

PHY 712 Electrodynamics

10-10:50 AM MWF in Olin 103

Notes for Lecture 10:

**Reading Chapter 4 in JDJ --
Dipolar fields and dielectrics**

A. Electric field due to a dipole

B. Electric polarization P

**C. Electric displacement D and
dielectric functions**

Course schedule for Spring 2023

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Mon: 01/9/2023	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/13/2023
2	Wed: 01/11/2023	Chap. 1	Electrostatic energy calculations	#2	01/18/2023
3	Fri: 01/13/2023	Chap. 1	Electrostatic energy calculations thanks to Ewald	#3	01/18/2023
	Mon: 01/16/2023		MLK Holiday -- no class		
4	Wed: 01/18/2023	Chap. 1 & 2	Electrostatic potentials and fields	#4	01/20/2023
5	Fri: 01/20/2023	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#5	01/23/2023
6	Mon: 01/23/2023	Chap. 1 - 3	Brief introduction to numerical methods	#6	01/25/2023
7	Wed: 01/25/2023	Chap. 2 & 3	Image charge constructions	#7	01/30/2023
8	Fri: 01/27/2023	Chap. 2 & 3	Cylindrical and spherical geometries		
9	Mon: 01/30/2023	Chap. 3 & 4	Spherical geometry and multipole moments	#8	02/01/2023
10	Wed: 02/01/2023	Chap. 4	Dipoles and Dielectrics	#9	02/03/2023
11	Fri: 02/03/2023	Chap. 4	Dipoles and Dielectrics		

PHY 712 -- Assignment #9

February 1, 2023

Continue reading Chapter 4 in **Jackson** .

1. Consider the localized charge given by

$$\rho(r,\theta)=\rho_0 r \exp(-ar) \cos(\theta)$$

where ρ_0 and a are positive constants. Find the electrostatic potential produced by this charge distribution. Examine your result in the limit $r \rightarrow 0$ and also in the limit $r \rightarrow \infty$ in terms of multipole moments and other interesting features.

Note that this problem is different from the one posted in the earlier version of these notes.

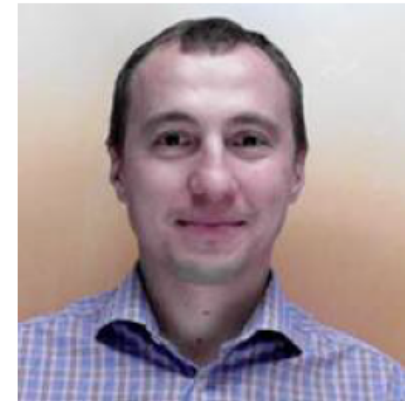
PHYSICS COLLOQUIUM

THURSDAY

FEBRUARY 2, 2023

MAGNETIC RESONANCE IN FAR INFRARED SPECTRAL RANGE.

Magnetic resonance is a physical phenomenon that occurs across a wide spectral range of the electromagnetic radiation. Far-infrared magneto spectroscopy (FIRMS) offers an experimental routine to probe the magnetic excitation spectrum between 0.4 and 22 THz (12- 720 cm^{-1}). In this talk, I will discuss the FIRMS technique employed at the National High Magnetic Field Laboratory, with a specific focus on how it has been successfully applied to the study of single-molecule magnets and low-dimensional spin systems. A short introduction includes a description of the measurement procedure, the data analysis and a recent sensitivity improvement. The advantages and limitations compared to other spectroscopic techniques are presented as well. The selected examples illustrate how the FIRMS technique allows direct measurements of zero-field splitting energy and enables



Mykhaylo Ozerov, Ph.D.
National High Magnetic Field
Laboratory

4:00 pm - Olin 101*


Note: For additional information on the seminar,
contact wfuphys@wfu.edu

Coming next week:

marking the
International
Day
of
Women
and
Girls
in Science
the Physics department will
host a panel discussion with
faculty and students

Friday, 2/10
11:30 am- 1:00 pm, Olin 105

Free
refreshments
served by



Ellen Ochoa

Chien-Shung Wu

Mae Jemison

Annie Easley

Review: General results for a multipole analysis of the electrostatic potential due to an isolated charge distribution:

General form of electrostatic potential with boundary value $\Phi(r \rightarrow \infty) = 0$ for confined charge density $\rho(\mathbf{r})$:

$$\begin{aligned}\Phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \\ &= \frac{1}{4\pi\epsilon_0} \int d^3r' \rho(\mathbf{r}') \left(\sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi') \right)\end{aligned}$$

