(S_\mathrm{Lor}^{-1})$ is independent of the type of spin-½ field. Next, we shall show that the interaction (3.432) is not invariant under parity unless $g = 0$ or $g' = 0$. Since ψ_p, ψ_Λ, and ϕ_π transform under parity as

$$\psi_\mathrm{p} \longrightarrow \eta_\mathrm{p}\gamma_4\psi_\mathrm{p}(-\mathbf{x}, t), \qquad \psi_\Lambda \longrightarrow \eta_\Lambda\gamma_4\psi_\Lambda(-\mathbf{x}, t), \qquad \phi_\pi \longrightarrow \eta_\pi\phi_\pi(-\mathbf{x}, t), \tag{3.437}$$

‡The use of a Hermitian Hamiltonian density is required since the fexpectation value o the Hamiltonian operator must be real. Furthermore, in Section 4–2 we shall show that the use of a non-Hermitian Hamiltonian violates probability conservation.

the interaction density (3.432) changes under the parity operation as follows:

$$\mathscr{H}_\mathrm{int}(\mathbf{x}, t) \longrightarrow \eta_\mathrm{p}^*\eta_\pi^*\eta_\Lambda\phi_\pi^\dagger(-\mathbf{x}, t)\bar{\psi}_\mathrm{p}(-\mathbf{x}, t)(g - g'\gamma_5)\psi_\Lambda(-\mathbf{x}, t) + \mathrm{Hc}. \tag{3.438}$$

Suppose $g' = 0$; then $\mathscr{H}_\mathrm{int}$ transforms like a scalar density,

$$\mathscr{H}_\mathrm{int}(\mathbf{x}, t) \longrightarrow \mathscr{H}_\mathrm{int}(-\mathbf{x}, t), \tag{3.439}$$

provided we choose the phase factors in (3.437) in such a way that $\eta_\mathrm{p}^*\eta_\pi^*\eta_\Lambda = 1$. Similarly for $g = 0$, $\mathscr{H}_\mathrm{int}$ again transforms like a scalar density provided we set $\eta_\mathrm{p}^*\eta_\pi^*\eta_\Lambda = -1$. However, if both g and g' are nonvanishing, then $\mathscr{H}_\mathrm{int}$ cannot be made to transform like a scalar density no matter how we choose η_p, η_π, and η_Λ. This is what we mean by saying that *the interaction is not invariant under parity.*

All this appears somewhat formal. We now wish to exhibit some of the physical manifestations of the parity-nonconserving interaction (3.432). For this purpose let us obtain the transition matrix element for this process. First, recall that the state vectors we have been using are time independent, while the interaction Hamiltonian density is made up of time-dependent field operators. In the language of time-dependent perturbation theory, discussed in Section 2–4, a state vector in our occupation-number space corresponds to a time-independent wave function $u_n(\mathbf{x})$ rather than to a time-dependent one $u_n(\mathbf{x})e^{-iE_nt/\hbar}$. Recalling the connection between the operators in the Heisenberg and the Schrödinger representations, we see that the matrix element of $\int \mathscr{H}_\mathrm{int}d^3x$ taken between the initial Λ state and the final $\pi^-\mathrm{p}$ state corresponds, in the language of Chapter 2, to the matrix element of $e^{iH_0t/\hbar}H_Ie^{-iH_0t/\hbar}$ taken between $u_i(\mathbf{x})$ and $u_f(\mathbf{x})$; the only new point is that our Hamiltonian can now change the nature of the particles. Using this fact and recalling the expression for $c^{(1)}$ given by (2.109) we see that $c^{(1)}(t)$ in our case is given by

$$c^{(1)}(t) = (-i/\hbar)\left\langle f\middle|\int_0^t dt'\int \mathscr{H}_\mathrm{int}(\mathbf{x}, t')\,d^3x\middle|i\right\rangle, \tag{3.440}$$

where the perturbation is assumed to be turned on at $t = 0$.‡

In the language of quantum field theory the initial and final states in Λ decay are given by

$$\begin{aligned} |i\rangle &= b_\mathbf{p}^{(\Lambda,s)\dagger}|0\rangle, \\ |f\rangle &= a^\dagger(\mathbf{p}_\pi)b_{\mathbf{p}'}^{(\mathrm{p},s')\dagger}|0\rangle, \end{aligned} \tag{3.441}$$

where $b_\mathbf{p}^{(\Lambda,s)\dagger}$, $b_{\mathbf{p}'}^{(\mathrm{p},s')\dagger}$, and $a^\dagger(\mathbf{p}_\pi)$ are the creation operators for the Λ hyperon, the proton, and the π^- meson. It is legitimate to replace ψ_Λ by its positive frequency part $\psi_\Lambda^{(+)}$ since the negative frequency part of ψ_Λ acting on the initial state would result in a $(\Lambda + \bar{\Lambda})$ state which, because of orthogonality, gives zero when multiplied by $\langle f|\phi_\pi^\dagger\bar{\psi}_\mathrm{p}$ from the left. Furthermore, the only part of $\psi_\Lambda^{(+)}$ which contributes is just $(1/\sqrt{V})\,b_\mathbf{p}^{(\Lambda,s)}u_\Lambda^{(s)}(\mathbf{p})\exp(i\mathbf{p}\cdot\mathbf{x}/\hbar - iEt/\hbar)$, since $b_{\mathbf{p}''}^{(\Lambda,s'')}b_\mathbf{p}^{(\Lambda,s)\dagger}|0\rangle = 0$, unless $\mathbf{p}'' = \mathbf{p}$, $s'' = s$. All this amounts to saying that it is all right to replace the

‡We shall present a more formal discussion of the connection between the transition matrix and the interaction Hamiltonian in Section 4–2, when we shall discuss the S-matrix expansion in the interaction representation.

field operator ψ_Λ by the wave function of the initial Λ particle multiplied by the corresponding annihilation operator. Likewise it is easy to see that $\phi_\pi^\dagger$ can be replaced by just

$$a^\dagger(\mathbf{p}_\pi)c\sqrt{\hbar/2\omega_\pi V}\exp(-i\mathbf{p}_\pi\cdot\mathbf{x}/\hbar + i\omega_\pi t)$$

(with $\hbar\omega_\pi = \sqrt{|\mathbf{p}_\pi|^2c^2 + m_\pi^2c^4}$), and $\bar{\psi}_\mathrm{p}$ can be replaced by

$$\sqrt{m_\mathrm{p}c^2/E_\mathrm{p}V}\,b_{\mathbf{p}'}^{(\mathrm{p},s')\dagger}\bar{u}_\mathrm{p}^{(s')}(\mathbf{p}')\exp(-i\mathbf{p}'\cdot\mathbf{x}/\hbar + iE_\mathrm{p}t/\hbar).$$

The result of all this gives

$$\begin{aligned}\left\langle f\left|\int \mathscr{H}_\mathrm{int}\,d^3x\right|i\right\rangle &= \langle f|a^\dagger(\mathbf{p}_\pi)b_{\mathbf{p}'}^{(\mathrm{p},s')\dagger}b_{\mathbf{p}=0}^{(\Lambda,s)}|i\rangle(c\sqrt{\hbar/2\omega_\pi})(\sqrt{m_\mathrm{p}c^2/E_\mathrm{p}})\bar{u}_\mathbf{p}^{(s')}(\mathbf{p}')(g+g'\gamma_5)u_\Lambda^{(s)}(0)\\ &\times\left[\frac{1}{\sqrt{V^3}}\int\exp\left(-\frac{i\mathbf{p}_\pi\cdot\mathbf{x}}{\hbar}-\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar}\right)d^3x\right]\exp\left(i\omega_\pi t+\frac{iE_\mathrm{p}t}{\hbar}-\frac{im_\Lambda c^2t}{\hbar}\right),\end{aligned} \tag{3.442}$$

where we have assumed that the initial Λ particle is at rest. Using

$$b_{\mathbf{p}=0}^{(\Lambda,s)}b_{\mathbf{p}=0}^{(\Lambda,s)\dagger}|0\rangle = (1 - b_{\mathbf{p}=0}^{(\Lambda,s)\dagger}b_{\mathbf{p}=0}^{(\Lambda,s)})|0\rangle,\quad\text{etc.},$$

it is not difficult to see that $\langle f|a^\dagger(\mathbf{p}_\pi)b_{\mathbf{p}'}^{(\mathrm{p},s')\dagger}b_{\mathbf{p}=0}^{(\Lambda,s)}|i\rangle$ can eventually be reduced to $\langle 0|0\rangle = 1$. The exponential time dependence in (3.442) is precisely the kind that appears in the derivation of the Golden Rule in time-dependent perturbation theory (cf. Eq. 2.113); if we assume that the perturbation is switched on at $t = 0$ and acts for a long time, the modulus squared of $\int_0^t \exp(i\omega_\pi t + iE_\mathrm{p}t'/\hbar - im_\Lambda c^2t'/\hbar)dt'$ leads to $2\pi\hbar t$ times the usual δ function that expresses energy conservation. Note also that the space integral in (3.442) simply tells us that the transition matrix element is zero unless momentum is conserved.

