PHY 712 Electrodynamics
10-10:50 AM MWF in Olin 103

Plan for Lecture 14:

Finish reading Chapter 5

1. Recap of hyperfine interaction
2. Macroscopic magnetization density M
3. H field and its relation to B
4. Magnetic boundary values

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Mon: 01/9/2023	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/13/2023
2	Wed: 01/11/2023	Chap. 1	Electrostatic energy calculations	#2	01/18/2023
3	Fri: 01/13/2023	Chap. 1	Electrostatic energy calculations thanks to Ewald	#3	01/18/2023
	Mon: 01/16/2023		MLK Holiday -- no class		
4	Wed: 01/18/2023	Chap. 1 & 2	Electrostatic potentials and fields	#4	01/20/2023
5	Fri: 01/20/2023	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#5	01/23/2023
6	Mon: 01/23/2023	Chap. 1 - 3	Brief introduction to numerical methods	#6	01/25/2023
7	Wed: 01/25/2023	Chap. 2 & 3	Image charge constructions	#7	01/30/2023
8	Fri: 01/27/2023	Chap. 2 & 3	Cylindrical and spherical geometries		
9	Mon: 01/30/2023	Chap. 3 & 4	Spherical geometry and multipole moments	#8	02/01/2023
10	Wed: 02/01/2023	Chap. 4	Dipoles and Dielectrics	#9	02/03/2023
11	Fri: 02/03/2023	Chap. 4	Dipoles and Dielectrics	#10	02/06/2023
12	Mon: 02/06/2023	Chap. 5	Magnetostatics	#11	02/08/2023
13	Wed: 02/08/2023	Chap. 5	Magnetic dipoles and hyperfine interaction	#12	02/10/2023
14	Fri: 02/10/2023	Chap. 5	Magnetic dipoles and dipolar fields	#13	02/13/2023
15	Mon: 02/13/2023	Chap. 6	Maxwell's Equations		
16	Wed: 02/15/2023	Chap. 6	Electromagnetic energy and forces		
17	Fri: 02/17/2023	Chap. 7	Electromagnetic plane waves		
18	Mon: 02/20/2023	Chap. 7	Electromagnetic plane waves		
19	Wed: 02/22/2023	Chap. 7	Optical effects of refractive indices		
20	Fri: 02/24/2023	Chap. 8	Brief introduction to wave guides		

PHY 712 -- Assignment #13

February 10, 2023

Finish reading Chapter 5 in **Jackson** .

1. Work through some of the details of magnetic shielding effects of the highly permeable spherical shell given in Eq. 5.121 of **Jackson** and/or the equivalent presentation in the lecture notes.

Comment on schedule --

February 2023

Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28	1	2	3	

Mid-term exam

Spring break


Ellen Ochoa
Chien-Shung Wu
Mae Jemison
Annie Easley
marking the
International Day of Women and Girls in Science
the Physics department will host a panel discussion with faculty and students
Friday, 2/10
11:30 am– 1:00 pm, Olin 105
Free refreshments served by
Panera BREAD

Your questions –

From Arezoo: Could you please explain more about J free and J total. Is it true that, net electric current includes J due to magnetization as well as the free macroscopic current J free due to electric conductance?

Summary of hyperfine interaction form:

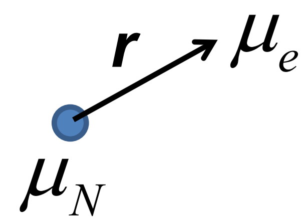
Interactions between magnetic dipoles

Sources of magnetic dipoles and other sources of magnetism in an atom:

- Intrinsic magnetic moment of a nucleus μ_N
- Intrinsic magnetic moment of an electron μ_e
- Magnetic field due to electron orbital current $\mathbf{J}_e(\mathbf{r})$

Interaction energy between a magnetic dipole ***m*** and a magnetic field **B:**

$$E_{int} = -\mathbf{m} \cdot \mathbf{B}$$



In this case:

$$E_{int} = -\mu_N \cdot \mathbf{B}_{\mu_e} - \mu_N \cdot \mathbf{B}_{\mathbf{J_e}}(0)$$

$$\mathbf{B}_{\boldsymbol{\mu}_e}(\mathbf{r}) = \frac{\mu_0}{4\pi} \left\{ \frac{3\hat{\mathbf{r}}(\boldsymbol{\mu}_e \cdot \hat{\mathbf{r}}) - \boldsymbol{\mu}_e}{r^3} + \frac{8\pi}{3} \boldsymbol{\mu}_e \delta^3(\mathbf{r}) \right\}$$