Suppose that $\rho(\mathbf{r}) = \sum_{lm} \rho_{lm}(r) Y_{lm}(\theta, \varphi)$

$$\Rightarrow \Phi(\mathbf{r}) = \frac{1}{\epsilon_0} \sum_{lm} \frac{1}{2l+1} Y_{lm}(\theta, \varphi) \left(\frac{1}{r^{l+1}} \int_0^r r'^{2+l} dr' \rho_{lm}(r') + r^l \int_r^\infty r'^{1-l} dr' \rho_{lm}(r') \right)$$

For $r \rightarrow \infty$:
$$\Phi(\mathbf{r}) = \frac{1}{\epsilon_0} \sum_{lm} \frac{1}{2l+1} Y_{lm}(\theta, \varphi) \underbrace{\frac{1}{r^{l+1}} \int_0^\infty r'^{2+l} dr' \rho_{lm}(r')}_{q_{lm}}$$

Your questions –

From Banasree: Does q_{lm} have any physical meaning?

Comment – It typically indicates the symmetry of a charge distribution and directly determines the long range form of the electrostatic potential.

Comment --

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

Acts like a projection operator



$$= \frac{1}{4\pi\epsilon_0} \int d^3r' \rho(\mathbf{r}') \left(\sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi') \right)$$

Suppose that $\rho(\mathbf{r}') = \sum_{lm} \rho_{lm}(r') Y_{lm}(\theta', \varphi')$

$$\Rightarrow \Phi(\mathbf{r}) = \frac{1}{\epsilon_0} \sum_{lm} \frac{1}{2l+1} Y_{lm}(\theta, \varphi) \left(\frac{1}{r^{l+1}} \int_0^r r'^{2+l} dr' \rho_{lm}(r') + r^l \int_r^\infty r'^{1-l} dr' \rho_{lm}(r') \right)$$

Why? -- Recall that

$$\int d\Omega' Y_{lm}^*(\theta', \varphi') Y_{\lambda\mu}(\theta', \varphi') = \delta_{l\lambda, m\mu}$$

The the multipole analysis has the following general behavior for $r \rightarrow \infty$:

For r outside the extent of $\rho(\mathbf{r})$:

$$\begin{aligned}\Phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \sum_{lm} \frac{4\pi}{2l+1} \frac{Y_{lm}(\theta, \varphi)}{r^{l+1}} \left(\int d^3r' r'^l Y_{lm}^*(\theta', \varphi') \rho(\mathbf{r}') \right) \\ &= \frac{1}{4\pi\epsilon_0} \sum_{lm} \frac{4\pi}{2l+1} \frac{q_{lm} Y_{lm}(\theta, \varphi)}{r^{l+1}}\end{aligned}\quad q_{lm} \equiv \int_0^\infty r'^{2+l} dr' \rho_{lm}(r')$$

In terms of Cartesian expansion :

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r} + \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} + \frac{1}{2} \sum_{i,j} Q_{ij} \frac{r_i r_j}{r^5} \dots \right)$$

Here q , p_i , and Q_{ij} are linearly proportional to the q_{lm} multipole values.

The multipole analysis also can be used to analyze the the electrostatic fields for $r \rightarrow 0$ as needed in the following example involving a very localized charge density $\rho(\mathbf{r})$ in a electrostatic field $\Phi(\mathbf{r})$ (such as a nucleus in the field produced by electrons in an atom).

charge density
within nucleus



electrostatic potential
due to electrons near
the nucleus.

$$W = \int d^3r \rho(\mathbf{r})\Phi(\mathbf{r})$$

$$\approx \int d^3r \rho(\mathbf{r}) \left(\Phi(0) + \mathbf{r} \cdot \nabla \Phi(\mathbf{r}) \Big|_{r=0} + \frac{1}{2} (\mathbf{r} \cdot \nabla)^2 \Phi(\mathbf{r}) \Big|_{r=0} + \dots \right)$$

$$= q\Phi(0) - \mathbf{p} \cdot \mathbf{E}(0) + \frac{1}{6} \sum_{i,j} Q_{ij} \frac{\partial^2 \Phi(0)}{\partial r_i \partial r_j} + \dots$$

The following results were mentioned on Monday and are presented here with greater detail.

$$\rho(\mathbf{r}) = \frac{q}{64\pi a^3} \left(\frac{r}{a}\right)^2 e^{-r/a} \sin^2 \theta$$