To sum up, the time-dependent matrix element that appears in the Golden Rule can be obtained immediately from the Hamiltonian operator $\int\mathscr{H}_\mathrm{int}d^3x$ just by replacing the quantized field operators in $\mathscr{H}_\mathrm{int}$ by the appropriate initial and final wave functions with their time dependence omitted. In other words, we get the correct results by pretending that the q-number density (3.432) made up of the field operators is a c-number density made up of the initial- and final-state wave functions.‡

Let us now simplify the spinor product in (3.442). We have

$$\begin{aligned}&\bar{u}_\mathrm{p}^{(s')}(\mathbf{p}')(g+g'\gamma_5)u_\Lambda^{(s)}(0)\\ &= \sqrt{\frac{m_\mathrm{p}c^2+E_\mathrm{p}}{2m_\mathrm{p}c^2}}\left(\chi^{(s')\dagger},\ -\chi^{(s')\dagger}\frac{(\boldsymbol{\sigma}\cdot\mathbf{p}'c)}{E_\mathrm{p}+m_\mathrm{p}c^2}\right)\begin{pmatrix}g & -g'\\ -g' & g\end{pmatrix}\begin{pmatrix}\chi^{(s)}\\ 0\end{pmatrix}\\ &= \sqrt{\frac{m_\mathrm{p}c^2+E_\mathrm{p}}{2m_\mathrm{p}c^2}}\chi^{(s')\dagger}\left(g+g'\frac{\boldsymbol{\sigma}\cdot\mathbf{p}'c}{E_\mathrm{p}+m_\mathrm{p}c^2}\right)\chi^{(s)}.\end{aligned} \tag{3.443}$$

‡In fact this kind of replacement is implicit in the discussions of beta decay that appear in most textbooks on nuclear physics, for example, Segrè (1964), Chapter 9, and Preston (1964), Chapter 15. We now understand why beta decay can be discussed at an elementary level without using the language of quantum field theory.

Assuming that the initial Λ is polarized with spin along the positive z-axis, we obtain

$$\left(g+g'\frac{\boldsymbol{\sigma}\cdot\mathbf{p}'c}{E_\mathrm{p}+mc^2}\right)\chi^{(s)} = (a_s + a_p\cos\theta)\begin{pmatrix}1\\0\end{pmatrix} + a_p\sin\theta\,e^{i\phi}\begin{pmatrix}0\\1\end{pmatrix}, \tag{3.444}$$

where

$$a_s = g,\qquad a_p = g'(|\mathbf{p}'|c/(E_\mathrm{p}+m_\mathrm{p}c^2)) \tag{3.445}$$

and the angles θ and ϕ characterize the orientation of $\mathbf{p}'$ relative to the Λ spin direction. The physical meaning of g and g' can now be seen as follows. If $g \neq 0$, $g' = 0$, the final state proton can be described by an $s_{\frac{1}{2}}$, $j_z = \frac{1}{2}$ wave function $\binom{1}{0}$. On the other hand, if $g = 0$, $g' \neq 0$, the final proton is in a pure $p_{\frac{1}{2}}$, $j_z = \frac{1}{2}$ state described by $\cos\theta\binom{1}{0} + \sin\theta\,e^{i\phi}\binom{0}{1}$. In other words, the scalar coupling (the g-term) gives rise to an $s_{\frac{1}{2}}\,\pi^-\mathrm{p}$ system, whereas the pseudoscalar (γ_5) coupling (the g'-term) gives rise to a $p_{\frac{1}{2}}\,\pi^-\mathrm{p}$ system. If g and g' are both nonvanishing, both $s_{\frac{1}{2}}$ and $p_{\frac{1}{2}}$ are allowed; in other words, the same initial state can go into final states of *opposite* parities. Recalling that for $g \neq 0$, $g' \neq 0$ the interaction density (3.432) is not invariant under parity, we see that an $\mathscr{H}_\mathrm{int}$ that does not transform like (3.439) under parity indeed gives rise to final states of opposite parities.

We shall digress here and examine the pseudoscalar (pseudoscalar) coupling of the pion to the nucleon (3.420) which gives rise to processes like $\mathrm{p}\longrightarrow\mathrm{p}+\pi^0$. We may argue that when the proton dissociates itself into a π^0 and a proton, the final $\pi\mathrm{p}$ system must be in $p_{\frac{1}{2}}$ state. Unfortunately, such dissociation processes are forbidden by energy momentum conservation if all the particles are free. However, taking advantage of a reaction in which the nucleon is bound (specifically $\pi^+ + \mathrm{d}\longrightarrow\mathrm{p}+\mathrm{p}$) it has been proved possible to show that the $\pi^\pm$ is pseudoscalar (with the convention that the proton and the neutron are both even), that is, $\phi(\mathbf{x},t)\longrightarrow -\phi(-\mathbf{x},t)$. Note that if there is just a γ_5 type coupling, parity is conserved; the "intrinsic" odd parity of the pion is compensated for by the odd orbital parity.‡

Coming back to Λ decay, we can now compute the decay-angular distribution. The relative probabilities of observing the proton with spin-up and spin-down can be obtained immediately from (3.444):

$$\begin{aligned}&\text{spin-up:}\quad &&|a_s + a_p\cos\theta|^2,\\ &\text{spin-down:}\quad &&|a_p|^2\sin^2\theta,\end{aligned} \tag{3.446}$$

which results in the decay-angular distribution of the proton

$$1 - \alpha\cos\theta, \tag{3.447}$$

where§

$$\alpha = -\frac{2\,\mathrm{Re}(a_s a_p^*)}{|a_s|^2 + |a_p|^2}. \tag{3.448}$$

‡An alternative way of understanding what is meant by the intrinsic odd parity of the pion is to visualize the pion as a very tightly bound state of a nucleon (intrinsically even) and an antinucleon (intrinsically odd) in a relative s-state.

§The minus sign in (3.447) arises from the fact that the experimentalists usually talk about the decay-angular distribution of the pion relative to the Λ spin: $1 + \alpha\cos\theta^{(\pi)}$ where $\cos\theta^{(\pi)} = -\cos\theta$.

Recalling that θ is measured from the Λ spin direction, we see that the angular distribution (3.447) implies that whenever $\mathrm{Re}(a_s a_p^*) \neq 0$ or equivalently $\mathrm{Re}(gg'^*) \neq 0$, there exists an observable effect that depends on $\langle \boldsymbol{\sigma}_\Lambda \rangle \cdot \mathbf{p}'$, a pseudoscalar quantity that changes its sign under parity.

To really understand the meaning of parity nonconservation in this decay process, it is instructive to work out the special decay configuration $\theta = 0$. This is shown in Fig. 3–9(a). The transition probability for this process is, according to (3.446), $|a_s + a_p|^2$ apart from kinematical factors. If we apply the parity operation to the decay configuration shown in Fig. 3–9(a), we obtain the decay configuration shown in Fig. 3–9(b) since, under parity, momentum changes but spin does not. However, according to (3.446), the transition probability for the physical situation described by Fig. 3–9(b) is $|a_s - a_p|^2$ since $\theta = \pi$. Thus the transition probability for $|i\rangle \longrightarrow |f\rangle$ is not the same as that for $\Pi|i\rangle \longrightarrow \Pi|f\rangle$ unless $\mathrm{Re}(a_s a_p^*) = 0$. Since the configuration in Fig. 3–9(b) is the mirror image of the configuration in Fig. 3–9(a) apart from a 180° rotation about an axis perpendicular to $\mathbf{p}'$, we conclude that the mirror image of our world looks different from our world if $\mathrm{Re}(gg'^*) \neq 0$.

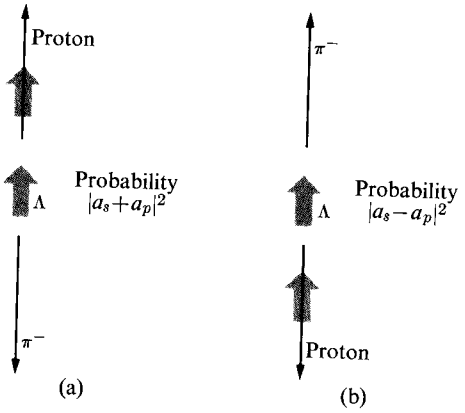


Fig. 3–9. Λ decay. Parity conservation would require that the two decay configurations (which go into each other under space inversion) be physically realizable with the same transition probability. The gray arrows indicate the spin direction.

Although we cannot prepare a polarized sample of Λ hyperons using a magnet, it turns out that the Λ hyperons produced in

$$\pi^- + \mathrm{p} \longrightarrow \Lambda + \mathrm{K}^0 \tag{3.449}$$

are strongly polarized in the direction $\mathbf{p}_{\pi\ \text{incident}} \times \mathbf{p}_\Lambda$. Parity nonconservation in Λ decay was unambiguously established, in 1957, by a Berkeley group and by a Columbia-Michigan-Pisa-Bologna collaboration group who showed that there are more decay pions emitted with $(\mathbf{p}_{\pi\ \text{incident}} \times \mathbf{p}_\Lambda) \cdot \mathbf{p}_{\pi\ \text{decay}} > 0$ than with $(\mathbf{p}_{\pi\ \text{incident}} \times \mathbf{p}_\Lambda) \cdot \mathbf{p}_{\pi\ \text{decay}} < 0$. Now, at last, it is possible to communicate even to intelligent beings in outer space that the incident pion direction, the Λ direction,

and the preferential direction of the decay pion, taken in that order, form the three axes of what we mean by a right-handed system.

Prior to 1956 practically everybody tacitly assumed that it was "illegal" to write parity-nonconserving interactions like (3.432). There was a good reason for this; the success of the parity selection rules in atomic and nuclear physics shows that the principle of parity conservation holds to a high degree of accuracy in both electromagnetic and strong interactions. In the years from 1954 to 1956, as various experimental groups studied the properties of "strange" mesons called τ^+ and θ^+ which decay via weak interactions as

$$\tau^+ \longrightarrow 2\pi^+ + \pi^-, \qquad \theta^+ \longrightarrow \pi^+ + \pi^0, \tag{3.450}$$

it soon became evident that τ^+ and θ^+ have the same mass and the same lifetime; it therefore appeared natural to assume that a τ-like decay event and a θ-like decay event simply represent different decay modes of the same parent particle (now called K^+). However, using an ingenious argument based only on parity and angular momentum conservation, R. H. Dalitz was able to show that the experimental energy and angular distributions of the pions from τ-decay strongly suggest that the τ and the θ could not possibly have the same spin parity. Since, at that time, people believed in parity conservation, this led to the famous τ-θ puzzle.‡ Faced with this dilemma, in the spring of 1956 T. D. Lee and C. N. Yang systematically investigated the validity of parity conservation in elementary particle interactions. Their conclusion was that in the realm of weak interactions parity conservation (which holds extremely well for the strong and electromagnetic interactions) was "only an extrapolated hypothesis unsupported by experimental evidence." Furthermore, they suggested a number of experiments that are really sensitive to the question of whether or not parity is conserved. (Their list of suggested experiments included the decay angular distribution of a polarized Λ hyperon which we have been discussing.) As is well known, subsequent experiments (beginning with the historic Co^{60} experiment of C. S. Wu and coworkers and the π-μ-e experiments of J. I. Friedman and V. L. Telegdi and of R. L. Garwin, L. M. Lederman, and M. Weinrich) have unequivocally supported the idea that weak interactions in general do not conserve parity.

Coming back to Λ decay, let us work out the decay rate using the Golden Rule and (3.442) through (3.446). The $\cos\theta$ term drops out as we integrate over all angles. For the reciprocal of the partial lifetime, we get

$$\begin{aligned}\frac{1}{\tau_{\Lambda\to \mathrm{p}\pi^-}} &= \frac{\Gamma(\Lambda \longrightarrow \mathrm{p}\pi^-)}{\hbar} \\ &= \frac{2\pi}{\hbar}\frac{c^2\hbar}{2\omega_\pi}\frac{m_\mathrm{p}c^2}{E_\mathrm{p}V}\frac{m_\mathrm{p}c^2 + E_\mathrm{p}}{2m_\mathrm{p}c^2}\left(|g|^2 + \frac{|g'|^2|\mathbf{p}|^2c^2}{E_\mathrm{p} + m_\mathrm{p}c^2}\right)\frac{4\pi V}{(2\pi\hbar)^3}\frac{|\mathbf{p}'|^2 d|\mathbf{p}'|}{d(E_\pi + E_\mathrm{p})} \\ &= \left(\frac{|g|^2}{4\pi\hbar c} + \frac{|g'|^2}{4\pi\hbar c}\frac{(E_\mathrm{p} - m_\mathrm{p}c^2)}{(E_\mathrm{p} + m_\mathrm{p}c^2)}\right)\frac{|\mathbf{p}'|(E_\mathrm{p} + m_\mathrm{p}c^2)}{\hbar m_\Lambda c},\end{aligned} \tag{3.451}$$

‡For detailed discussions of the τ-θ puzzle see, for example, Nishijima (1964), pp. 315–323; Sakurai (1964), pp. 47–51.

where we have used

$$\frac{d|\mathbf{p}'|}{d(E_p + E_\pi)} = \frac{d|\mathbf{p}'|}{[(|\mathbf{p}'|c^2/E_p) + (|\mathbf{p}'|c^2/E_\pi)]\, d|\mathbf{p}'|} = \frac{\hbar\omega_\pi E_p}{m_\Lambda |\mathbf{p}'| c^4}. \tag{3.452}$$

If we insert the experimentally measured mean life of the Λ particle (2.6×10^{-10} sec) and the branching ratio into the $\pi^- p$ decay mode (known to be about $\frac{2}{3}$), we obtain

$$\frac{|g|^2}{4\pi\hbar c} + 0.003\,\frac{|g'|^2}{4\pi\hbar c} \simeq 2 \times 10^{-14}. \tag{3.453}$$

Thus the dimensionless coupling constants

$$|g|^2/(4\pi\hbar c) \qquad \text{and} \qquad |g'|^2/(4\pi\hbar c)$$

for Λ decay are seen to be small compared to $e^2/(4\pi\hbar c) \simeq \frac{1}{137}$ by many orders of magnitude. The interaction responsible for Λ decay is indeed "weak."

Fermi theory of beta decay. Historically the theory of weak interactions started when E. Fermi wrote, in 1932, a Hamiltonian density that involves the proton, neutron, electron, and neutrino fields to account for nuclear beta decay:

$$\mathrm{n} \longrightarrow \mathrm{p} + \mathrm{e}^- + \bar{\nu}. \tag{3.454}$$

Fermi assumed for simplicity that the derivatives of the field operators do not appear. With this hypothesis the most general interaction density invariant under proper orthochronous Lorentz transformations has the form

$$\mathscr{H}_{\text{int}} = \sum_i (\bar{\psi}_p \Gamma_i \psi_n)[\bar{\psi}_e \Gamma_i (C_i + C_i' \gamma_5)\psi_\nu] + \mathrm{Hc}, \tag{3.455}$$

where‡

$$\Gamma_i = 1,\ \gamma_\lambda,\ \sigma_{\lambda\sigma},\ i\gamma_5\gamma_\lambda,\ \gamma_5. \tag{3.456}$$

We have subscribed to the usual convention according to which the light neutral particle emitted together with the e^- in (3.454) is an "antineutrino" ($\bar{\nu}$), not a "neutrino" (ν), and the field operator ψ_ν annihilates neutrinos and creates antineutrinos. Evidently the explicitly written part of (3.455) can account for the neutrino induced reaction

$$\nu + \mathrm{n} \longrightarrow \mathrm{e}^- + \mathrm{p} \tag{3.457}$$

as well as for β^- decay (3.454). The Hc in (3.455) can describe β^+ decay, K (electron) capture, etc.:

$$\begin{aligned} \mathrm{p} &\longrightarrow \mathrm{n} + \mathrm{e}^+ + \nu, \\ \mathrm{e}^- + \mathrm{p} &\longrightarrow \mathrm{n} + \nu, \end{aligned} \tag{3.458}$$

since, when explicitly written, it contains the annihilation operators for protons and electrons and the creation operators for neutrons, positrons, and neutrinos.