Note that: $\sqrt{\frac{4\pi}{5}} Y_{20}(\theta, \phi) = \frac{3}{2} \cos^2 \theta - \frac{1}{2} = 1 - \frac{3}{2} \sin^2 \theta$

$$\sin^2 \theta = \frac{2}{3} - \frac{2}{3} \sqrt{\frac{4\pi}{5}} Y_{20}(\theta, \phi) = \frac{2}{3} \sqrt{\frac{4\pi}{1}} Y_{00}(\theta, \phi) - \frac{2}{3} \sqrt{\frac{4\pi}{5}} Y_{20}(\theta, \phi)$$

$$\Rightarrow \rho(\mathbf{r}) = \rho_{00}(r) Y_{00}(\theta, \phi) + \rho_{20}(r) Y_{20}(\theta, \phi)$$

$$\Phi(\mathbf{r}) = \Phi_{00}(r) Y_{00}(\theta, \phi) + \Phi_{20}(r) Y_{20}(\theta, \phi)$$

$$\Phi_{lm} = \frac{1}{4\pi\epsilon_0} \frac{4\pi}{2l+1} \left(\frac{1}{r^{l+1}} \int_0^r r'^{2+l} dr' \rho_{lm}(r') + r^l \int_r^\infty r'^{1-l} dr' \rho_{lm}(r') \right)$$

$$\rho_{00}(r) = \frac{2}{3} \sqrt{4\pi} \frac{q}{64\pi a^3} \left(\frac{r}{a}\right)^2 e^{-r/a} \quad \rho_{20}(r) = -\frac{2}{3} \sqrt{\frac{4\pi}{5}} \frac{q}{64\pi a^3} \left(\frac{r}{a}\right)^2 e^{-r/a}$$

Writing out the details of the potential from evaluating integrals

$$\Phi_{00}(r) = \frac{1}{4\pi\epsilon_0} \sqrt{4\pi} \frac{q}{r} \left(1 - e^{-r/a} \left(1 + \frac{3r}{4a} + \frac{r^2}{4a^2} + \frac{r^3}{24a^3} \right) \right)$$

$$\Phi_{20}(r) = -\frac{6}{4\pi\epsilon_0} \sqrt{\frac{4\pi}{5}} \frac{qa^2}{r^3} \left(1 - e^{-r/a} \left(1 + \frac{r}{a} + \frac{r^2}{2a^2} + \frac{r^3}{6a^3} + \frac{r^4}{24a^3} + \frac{r^5}{144a^5} \right) \right)$$

For $r \rightarrow \infty$; in terms for Legendre polynomials:

$$\Phi(\mathbf{r}) \rightarrow \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r} - \frac{6a^2}{r^3} P_2(\cos\theta) \right) \quad Y_{l0}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)$$

For $r \rightarrow 0$; in terms for Legendre polynomials :

$$\Phi(\mathbf{r}) \rightarrow \frac{q}{4\pi\epsilon_0} \left(\frac{1}{4a} - \frac{r^2}{120a^3} P_2(\cos\theta) \right)$$

More details continued --

For $r \rightarrow 0$; in terms for Legendre polynomials :

$$\Phi(\mathbf{r}) \rightarrow \frac{q}{4\pi\epsilon_0} \left(\frac{1}{4a} - \frac{r^2}{120a^3} P_2(\cos\theta) \right)$$

Implications for electric quadrupole interaction :

$$W = \frac{1}{6} \sum_{i,j} Q_{ij} \frac{\partial^2 \Phi(0)}{\partial r_i \partial r_j} + \dots \quad P_2(\cos\theta) = \frac{3}{2} \cos^2 \theta - \frac{1}{2} = \frac{1}{2r^2} (3z^2 - r^2) \\ = \frac{1}{2r^2} (2z^2 - x^2 - y^2)$$

For $r \rightarrow 0$; in terms of Cartesian coordinates

$$\Phi(\mathbf{r}) \rightarrow \frac{q}{4\pi\epsilon_0} \left(\frac{1}{4a} - \frac{2z^2 - x^2 - y^2}{240a^3} \right)$$

$$\frac{\partial^2 \Phi(0)}{\partial x^2} = \frac{\partial^2 \Phi(0)}{\partial y^2} = -\frac{1}{2} \frac{\partial^2 \Phi(0)}{\partial z^2} = \frac{q}{4\pi\epsilon_0} \frac{1}{120a^3}$$

Example of multipole distribution continued --

Electric quadrupole interaction:

$$W = \frac{1}{6} \sum_{i,j} Q_{ij} \frac{\partial^2 \Phi(0)}{\partial r_i \partial r_j} = \frac{1}{6} \left(Q_{xx} \frac{\partial^2 \Phi(0)}{\partial x^2} + Q_{yy} \frac{\partial^2 \Phi(0)}{\partial y^2} + Q_{zz} \frac{\partial^2 \Phi(0)}{\partial z^2} \right)$$

For symmetric nuclei, $Q_{zz} \equiv Qq = -\frac{1}{2} Q_{xx} = -\frac{1}{2} Q_{yy}$

$$W \approx -\frac{q^2}{4\pi\epsilon_0} \frac{Q}{240a^3}$$

Here q stands for the elementary charge
 $q=e=1.602176634 \times 10^{-19} \text{C}$

This result is discussed in Jackson's problem #4.7. In that example Q is approximately 10^{-28} m^2 as a result of the internal structure of the nucleus.

Summary -- Notion of multipole moment:

In the spherical harmonic representation --

define the moment q_{lm} of the (confined) charge distribution $\rho(\mathbf{r})$:

$$q_{lm} \equiv \int d^3 r' r'^l Y_{lm}^*(\theta', \phi') \rho(\mathbf{r}')$$

In the Cartesian representation --

define the monopole moment q :

$$q \equiv \int d^3 r' \rho(\mathbf{r}')$$

define the dipole moment \mathbf{p} :

$$\mathbf{p} \equiv \int d^3 r' \mathbf{r}' \rho(\mathbf{r}')$$

define the quadrupole moment components Q_{ij} ($i, j \rightarrow x, y, z$):

$$Q_{ij} \equiv \int d^3 r' \left(3r'_i r'_j - r'^2 \delta_{ij} \right) \rho(\mathbf{r}')$$

General form of electrostatic potential in terms of multipole moments:

For r outside the extent of $\rho(\mathbf{r})$:

$$\begin{aligned}\Phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \sum_{lm} \frac{4\pi}{2l+1} \frac{Y_{lm}(\theta, \varphi)}{r^{l+1}} \left(\int d^3r' r'^l Y_{lm}^*(\theta', \varphi') \rho(\mathbf{r}') \right) \\ &= \frac{1}{4\pi\epsilon_0} \sum_{lm} \frac{4\pi q_{lm}}{2l+1} \frac{Y_{lm}(\theta, \varphi)}{r^{l+1}}\end{aligned}$$

In terms of Cartesian expansion :

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r} + \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} + \frac{1}{2} \sum_{i,j} Q_{ij} \frac{r_i r_j}{r^5} \dots \right)$$


Focus on dipolar contributions:

For r outside the extent of $\rho(\mathbf{r})$:

Electrostatic potential:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{\mathbf{p} \cdot \mathbf{r}}{r^3} \right)$$

Electrostatic field:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{3\mathbf{r} (\mathbf{p} \cdot \mathbf{r}) - r^2 \mathbf{p}}{r^5} - \frac{4\pi}{3} \mathbf{p} \delta^3(\mathbf{r}) \right)$$


Poorly defined for $r \rightarrow 0$

Correct value for $r \rightarrow 0$

“Justification” of surprising δ -function term in dipole electric field -- Assuming dipole is located at $r=0$, we need to need to evaluate the electrostatic field near $r=0$:

We will use the approximation:

$$\mathbf{E}(\mathbf{r} \approx \mathbf{0}) \approx \left(\int_{\text{sphere}} \mathbf{E}(\mathbf{r}) d^3 r \right) \delta^3(\mathbf{r}).$$

First we note that:

$$\int_{r \leq R} \mathbf{E}(\mathbf{r}) d^3 r = -R^2 \int_{r=R} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\Omega.$$

Some details -- amplifying discussion in JDJ:

$$\int_{r \leq R} \mathbf{E}(\mathbf{r}) d^3 r = -R^2 \int_{r=R} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\Omega.$$

This result follows from the divergence theorem:

$$\int_{\text{vol}} \nabla \cdot \mathcal{V} d^3 r = \int_{\text{surface}} \mathcal{V} \cdot d\mathbf{A}.$$

In our case, this theorem can be used for each cartesian coordinate if we choose $\mathcal{V} \equiv \hat{\mathbf{x}}\Phi(\mathbf{r})$ for the x component, etc.

$$\int_{r \leq R} \nabla \Phi(\mathbf{r}) d^3 r = \hat{\mathbf{x}} \int_{r \leq R} \nabla \cdot (\hat{\mathbf{x}}\Phi) d^3 r + \hat{\mathbf{y}} \int_{r \leq R} \nabla \cdot (\hat{\mathbf{y}}\Phi) d^3 r + \hat{\mathbf{z}} \int_{r \leq R} \nabla \cdot (\hat{\mathbf{z}}\Phi) d^3 r,$$

which is equal to:

$$\int_{r=R} \Phi(\mathbf{r}) R^2 d\Omega ((\hat{\mathbf{x}} \cdot \hat{\mathbf{r}})\hat{\mathbf{x}} + (\hat{\mathbf{y}} \cdot \hat{\mathbf{r}})\hat{\mathbf{y}} + (\hat{\mathbf{z}} \cdot \hat{\mathbf{r}})\hat{\mathbf{z}}) = \int_{r=R} \Phi(\mathbf{r}) R^2 d\Omega \hat{\mathbf{r}}.$$