‡We avoid the indices μ and ν to prevent possible confusions with muon and neutrino.

The constants C_i and C_i' characterize the strength of the interactions of type i (scalar, vector, tensor, axial vector, and pseudoscalar); they have the dimension of energy times volume. From our earlier discussion on Λ decay, it is evident that parity conservation requires either

$$\begin{aligned} C_i' &= 0 \qquad \text{for all } i \quad (\eta_p^* \eta_n \eta_e^* \eta_\nu = 1), \\ \text{or} \\ C_i &= 0 \qquad \text{for all } i \quad (\eta_p^* \eta_n \eta_e^* \eta_\nu = -1). \end{aligned} \tag{3.459}$$

If neither of the two possibilities is satisfied, then the interaction density (3.455) is not invariant under parity.

As it stands, (3.455) contains 10 arbitrary constants (which need not be purely real). About a quarter-century after the appearance of Fermi's paper, it finally became evident that the correct Hamiltonian density that phenomenologically described nuclear beta decay was

$$\mathscr{H}_{\text{int}} = C_V(\bar{\psi}_p \gamma_\lambda \psi_n)[\bar{\psi}_e \gamma_\lambda (1 + \gamma_5)\psi_\nu] + C_A(\bar{\psi}_p i\gamma_5 \gamma_\lambda \psi_n)[\bar{\psi}_e i\gamma_5 \gamma_\lambda (1 + \gamma_5)\psi_\nu] + \mathrm{Hc} \tag{3.460}$$

with

$$\begin{aligned} C_V &= 6.2 \times 10^{-44}\ \text{MeV cm}^3 \simeq (10^{-5}/\sqrt{2})\, m_p c^2 (\hbar/m_p c)^3, \\ C_A/C_V &\simeq -1.2. \end{aligned} \tag{3.461}$$

The interaction (3.460) with $C_V \simeq -C_A$ is known as the $V - A$ interaction; it was written, on aesthetic grounds, by E. C. G. Sudarshan and R. E. Marshak, by R. P. Feynman and M. Gell-Mann, and by J. J. Sakurai in advance of the confirming experiments. Since the nucleon can be assumed to be nonrelativistic, only the time component of the vector covariant and the space components of the axial vector covariant contribute (γ_k and $i\gamma_5\gamma_4$ are "small") unless the symmetry of the initial and final nuclear states is such that the expectation values of 1 (the nonrelativistic limit of $\bar{\psi}_p \gamma_4 \psi_n$) and σ_k (the nonrelativistic limit of $\psi_p i\gamma_5 \gamma_k \psi_n$) are both zero. The vector interaction gives the Fermi selection rule $\Delta J = 0$, no parity change, while the axial-vector interaction gives the Gamow-Teller selection rule $\Delta J = 0, \pm 1$, no parity change, for the *nuclear* states.

We shall not discuss in detail the various aspects of nuclear beta decay: the electron spectrum, the *ft*-values, forbidden transitions, the electron-neutrino angular correlation, the angular distributions of electrons from polarized nuclei, etc. They are treated in standard textbooks on nuclear physics.‡ We concentrate on just one aspect of (3.460), namely the physical meaning of $(1 + \gamma_5)\psi_\nu$.

Two-component neutrino. The neutrino field ψ_ν can be expanded just as in (3.392). The only difference is that its mass is consistent with zero, $m_\nu < 200\ \text{eV}/c^2$ experimentally. The positive frequency part of ψ_ν is linear in the free-particle spinor for an annihilated neutrino. So let us investigate the effect of $(1 + \gamma_5)$ on $u^{(s)}(\mathbf{p})$.

‡See, for example, Preston (1962), Chapter 15; Källén (1964), Chapter 13.

As $m_\nu \longrightarrow 0$, we have

$$\gamma_5 \begin{pmatrix} \chi^{(s)} \\ (\boldsymbol{\sigma}\cdot\mathbf{p}c/E)\chi^{(s)} \end{pmatrix} = -\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} \chi^{(s)} \\ \boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\chi^{(s)} \end{pmatrix} = -\begin{pmatrix} \boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\chi^{(s)} \\ \chi^{(s)} \end{pmatrix}. \tag{3.462}$$

Meanwhile

$$-\boldsymbol{\Sigma}\cdot\hat{\mathbf{p}}\begin{pmatrix} \chi^{(s)} \\ (\boldsymbol{\sigma}\cdot\mathbf{p}c/E)\chi^{(s)} \end{pmatrix} = -\begin{pmatrix} \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} \end{pmatrix}\begin{pmatrix} \chi^{(s)} \\ \boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\ \chi^{(s)} \end{pmatrix} = -\begin{pmatrix} \boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\chi^{(s)} \\ \chi^{(s)} \end{pmatrix}. \tag{3.463}$$

This means that as $m_\nu \longrightarrow 0$, the γ_5 operator (sometimes called the *chirality* operator‡) and the negative of the helicity operator have the same effect on a free-particle spinor $u^{(s)}(\mathbf{p})$. In particular, eigenspinors of $\boldsymbol{\Sigma}\cdot\hat{\mathbf{p}}$ are also eigenspinors of γ_5 with opposite eigenvalues, and $(1+\gamma_5)$ acting on the free-particle spinor for a right-handed particle gives zero as $m_\nu \longrightarrow 0$. It then follows that $(1+\gamma_5)\psi_\nu^{(+)}$ annihilates only left-handed (helicity $= -1$) neutrinos, denoted by ν_L. Likewise it is easy to see that

$$\begin{aligned} &(1+\gamma_5)\psi_\nu^{(-)} && \text{creates} && \bar{\nu}_R, \\ &\bar{\psi}_\nu^{(+)}(1-\gamma_5) && \text{annihilates} && \bar{\nu}_R, \\ &\bar{\psi}_\nu^{(-)}(1-\gamma_5) && \text{creates} && \nu_L, \end{aligned}$$

where $\bar{\nu}_R$ stands for a right-handed antineutrino. Now the Hc in (3.460) contains $\bar{\psi}_\nu(1-\gamma_5)$ since

$$\left[\bar{\psi}_e\begin{Bmatrix} \gamma_\lambda \\ i\gamma_5\gamma_\lambda \end{Bmatrix}(1+\gamma_5)\psi_\nu\right]^\dagger = \psi_\nu^\dagger(1+\gamma_5)\begin{Bmatrix} \gamma_\lambda \\ -i\gamma_\lambda\gamma_5 \end{Bmatrix}\gamma_4\psi_e = \bar{\psi}_\nu(1-\gamma_5)\begin{Bmatrix} \mp\gamma_\lambda \\ \pm i\gamma_5\gamma_\lambda \end{Bmatrix}\psi_e, \tag{3.464}$$

where the upper signs are for the space components, the lower for the time components. An immediate consequence of this is that the helicity of the neutrino emitted in K (electron) capture [the second of (3.458)] is -1. This has indeed been shown to be the case by a beautiful experiment of M. Goldhaber, L. Grodzins, and A. W. Sunyar. The positive (negative) helicity of the antineutrino (neutrino) emitted in β^- (β^+) decay has also been inferred from the electron (positron) polarization and the $e^-\bar{\nu}(e^+\nu)$ angular correlation (cf. Problem 3–14). Parity conservation would require that in a physically realizable process the emission probability for a right-handed particle be the same as that for a left-handed particle since the helicity changes sign under parity. It is therefore evident that the interaction (3.460) which produces only $\nu_L(\bar{\nu}_R)$ in β^+ (β^-) decay is incompatible with the principle of parity conservation.

‡This expression is derived from the Greek word $\chi\varepsilon\iota\rho$ meaning "hand." The term "chirality" was first used by Lord Kelvin in a somewhat different context.

There is another (simpler) way to see the connection between helicity and chirality for a massless particle. Let us go back to (3.26) which was obtained by linearizing the Waerden equation. When the fermion mass is zero, the two equations are completely decoupled,‡

$$\left(i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla} - i\frac{\partial}{\partial x_0}\right)\phi^{(L)} = 0, \qquad \left(-i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla} - i\frac{\partial}{\partial x_0}\right)\phi^{(R)} = 0. \tag{3.465}$$

Let us now postulate that only the first of (3.465) has to do with physical reality. Evidently a free-particle solution of the $\phi^{(L)}$ equation in the c-number (wavefunction) language satisfies

$$\boldsymbol{\sigma}\cdot\mathbf{p} = -E/c, \tag{3.466}$$

where E can be positive or negative. This means that the helicity of a positive-energy neutrino is negative, while the helicity of a negative-energy neutrino is positive. Using the hole theory, we infer that the helicity of a (positive-energy) antineutrino is also positive (cf. Table 3–4). We should emphasize that the assertion that the neutrino (antineutrino) is always left-handed (right-handed) makes sense only if the mass is *strictly* zero. Otherwise we can easily perform a Lorentz transformation that changes a left-handed particle into a right-handed one.

Meanwhile, according to Problem 3–5, in the Weyl representation of the Dirac equation we have

$$\psi = \begin{pmatrix} \phi^{(R)} \\ \phi^{(L)} \end{pmatrix}, \tag{3.467}$$

and

$$\gamma_5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}. \tag{3.468}$$

It is then evident that $(1+\gamma_5)\psi_\nu$ selects just $\phi^{(L)}$, that is, only the lower two of the four components of ψ in the Weyl representation:

$$(1+\gamma_5)\psi = 2\begin{pmatrix} 0 \\ \phi^{(L)} \end{pmatrix}. \tag{3.469}$$

‡It is amusing that the Maxwell equations applying to empty space can also be written in a form similar to (3.465). The reader may show that (1.57) and (1.58) with $\rho = 0$, $\mathbf{j} = 0$ can be combined to give

$$\left(-i\mathbf{S}\cdot\boldsymbol{\nabla} - i\frac{\partial}{\partial x_0}\right)\Phi = 0,$$

where

$$S_1 = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad S_2 = i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad S_3 = i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

and

$$\Phi = \begin{pmatrix} B_1 - iE_1 \\ B_2 - iE_2 \\ B_3 - iE_3 \end{pmatrix}.$$

Suppose we demand that only

$$(1+\gamma_5)\psi_\nu \qquad \text{and} \qquad \bar{\psi}_\nu(1-\gamma_5) = [(1+\gamma_5)\psi_\nu]^\dagger\gamma_4$$

appear in the beta-decay interaction. Because of (3.469) this requirement amounts to saying that only $\phi^{(L)}$ and $\phi^{(L)\dagger}$ appear, hence the neutrinos (antineutrinos) emitted in β^+ (β^-) decay are necessarily left- (right-) handed. We may argue that the appearance of only $\phi^{(L)}$ is due to either (a) an intrinsic property of the *free* neutrino itself, or (b) a property of the parity-nonconserving beta-decay *interaction* that just happens to select $\phi^{(L)}$ (and also of other interactions in which the neutrino participates). If the neutrino mass is strictly zero, the two points of view are, in practice, completely indistinguishable.‡ A theory of neutrinos based on $\phi^{(L)} \neq 0$, $\phi^{(R)} = 0$ (or vice versa) is called the two-component theory of the neutrino.

Historically, the idea that the neutrino is described by $\phi^{(L)}$ only (or $\phi^{(R)}$ only) was advanced by H. Weyl in 1929; it was rejected by W. Pauli in his *Handbuch* article on the grounds that the wave equation for $\phi^{(L)}$ only (called the Weyl equation) is not manifestly covariant under space inversion (cf. Problem 3–5). With the advent of parity nonconservation in 1957 the Weyl equation was revived by A. Salam, by L. D. Landau, and by T. D. Lee and C. N. Yang.

Muon capture,

$$\mu^- + \mathrm{p} \longrightarrow \mathrm{n} + \nu', \tag{3.470}$$

can also be described by an interaction of the form (3.455); ψ_e and ψ_ν are now replaced by ψ_μ and $\psi_{\nu'}$. The particle ν' whose mass is also consistent with zero turns out again to be left-handed (its antiparticle $\bar{\nu}'$ is right-handed). For some years it was generally believed that ν and ν' were identical; however, motivated by the experimental absence of $\mu^+ \longrightarrow \mathrm{e}^+ + \gamma$ (which can be best understood if $\nu \neq \nu'$), B. Pontecorvo and others proposed an experiment to test the assumption that ν is the same as ν'. Noting that ν' also appears in

$$\pi^+ \longrightarrow \mu^+ + \nu' \tag{3.471}$$

(because π^+ can virtually disintegrate into a proton and an antineutron), we can settle the question of the identity or the nonidentity of ν and ν' by examining whether or not the neutral particle from pion decay (3.471) can induce a high-energy neutrino reaction, that is,

$$\nu' + \mathrm{n} \longrightarrow \mathrm{e}^- + \mathrm{p}. \tag{3.472}$$

In 1962 M. Schwartz and collaborators established experimentally that (3.472) is forbidden, while

$$\nu' + \mathrm{n} \longrightarrow \mu^- + \mathrm{p} \tag{3.473}$$

is fully allowed for ν' from pion decay; these experimental facts are in agreement with the idea that ν and ν' are different.

‡In general, you may introduce as many fields as you like without changing the physical content, so long as the fields introduced are coupled to nothing.

Pion decay and the CPT theorem. As a final example, let us make a comparison between

$$\pi^- \longrightarrow \mu^- + \bar{\nu}' \tag{3.474}$$

and

$$\pi^- \longrightarrow \mathrm{e}^- + \bar{\nu}. \tag{3.475}$$

It is appealing to assume that the interactions responsible for (3.474) and (3.475) are the same in form as well as in strength, that is, the interactions are invariant under

$$\mathrm{e}^\pm \longleftrightarrow \mu^\pm, \qquad \nu \longleftrightarrow \nu', \qquad \bar{\nu} \longleftrightarrow \bar{\nu}'. \tag{3.476}$$

In addition it is reasonable to postulate that in (3.475) the field operators ψ_e and ψ_ν enter in the same combination as in nuclear beta decay, viz. in the combination $i\bar{\psi}_\mathrm{e}\gamma_5\gamma_\lambda(1+\gamma_5)\psi_\nu$ [which is also equal to $-i\bar{\psi}_\mathrm{e}\gamma_\lambda(1+\gamma_5)\psi_\nu$]. With these two hypotheses, the interaction densities may be taken as

$$\mathscr{H}_\mathrm{int} = if\frac{\hbar}{m_\pi c}\frac{\partial\phi_\pi}{\partial x_\lambda}[\bar{\psi}_\mathrm{e}\gamma_5\gamma_\lambda(1+\gamma_5)\psi_\nu] + \mathrm{Hc}, \tag{3.477}$$

for (3.475) and

$$\mathscr{H}_\mathrm{int} = if\frac{\hbar}{m_\pi c}\frac{\partial\phi_\pi}{\partial x_\lambda}[\bar{\psi}_\mu\gamma_5\gamma_\lambda(1+\gamma_5)\psi_{\nu'}] + \mathrm{Hc}, \tag{3.478}$$

for (3.474). The constant f is assumed to be universal, that is, the same for both (3.474) and (3.475); we have inserted the factor $\hbar/m_\pi c$ just to make $(f^2/4\pi\hbar c)$ dimensionless. Using the same argument as in Λ decay, we obtain the following for the π-e process:

$$c^{(1)} = -\frac{i}{\hbar}\frac{f\hbar}{m_\pi c}\left(c\sqrt{\frac{\hbar}{2\omega_\pi}}\right)\left(\sqrt{\frac{m_\nu c^2}{E_\nu}}\right)\left(\sqrt{\frac{m_\mathrm{e}c^2}{E_\mathrm{e}}}\right)[i\bar{u}_\mathrm{e}\gamma_5\gamma_\lambda(1+\gamma_5)v_\nu]$$
$$\times\frac{1}{V^{3/2}}\int_0^t dt'\int d^3x'\left[\left(\frac{\partial}{\partial x'_\lambda}\exp\left[i\frac{p^{(\pi)}\cdot x'}{\hbar}\right]\right)\left(\exp\left[-\frac{ip^{(\bar{\nu})}\cdot x'}{\hbar}-\frac{ip^{(\mathrm{e})}\cdot x'}{\hbar}\right]\right)\right], \tag{3.479}$$

where we have suppressed the momentum and spin indices for the free-particle spinors. The four-gradient acting on the pion plane wave just brings down $ip_\lambda^{(\pi)}/\hbar$; because of energy-momentum conservation we get

$$\begin{aligned} ip_\lambda^{(\pi)}\bar{u}_\mathrm{e}\gamma_5\gamma_\lambda(1+\gamma_5)v_\nu &= -ip_\lambda^{(\pi)}\bar{u}_\mathrm{e}\gamma_\lambda(1+\gamma_5)v_\nu \\ &= -i\bar{u}_\mathrm{e}(\gamma\cdot p^{(\mathrm{e})}+\gamma\cdot p^{(\bar{\nu})})(1+\gamma_5)v_\nu \\ &= m_\mathrm{e}c\bar{u}_\mathrm{e}(1+\gamma_5)v_\nu, \end{aligned} \tag{3.480}$$

where we have used (3.388) and (3.390) with $m_\nu = 0$. Let us now work in the rest system of the decaying pion with the z-axis along the antineutrino momentum denoted by $\mathbf{p}$. Using the explicit forms of free-particle spinors (cf. Eqs. 3.115 and 3.384), we can easily verify that for $\mathbf{p}$ along the quantization axis,

$$(1+\gamma_5)v_\nu^{(1)}(\mathbf{p}) = 2v_\nu^{(1)}(\mathbf{p}), \qquad (1+\gamma_5)v_\nu^{(2)}(\mathbf{p}) = 0 \tag{3.481}$$

as $m_\nu \longrightarrow 0$, which supports our earlier assertion that $(1+\gamma_5)\psi_\nu^{(-)}$ creates right-handed antineutrinos only. The amplitude for the production of a right-handed