Therefore --

$$\int_{r \leq R} \mathbf{E}(\mathbf{r}) d^3 r = -\int_{r \leq R} \nabla \Phi(\mathbf{r}) d^3 r = -R^2 \int_{r=R} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\Omega.$$

More details

$$\int_{r \leq R} \mathbf{E}(\mathbf{r}) d^3 r = -R^2 \int_{r=R} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\Omega.$$

Now, we notice that the electrostatic potential can be determined from the charge density $\rho(\mathbf{r})$ according to:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3 r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{4\pi\epsilon_0} \sum_{lm} \frac{4\pi}{2l+1} \int d^3 r' \rho(\mathbf{r}') \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}^*(\hat{\mathbf{r}}) Y_{lm}(\hat{\mathbf{r}}').$$

We also note that the unit vector can be written in terms of spherical harmonic functions:

$$\hat{\mathbf{r}} = \begin{cases} \sin(\theta) \cos(\phi) \hat{\mathbf{x}} + \sin(\theta) \sin(\phi) \hat{\mathbf{y}} + \cos(\theta) \hat{\mathbf{z}} \\ \sqrt{\frac{4\pi}{3}} \left(Y_{1-1}(\hat{\mathbf{r}}) \frac{\hat{\mathbf{x}} + i\hat{\mathbf{y}}}{\sqrt{2}} + Y_{11}(\hat{\mathbf{r}}) \frac{-\hat{\mathbf{x}} + i\hat{\mathbf{y}}}{\sqrt{2}} + Y_{10}(\hat{\mathbf{r}}) \hat{\mathbf{z}} \right) \end{cases}$$

$$\begin{aligned} \int_{r=R} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\Omega &= \frac{1}{3\epsilon_0} \int d^3 r' \rho(\mathbf{r}') \frac{r_{<}}{r_{>}^2} \sqrt{\frac{4\pi}{3}} \left(Y_{1-1}(\hat{\mathbf{r}}') \frac{\hat{\mathbf{x}} + i\hat{\mathbf{y}}}{\sqrt{2}} + Y_{11}(\hat{\mathbf{r}}') \frac{-\hat{\mathbf{x}} + i\hat{\mathbf{y}}}{\sqrt{2}} + Y_{10}(\hat{\mathbf{r}}') \hat{\mathbf{z}} \right) \\ &= \frac{1}{3\epsilon_0} \int d^3 r' \rho(\mathbf{r}') \frac{r_{<}}{r_{>}^2} \hat{\mathbf{r}}' \end{aligned}$$

More details continued --

When we evaluate the integral over solid angle $d\Omega$, only the $l = 1$ terms contribute, and the result of the integration reduces to:

$$-R^2 \int_{r=R} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\Omega = -\frac{1}{4\pi\epsilon_0} \frac{4\pi R^2}{3} \int d^3 r' \rho(\mathbf{r}') \frac{r_{<}}{r_{>}^2} \hat{\mathbf{r}}'.$$

The choice of $r_{<}$ and $r_{>}$ is a choice between the integration variables r' and the sphere radius R . If the sphere encloses the charge distribution, $\rho(\mathbf{r}')$, then $r_{<} = r'$ and $r_{>} = R$ so that the result is:

$$-R^2 \int_{r=R} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\Omega = -\frac{1}{4\pi\epsilon_0} \frac{4\pi R^2}{3} \frac{1}{R^2} \int d^3 r' \rho(\mathbf{r}') r' \hat{\mathbf{r}}' \equiv -\frac{\mathbf{p}}{3\epsilon_0}.$$

Otherwise, if the charge distribution $\rho(\mathbf{r}')$ lies outside of the sphere, then $r_{<} = R$ and $r_{>} = r'$ and the result is:

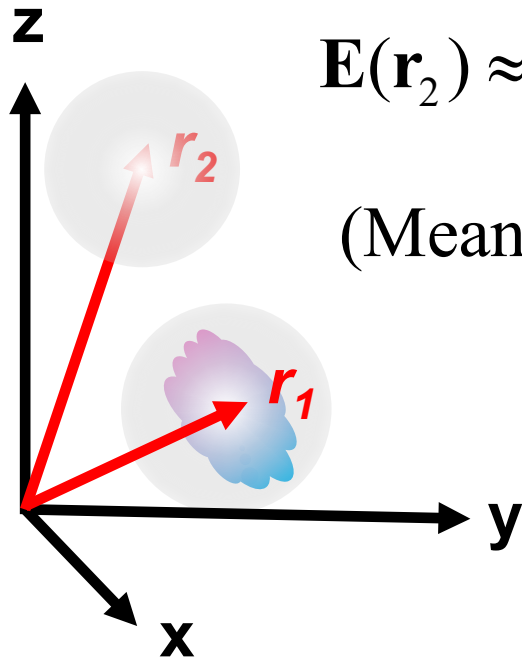
$$-R^2 \int_{r=R} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\Omega = -\frac{1}{4\pi\epsilon_0} \frac{4\pi R^2}{3} R \int d^3 r' \frac{\rho(\mathbf{r}')}{r'^2} \hat{\mathbf{r}}' \equiv \frac{4\pi R^3}{3} \mathbf{E}(0).$$

In summary --

Electrostatic dipolar field for dipole moment \mathbf{p} at $\mathbf{r}=0$:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{3\mathbf{r} (\mathbf{p} \cdot \mathbf{r}) - r^2 \mathbf{p}}{r^5} - \frac{4\pi}{3} \mathbf{p} \delta^3(\mathbf{r}) \right)$$

Summary of key argument:



$$\mathbf{E}(\mathbf{r}_2) \approx \frac{3}{4\pi R^3} \int_{r \leq R} d^3 r \mathbf{E}(\mathbf{r}_2 + \mathbf{r}) = \mathbf{E}(\mathbf{r}_2)$$

(Mean value theorem for Laplace equation)

$$\mathbf{E}(\mathbf{r}_1) \approx \frac{3}{4\pi R^3} \int_{r \leq R} d^3 r \mathbf{E}(\mathbf{r}_1 + \mathbf{r})$$

$$\approx \frac{3}{4\pi R^3} \left(-\frac{\mathbf{p}}{3\epsilon_0} \right) \Rightarrow -\frac{\mathbf{p}}{3\epsilon_0} \delta^3(\mathbf{r} - \mathbf{r}_1)$$

Summary:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{3\mathbf{r} (\mathbf{p} \cdot \mathbf{r}) - r^2 \mathbf{p}}{r^5} - \frac{4\pi}{3} \mathbf{p} \delta^3(\mathbf{r}) \right)$$

Coarse grain representation of macroscopic distribution of dipoles:

Electric polarization $\mathbf{P}(\mathbf{r})$ due to collection of dipoles :

$$\mathbf{P}(\mathbf{r}) \equiv \sum_i \mathbf{p}_i \delta^3(\mathbf{r} - \mathbf{r}_i)$$

Monopole electric charge density $\rho_{\text{mono}}(\mathbf{r})$:

$$\rho_{\text{mono}}(\mathbf{r}) \equiv \sum_i q_i \delta^3(\mathbf{r} - \mathbf{r}_i)$$

Electrostatic potential for a single monopole charge q
and a single dipole \mathbf{p} :

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r} + \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} \right)$$

Coarse grain representation of macroscopic distribution of dipoles -- continued:

Electrostatic potential for a single monopole charge q
and a single dipole \mathbf{p} :

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r} + \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} \right)$$

Electrostatic potential for collections of monopole charges q_i
and dipoles \mathbf{p}_i :

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\int d^3r' \frac{\rho_{mono}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \int d^3r' \frac{\mathbf{P}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \right)$$

Note:
$$\int d^3r' \frac{\mathbf{P}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} = \int d^3r' \mathbf{P}(\mathbf{r}') \cdot \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} = - \int d^3r' \frac{\nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

Coarse grain representation of macroscopic distribution of dipoles -- continued:

Electrostatic potential for collections of monopole charges q_i and dipoles \mathbf{p}_i :

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\int d^3r' \frac{\rho_{mono}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} - \int d^3r' \frac{\nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right)$$

$$-\nabla^2 \Phi(\mathbf{r}) = \nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{1}{\epsilon_0} (\rho_{mono}(\mathbf{r}) - \nabla \cdot \mathbf{P}(\mathbf{r}))$$

$$\Rightarrow \nabla \cdot (\epsilon_0 \mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r})) = \rho_{mono}(\mathbf{r})$$

Define Displacement field: $\mathbf{D}(\mathbf{r}) \equiv \epsilon_0 \mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r})$

Macroscopic form of Gauss's law: $\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho_{mono}(\mathbf{r})$

Coarse grain representation of macroscopic distribution of dipoles -- continued:

Many materials are polarizable and produce a polarization field in the presence of an electric field with a proportionality constant χ_e :

$$\mathbf{P}(\mathbf{r}) = \varepsilon_0 \chi_e \mathbf{E}(\mathbf{r})$$

$$\mathbf{D}(\mathbf{r}) \equiv \varepsilon_0 \mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r}) = \varepsilon_0 (1 + \chi_e) \mathbf{E}(\mathbf{r}) \equiv \varepsilon \mathbf{E}(\mathbf{r})$$

ε represents the dielectric function of the material

Boundary value problems in dielectric materials

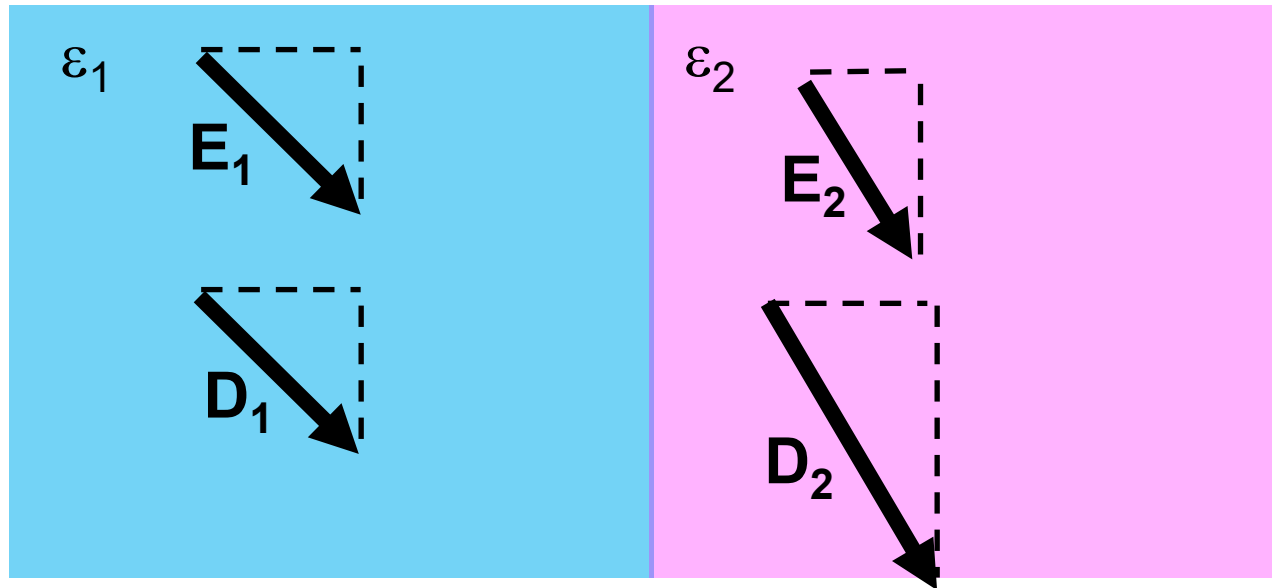
For $\rho_{\text{mono}}(\mathbf{r}) = 0$

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = 0 \quad \text{and} \quad \nabla \times \mathbf{E}(\mathbf{r}) = 0$$

\Rightarrow At a surface between two dielectrics, in terms of surface normal $\hat{\mathbf{r}}$:

$$\hat{\mathbf{r}} \cdot \mathbf{D}(\mathbf{r}) = \text{continuous} = \hat{\mathbf{r}} \times \mathbf{E}(\mathbf{r})$$

Boundary value problems in the presence of dielectrics – example:



$$\text{For } \frac{\epsilon_2}{\epsilon_1} = 2$$

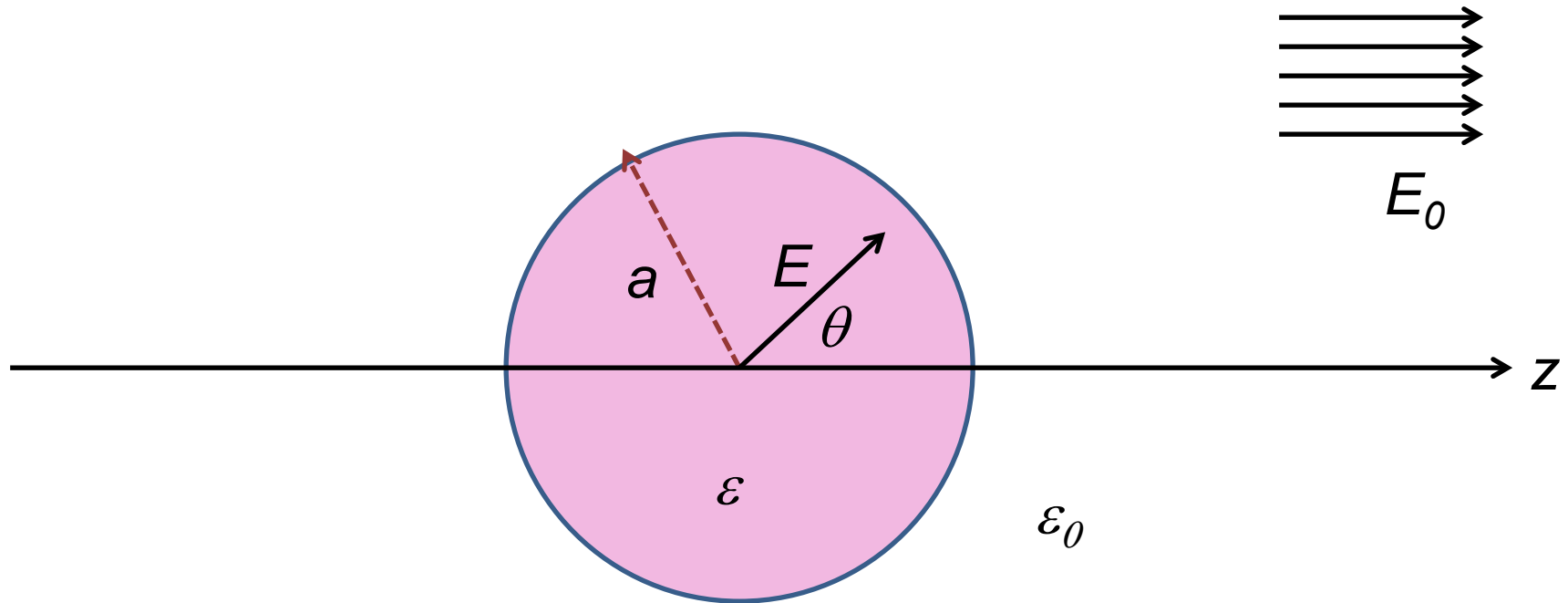
For isotropic dielectrics:

$$D_{1n} = D_{2n} \quad \epsilon_1 E_{1n} = \epsilon_2 E_{2n}$$

$$E_{1t} = E_{2t} \quad \frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2}$$

Boundary value problems in the presence of dielectrics

– example:



$$\nabla \cdot \mathbf{D}(\mathbf{r}) = 0 \quad \text{and} \quad \nabla \times \mathbf{E}(\mathbf{r}) = 0 \quad \text{At } r = a: \quad \epsilon \frac{\partial \Phi_{<}(\mathbf{r})}{\partial r} = \epsilon_0 \frac{\partial \Phi_{>}(\mathbf{r})}{\partial r}$$

$$\text{For } r \leq a \quad \mathbf{D}(\mathbf{r}) = -\epsilon \nabla \Phi(\mathbf{r})$$

$$\text{For } r > a \quad \mathbf{D}(\mathbf{r}) = -\epsilon_0 \nabla \Phi(\mathbf{r})$$

$$\frac{\partial \Phi_{<}(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_{>}(\mathbf{r})}{\partial \theta}$$

Boundary value problems in the presence of dielectrics

– example -- continued:

$$\Phi_{<}(\mathbf{r}) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta)$$

$$\Phi_{>}(\mathbf{r}) = \sum_{l=0}^{\infty} \left(B_l r^l + \frac{C_l}{r^{l+1}} \right) P_l(\cos \theta)$$

$$\text{At } r = a: \quad \epsilon \frac{\partial \Phi_{<}(\mathbf{r})}{\partial r} = \epsilon_0 \frac{\partial \Phi_{>}(\mathbf{r})}{\partial r}$$

$$\frac{\partial \Phi_{<}(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_{>}(\mathbf{r})}{\partial \theta}$$

$$\text{For } r \rightarrow \infty \quad \Phi_{>}(\mathbf{r}) = -E_0 r \cos \theta$$

Solution -- only $l = 1$ contributes

$$B_1 = -E_0$$

$$A_1 = -\left(\frac{3}{2 + \epsilon / \epsilon_0} \right) E_0$$

$$C_1 = \left(\frac{\epsilon / \epsilon_0 - 1}{2 + \epsilon / \epsilon_0} \right) a^3 E_0$$

Boundary value problems in the presence of dielectrics – example -- continued:

$$\Phi_{<}(\mathbf{r}) = -\left(\frac{3}{2 + \epsilon / \epsilon_0}\right) E_0 r \cos \theta$$

$$\Phi_{>}(\mathbf{r}) = -\left(r - \left(\frac{\epsilon / \epsilon_0 - 1}{2 + \epsilon / \epsilon_0}\right) \frac{a^3}{r^2}\right) E_0 \cos \theta$$