(spin-*down*) electron with momentum $\mathbf{p}_e = -\mathbf{p}$ is proportional to

$$\sqrt{\frac{m_\nu c^2}{E_\nu}}\sqrt{\frac{m_e c^2}{E_e}}\,\bar{u}_e^{(2)}(-\mathbf{p})(1+\gamma_5)v_\nu^{(1)}(\mathbf{p})$$

$$= -2\sqrt{\frac{m_\nu c^2+E_\nu}{2E_\nu}}\sqrt{\frac{m_e c^2+E_e}{2E_e}}\left(0,1,0,-\frac{|\mathbf{p}|c}{E_e+m_e c^2}\right)\begin{pmatrix}0\\-1\\0\\1\end{pmatrix}$$

$$= \frac{E_e+m_e c^2+|\mathbf{p}|c}{\sqrt{E_e(E_e+m_e c^2)}} = \sqrt{\frac{m_\pi c^2}{2E_e}}, \tag{3.482}$$

as $m_\nu \to 0$, where we have used

$$|\mathbf{p}| = \frac{E_\nu}{c} = \frac{(m_\pi^2-m_e^2)c}{2m_\pi},$$

$$E_e + |\mathbf{p}|c = m_\pi c^2. \tag{3.483}$$

Note that the use of the normalization convention (3.389) does not cause any trouble in the $m_\nu \to 0$ limit. For the other (left-handed) spin state of the electron we can readily show that

$$\bar{u}_e^{(1)}(-\mathbf{p})(1+\gamma_5)v_\nu^{(1)}(\mathbf{p}) = 0, \tag{3.484}$$

which we could actually have guessed from angular momentum conservation. (The pion is spinless and the orbital angular momentum cannot have a nonvanishing component along $\mathbf{p}$.) The only other thing we must evaluate is the phase-space factor

$$\frac{|\mathbf{p}|^2 d|\mathbf{p}|}{d(E_\nu+E_e)} = \frac{E_e|\mathbf{p}|^2}{m_\pi c^3} = \frac{E_e(m_\pi^2-m_e^2)^2}{4m_\pi^3 c}. \tag{3.485}$$

Collecting all the factors and using the Golden Rule, we finally get

$$\frac{1}{\tau(\pi^- \longrightarrow e^- + \bar{\nu})} = \left(\frac{f^2}{4\pi\hbar c}\right)\frac{m_e^2(m_\pi^2-m_e^2)^2c^2}{\hbar m_\pi^5}. \tag{3.486}$$

Similarly

$$\frac{1}{\tau(\pi^- \longrightarrow \mu^- + \bar{\nu}')} = \left(\frac{f^2}{4\pi\hbar c}\right)\frac{m_\mu^2(m_\pi^2-m_\mu^2)^2c^2}{\hbar m_\pi^5}. \tag{3.487}$$

What is more significant, the ratio of (3.475) to (3.474) is

$$\frac{\Gamma(\pi^- \longrightarrow e^- + \bar{\nu})}{\Gamma(\pi^- \longrightarrow \mu^- + \bar{\nu}')} = \left(\frac{m_e}{m_\mu}\right)^2\frac{(m_\pi^2-m_e^2)^2}{(m_\pi^2-m_\mu^2)^2} \simeq 1.3\times 10^{-4}. \tag{3.488}$$

Note that the muonic decay mode is much more frequent despite the smaller Q-value available; it is hard to beat the factor $(m_e/m_\mu)^2 \simeq 2.3\times 10^{-5}$.

Actually the fact that the transition probability computed with (3.477) is proportional to m_e^2 is not surprising. To see this we first note that, if $m_e = 0$, then

$$\bar{\psi}_e\gamma_5\gamma_\lambda(1+\gamma_5)\psi_\nu = \bar{\psi}_e(1-\gamma_5)\gamma_5\gamma_\lambda\psi_\nu$$

would create only left-handed electrons. But the simultaneous creation of a left-handed electron and a right-handed $\bar{\nu}$ in pion decay is strictly forbidden by angular momentum conservation. It is therefore expected that the amplitude for (3.473) is proportional to m_e.

Although the result (3.488) was first obtained by M. Ruderman and R. Finkelstein‡ as early as 1949, only two years after the discovery of the pion, for a very long time the experimentalists failed to find the electronic decay mode; the quoted upper limit was an order of magnitude *lower* than (3.488). When it became evident that the electron and neutrino enter in the combination $\bar{\psi}_e\gamma_\lambda(1+\gamma_5)\psi_\nu$ in beta decay and also in muon decay, the idea of describing pion decay by (3.477) and (3.478) became so attractive that some theoreticians even said, "The experiments must be wrong." Truly enough, a number of experiments performed since 1958 at CERN and other places brilliantly confirmed the predicted ratio (3.488). We mention this example just to emphasize that the power of relativistic quantum mechanics in making quantitative predictions is not limited to the domain of electromagnetic interactions.

As a by-product of our calculation we obtain

$$(f^2/4\pi\hbar c) \simeq 1.8\times 10^{-15}, \tag{3.489}$$

when the observed pion lifetime is inserted in (3.487). So this is again an example of "weak" interactions. We may also mention that a similar calculation with Hc in (3.478) gives the following result:

$$\Gamma(\pi^+ \longrightarrow \mu^+ + \nu_L') = \Gamma(\pi^- \longrightarrow \mu^- + \bar{\nu}_R') \neq 0,$$
$$\Gamma(\pi^+ \longrightarrow \mu^+ + \nu_R') = \Gamma(\pi^- \longrightarrow \mu^- + \bar{\nu}_L') = 0. \tag{3.490}$$

This means that charge conjugation invariance [which demands $\Gamma(\pi^+ \longrightarrow \mu^+ + \nu_L') = \Gamma(\pi^- \longrightarrow \mu^- + \bar{\nu}_L')$, etc.] is violated as well as invariance under parity.

It is very important to note that the equality of the π^+ and the π^- lifetime holds despite the breakdown of charge conjugation invariance.

Following E. P. Wigner, we shall define the time-reversal operation which reverses both momentum and spin. This definition of time reversal agrees with our intuitive notion of the "reversal of motion."§ Consider now the product

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‡The original calculation of Ruderman and Finkelstein was based on the parity conserving interaction $if(\hbar/m_\pi c)(\partial\phi_\pi/\partial x_\lambda)\bar{\psi}_e\gamma_5\gamma_\lambda\psi_\nu$. The numerical result for the ratio of the two decay modes is unchanged since

$$\bar{\psi}_e\gamma_5\gamma_\lambda\psi_\nu = \tfrac{1}{2}[\bar{\psi}_e\gamma_5\gamma_\lambda(1+\gamma_5)\psi_\nu + \bar{\psi}_e\gamma_5\gamma_\lambda(1-\gamma_5)\psi_\nu],$$

and the calculation for the emission of a right-handed ν proceeds in the same way as that for the emission of a left-handed ν.

§If the final state is not described by a plane wave, a much more careful discussion of time reversal is needed. This is because time reversal interchanges the roles of an incoming and an outgoing state. For a detailed discussion of time reversal in both nonrelativistic and relativistic quantum mechanics, see Sakurai (1964), Chapter 4.

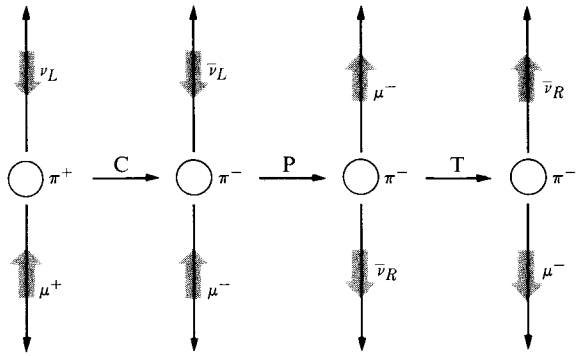

Fig. 3–10. CPT operation applied to pion decay. The gray arrows indicate the spin direction.

Table 3–5

TRANSFORMATION PROPERTIES UNDER CHARGE CONJUGATION, PARITY, TIME REVERSAL, AND CPT

	Q	p	J	Helicity
Charge conjugation‡	−	+	+	+
Parity	+	−	+	−
Time reversal	+	−	−	+
CPT	−	+	−	−

‡By charge conjugation we mean "particle-antiparticle conjugation."

of charge conjugation, parity, and time reversal, denoted by CPT. The relations (3.490) are completely consistent with invariance under CPT; the decay configuration obtained by applying CPT is seen to be a physically realizable configuration with the same transition probability (Fig. 3–10 and Table 3–5).

In the formalism developed in this chapter, the invariance under CPT can be regarded as a consequence of the use of a Hermitian Hamiltonian. (Actually the pion decay interaction we have written turns out also to be invariant under CP, the product of charge conjugation and parity, but it is easy to write down a Hermitian Hamiltonian that violates CP invariance; for example, the electric-dipole-moment interaction of Problem 3–16.) In a more axiomatic formulation of quantum field theory it can be shown that it is impossible to violate CPT invariance without drastically altering the structure of the theory. The fact that a wide class of quantum field theories is invariant under CPT was first demonstrated by G. Lüders and W. Pauli; its physical implications have been extensively discussed by T. D. Lee, R. Oehme, and C. N. Yang.§

§For a more complete discussion of CPT invariance see Nishijima (1964), pp. 329–339; Sakurai (1964), Chapter 6; Streater and Wightman (1963), pp. 142–146.

PROBLEMS

3–1. Prove Eq. 3.13.

3–2. Consider an electron in a uniform and constant magnetic field **B** along the z-axis. Obtain the most general four-component positive-energy eigenfunctions. Show that the energy eigenvalues are given by

$$E = \sqrt{m^2c^4 + c^2p_3^2 + 2n\hbar c\,|e\mathbf{B}|}$$

with $n = 0, 1, 2, \ldots$ List all the constants of the motion.

3–3. (a) Construct the normalized wave functions for $E > 0$ plane waves which are eigenstates of the helicity operator. (The momentum **p** is not necessarily assumed to be in the z-direction.) Evaluate the expectation values of $\boldsymbol{\Sigma}\cdot\hat{\mathbf{p}}$ and $i\gamma_5\boldsymbol{\gamma}\cdot\hat{\mathbf{p}} = \beta\boldsymbol{\Sigma}\cdot\hat{\mathbf{p}}$, that is, $\int \psi^\dagger\boldsymbol{\Sigma}\cdot\hat{\mathbf{p}}\psi\, d^3x = \int \bar{\psi}i\gamma_5\boldsymbol{\gamma}\cdot\hat{\mathbf{p}}\psi\, d^3x$ and $\int \psi^\dagger i\gamma_5\boldsymbol{\gamma}\cdot\hat{\mathbf{p}}\psi\, d^3x = \int \bar{\psi}\boldsymbol{\Sigma}\cdot\mathbf{p}\psi\, d^3x$.

(b) Construct the normalized wave function for an $E > 0$ transversely polarized plane wave whose propagation and spin ($\langle\boldsymbol{\Sigma}\rangle$) directions are along the positive z- and the positive x-axes respectively. Evaluate the expectation values of Σ_1 and $i\gamma_5\gamma_1 = \beta\Sigma_1$.

3–4. Let $\psi(\mathbf{x}, t)$ be the wave function for an $E > 0$ plane wave.

(a) Show that $\gamma_4\psi(-\mathbf{x}, t)$ is indeed the wave function for the $E > 0$ plane wave with momentum reversed and the spin direction unchanged.

(b) Show that $i\Sigma_2\psi^*(\mathbf{x}, -t)$ is the wave function for the "time-reversed state" in the sense that the momentum and spin directions are both reversed while the sign of the energy is unchanged.

3–5. Consider the coupled two-component equations (cf. Eq. 3.26)

$$-i\frac{\partial}{\partial x_\mu}\sigma_\mu^{(R)}\phi^{(R)} = -\frac{mc}{\hbar}\phi^{(L)},$$

$$i\frac{\partial}{\partial x_\mu}\sigma_\mu^{(L)}\phi^{(L)} = -\frac{mc}{\hbar}\phi^{(R)},$$

$$\sigma_\mu^{(R)} = (\boldsymbol{\sigma}, i), \qquad \sigma_\mu^{(L)} = (\boldsymbol{\sigma}, -i).$$

(a) Write the above wave equations in the form

$$\left(\gamma'_\mu\frac{\partial}{\partial x_\mu} + \frac{mc}{\hbar}\right)\psi' = 0,$$

where

$$\psi' = \begin{pmatrix}\phi^{(R)}\\ \phi^{(L)}\end{pmatrix}.$$

Obtain the explicit forms of γ'_μ and $\gamma'_5 = \gamma'_1\gamma'_2\gamma'_3\gamma'_4$ and check $\{\gamma'_\mu, \gamma'_\nu\} = 2\delta_{\mu\nu}$ for $\mu, \nu = 1, \ldots, 5$. Find a unitary S that relates the new (Weyl) set $\{\gamma'_\mu\}$ and the standard (Dirac-Pauli) set $\{\gamma_\mu\}$ via $\gamma'_\mu = S\gamma_\mu S^{-1}$.

(b) Without using the invariance properties of the Dirac theory, discussed in Section 3–4, establish directly the relativistic covariance of the coupled two-component equations. In particular, find 2×2 matrices $S^{(R)}$ and $S^{(L)}$ such that

$$\phi^{(R,L)'}(x') = S^{(R,L)}\phi^{(R,L)}(x),$$

where x and x' are related by a Lorentz transformation along the x_1-axis. Discuss

also the covariance of the coupled wave equations under parity; $\phi^{(R,L)'}(x') = ?$, where $x' = (-\mathbf{x}, ict)$.

3–6. (a) Reduce

$$\bar{u}^{(r')}(\mathbf{p}')\sigma_{k4}u^{(r)}(\mathbf{p}) = iu^{(r')\dagger}(\mathbf{p}')\gamma_k u^{(r)}(\mathbf{p})$$

to the form $\chi^{(s')\dagger}\mathcal{O}\chi^{(s)}$ where $u^{(r)}(\mathbf{p})$ and $u^{(r')}(\mathbf{p}')$ are positive-energy free particle spinors and $\chi^{(s)}$ and $\chi^{(s')}$ are the corresponding two-component Pauli spinors. Assume $|\mathbf{p}| = |\mathbf{p}'|$.

(b) The interaction of a neutron with the electromagnetic field can be represented at low energies by the phenomenological Hamiltonian density

$$\mathscr{H}_{\text{int}} = -(\kappa|e|\hbar/2m_n c)[\tfrac{1}{2}F_{\mu\nu}\bar{\psi}\sigma_{\mu\nu}\psi], \qquad \kappa = -1.91.$$

Using the expression obtained in (a), compute the differential cross section for the scattering of a slow neutron by an electrostatic field in Born approximation. Interpret your results physically. Show, in particular, that even a very slow neutron can have an attractive short-ranged ($\delta^3(\mathbf{x})$-like) interaction (known as the *Foldy interaction*) with an electron. Show also that if this interaction is represented by an equivalent spherical potential well of radius $r_0 = (e^2/4\pi m_e c^2)$, then the depth of the potential is 4.08 keV.‡

3–7. Using the uncertainty principle, N. Bohr argued that it is impossible to prepare a beam of free electrons with all spins pointing in the same direction by means of a Stern-Gerlach type experiment or, more generally, by a selection mechanism that takes advantage of the classical concept of a particle trajectory.§ Justify Bohr's thesis from the point of view of the Dirac theory of the electron. Would this still be true even if the electron had a large anomalous magnetic moment?

3–8. Consider the unitary operator

$$U = \sqrt{\frac{mc^2 + |E|}{2|E|}} + \frac{\beta\boldsymbol{\alpha}\cdot\mathbf{p}c}{\sqrt{2|E|(mc^2 + |E|)}}$$

where $|E|$ is to be understood as a "square root operator," $\sqrt{c^2\mathbf{p}^2 + m^2c^4}$, which just gives $\sqrt{|\mathbf{p}|^2c^2 + m^2c^4}$ as its eigenvalue when it acts on the wave function for a free-particle plane wave.

(a) Show that the application of U to the wave function for a positive- (negative-) energy plane wave results in a wave function whose lower (upper) two components are missing.

(b) The above transformation (first considered in 1948 by M. H. Pryce‖) can be regarded as a change in the representation of the Dirac matrices. Show that the operator in the usual representation that corresponds to $\mathbf{x}$ in the new representation is

$$\mathbf{X} = \mathbf{x} + \frac{ic\hbar\beta\boldsymbol{\alpha}}{2|E|} - \frac{ic^3\hbar\beta(\boldsymbol{\alpha}\times\mathbf{p})\mathbf{p}}{2|E|^2(|E| + mc^2)} - \frac{c^2\hbar(\boldsymbol{\sigma}\times\mathbf{p})}{2|E|(|E| + mc^2)}.$$

(The operator $\mathbf{X}$ is sometimes called the *mean position operator*.)

‡Although the Foldy interaction is much weaker than the nuclear interaction, it can be detected experimentally as thermal neutrons are scattered coherently by high-Z atoms.

§The original argument of Bohr can be found in Mott and Massey (1949), p. 61.

‖This transformation is a special case of a class of transformations considered extensively by L. L. Foldy and S. A. Wouthuysen, and by S. Tani.

(c) Prove

$$\dot{\mathbf{X}} = \pm c^2\mathbf{p}/|E|.$$

(Note that the expression for $\dot{\mathbf{X}}$ is free from *Zitterbewegung*, but the nonlocality of $\mathbf{X}$ is the price we must pay.)

3–9. At some instant of time (say, $t = 0$) the normalized wave function for an electron is known to be

$$\psi(\mathbf{x}, 0) = \frac{1}{\sqrt{V}}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} e^{ip_3x_3/\hbar}$$

where a, b, c, and d are independent of the space-time coordinates and satisfy $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$. Find the probabilities of observing the electron with (i) $E > 0$ spin-up, (ii) $E > 0$ spin-down, (iii) $E < 0$ spin-up, and (iv) $E < 0$ spin-down.

3–10. Consider a Dirac particle subject to a (three-dimensional) spherical well potential

$$V(r) = -V_0 < 0 \qquad \text{for} \quad r < r_0,$$
$$V(r) = 0 \qquad\qquad\ \ \text{for} \quad r > r_0.$$

(a) Obtain the exact four-component energy eigenfunctions for $j = \frac{1}{2}$ ("even") bound states, where "even" means even orbital parity for the upper two components.

(b) Set up an equation that determines the energy eigenvalues.

(c) What happens if the strength of the potential is increased so that V_0 becomes comparable to or larger than $2mc^2$?

3–11. Discuss how the numbers of nodes of the radial functions $G(r)$ and $F(r)$ of the hydrogen atom are related to the quantum numbers n, j, and l.

3–12. Consider a positive-energy electron at rest with spin-up. Suppose at $t = 0$ we apply an external (classical vector potential) represented by

$$\mathbf{A} = \hat{\mathbf{n}}_3 a\cos\omega t$$

(where a is space-time independent, and $\hat{\mathbf{n}}_3$ stands for a unit vector in the positive z-direction). Show that for $t > 0$ there is a finite probability of finding the electron in a negative-energy state if the negative-energy states are assumed to be initially empty. In particular work out quantitatively the following two cases: $\hbar\omega \ll 2mc^2$ and $\hbar\omega \approx 2mc^2$.

3–13. (a) Prove

$$\sum_{r=1}^{4} u_\alpha^{(r)}(\mathbf{p})u_\beta^{(r)\dagger}(\mathbf{p}) = \sum_{s=1}^{2}[u_\alpha^{(s)}(\mathbf{p})u_\beta^{(s)\dagger}(\mathbf{p}) + v_\alpha^{(s)}(-\mathbf{p})v_\beta^{(s)\dagger}(-\mathbf{p})]$$
$$= \delta_{\alpha\beta}(|E|/mc^2).$$

(b) Using the above relation, prove the equal-time anticommutation relation between $\psi_\alpha(x)$ and $\psi_\beta^\dagger(x)$ (3.409).

3–14. Consider an allowed pure Fermi β^+ decay. Only the vector interaction contributes, and it is legitimate to replace γ_μ by γ_4. Without using the trace techniques to be introduced in the next chapter show that the positron-neutrino angular correlation is given by

$$1 + (v/c)_{e^+}\cos\theta,$$

where θ is the angle between the momenta of e^+ and ν.‡ Compute also the expectation value of the positron helicity, and interpret your result, using angular momentum conservation.

3–15. We showed in Section 2–3 that in atomic physics, favored radiative transitions take place between states of opposite parities, that is, parity must *change*. Meanwhile, we know that the fundamental electromagnetic interaction *conserves* parity. Resolve this paradox.

3–16. (a) Suppose the electron had a static electric dipole moment analogous to the magnetic moment. Write a Hamiltonian density that represents the interaction of the electric dipole moment with the electromagnetic field and prove that it is not invariant under parity.

(b) Show that the electric dipole moment interaction would lead to a mixing (in the quantum-mechanical sense) between the $2s_{\frac{1}{2}}$ and $2p_{\frac{1}{2}}$ states of the hydrogen atom. From the fact that the observed and the calculated Lamb shift agree within 0.5 Mc, obtain an upper limit on the magnitude of the electric dipole moment of the electron. (*Caution:* the relevant matrix element vanishes if the nonrelativistic wave functions for $2s_{\frac{1}{2}}$ and $2p_{\frac{1}{2}}$ are used.)

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‡Because $\gamma_4 u(\mathbf{p}) = u(-\mathbf{p})$, it is easy to see that the angular coefficient would be just opposite in sign if the interaction were scalar. Historically the vector nature of Fermi transitions was first established by J. S. Allen and collaborators, who showed that the positron and the neutrino tend to be emitted in the same direction in the β^+ decay of A^{35} (almost pure Fermi).

CHAPTER 4

COVARIANT PERTURBATION THEORY

4–1. NATURAL UNITS AND DIMENSIONS

In relativistic quantum theory it is most convenient to use units in which action (energy times time) is measured in $\hbar$, and length divided by time is measured in c. This system of units is referred to as *natural units*. When we work in natural units (as we shall for the remaining part of this book) the symbol m_e may mean not only the electron mass but also any one of the following:

a) reciprocal length

$$m_e\left(=\frac{1}{\hbar/m_e c}\right)=\frac{1}{3.86\times10^{-11}\text{ cm}}, \tag{4.1}$$

b) reciprocal time

$$m_e\left(=\frac{1}{\hbar/m_e c^2}\right)=\frac{1}{1.29\times10^{-21}\text{ sec}}, \tag{4.2}$$

c) energy

$$m_e(=m_e c^2)=0.511\text{ MeV}, \tag{4.3}$$

d) momentum

$$m_e(=m_e c)=0.511\text{ MeV}/c. \tag{4.4}$$

In natural units the fine-structure constant is simply

$$\alpha=\frac{e^2}{4\pi}\simeq\frac{1}{137.04}. \tag{4.5}$$

In the "pure" electrodynamics of electrons and photons, apart from m_e there is no other constant that has the dimension of mass or reciprocal length. So, from purely dimensional considerations, we may argue that the cross section for any electrodynamic process which is of order e^2 in the amplitude must be of the order of

$$\frac{\alpha^2}{m_e^2}=\left(\frac{e^2}{4\pi}\right)^2\frac{1}{m_e^2}=r_0^2, \tag{4.6}$$

r_0 being the classical radius of the electron. Indeed, apart from the numerical factor $8\pi/3$, (4.6) is precisely the Thomson cross section computed in Chapter 2. Working with α and m_e, we can form other constants which are familiar from atomic physics. For instance, the Bohr radius is given by

$$a_0=1/\alpha m_e, \tag{4.7}$$

so that

$$a_0:\frac{1}{m_e}:r_0\simeq137:1:\frac{1}{137}. \tag{4.8}$$

The Rydberg energy is simply

$$\text{Ry}_\infty=\alpha^2 m_e/2=13.6\text{ eV}. \tag{4.9}$$