Hyperfine interaction energy: -- continued

$$E_{int} = -\mu_N \cdot \mathbf{B}_{\mu_e} - \mu_N \cdot \mathbf{B}_{\mathbf{J}_e}(0)$$

Here we assume that nuclear position is **r**=*0.*

Evaluation of the magnetic field at the nucleus due to the electron current density:

The vector potential associated with an electron in a bound state of an atom as described by a quantum mechanical wavefunction $\psi_{nlm_l}(\mathbf{r})$ can be written:

$$\mathbf{A}_{\mathbf{J}_e}(\mathbf{r}) = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \int d^3r' \, \frac{\hat{\mathbf{z}} \times \mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \frac{\left|\psi_{nlm_l}(\mathbf{r}')\right|^2}{r'^2 \sin^2\theta'}$$

We want to evaluate the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ in the vicinity of the nucleus $(\mathbf{r} \to 0)$.

Hyperfine interaction energy: -- continued

$$\mathbf{B}_{\mathbf{J}_e}(0) = \nabla \times \mathbf{A}_{\mathbf{J}_e}\Big|_{\mathbf{r}\to 0} = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \int d^3r' \nabla \times \frac{\hat{\mathbf{z}} \times \mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \frac{\left|\psi_{nlm_l}(\mathbf{r}')\right|^2}{r'^2 \sin^2\theta'}\Bigg|_{\mathbf{r}\to 0}$$

$$\mathbf{B}_{\mathbf{o}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \int d^3r' \frac{(\mathbf{r}-\mathbf{r}')\times(\hat{\mathbf{z}}\times\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \frac{\left|\psi_{nlm_l}(\mathbf{r}')\right|^2}{r'^2 \sin^2\theta'}\Bigg|_{\mathbf{r}\to 0}$$

$$\mathbf{B}_{\mathbf{o}}(\mathbf{0}) = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \int d^3r' \frac{\mathbf{r}'\times(\hat{\mathbf{z}}\times\mathbf{r}')}{r'^3} \frac{\left|\psi_{nlm_l}(\mathbf{r}')\right|^2}{r'^2 \sin^2\theta'}$$

$$\hat{\mathbf{r}}' \times (\hat{\mathbf{z}}\times\hat{\mathbf{r}}') = \hat{\mathbf{z}}(\mathbf{1}-\cos^2\boldsymbol{\theta}') - \hat{\mathbf{x}}\cos\theta'\sin\theta'\cos\phi' - \hat{\mathbf{y}}\cos\theta'\sin\theta'\sin\phi').$$

$$\mathbf{B}_{\mathbf{o}}(\mathbf{0}) = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \int d^3r' \frac{\hat{\mathbf{z}} r'^2 \sin^2\theta'}{r'^3} \frac{\left|\psi_{nlm_l}(\mathbf{r}')\right|^2}{r'^2 \sin^2\theta'} = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \hat{\mathbf{z}} \int d^3r' \frac{\left|\psi_{nlm_l}(\mathbf{r}')\right|^2}{r'^3}$$

$$= -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \hat{\mathbf{z}} \left\langle \frac{1}{r'^3} \right\rangle$$

Note that this field at the nucleus site is due to the electronic orbital angular momentum.

Hyperfine interaction energy: -- continued

$$E_{int} \equiv H_{HF} = -\mu_N \cdot \mathbf{B}_{\mu_e} - \mu_N \cdot \mathbf{B}_{\mathbf{J_e}}(0)$$

Putting all of the terms together:

$$H_{\mathrm{HF}} = -\frac{\mu_0}{4\pi}\left(\left\langle \frac{3(\boldsymbol{\mu}_\mathbf{N} \cdot \hat{\mathbf{r}})(\boldsymbol{\mu}_\mathbf{e} \cdot \hat{\mathbf{r}}) - \boldsymbol{\mu}_\mathbf{N} \cdot \boldsymbol{\mu}_\mathbf{e}}{r^3} + \frac{8\pi}{3}\boldsymbol{\mu}_\mathbf{N} \cdot \boldsymbol{\mu}_\mathbf{e} \delta^3(\mathbf{r}) \right\rangle + \frac{e}{m_e}\left\langle \frac{\mathbf{L} \cdot \boldsymbol{\mu}_\mathbf{N}}{r^3} \right\rangle \right).$$

In this expression the brackets $\langle \ \rangle$ indicate evaluating the expectation value relative to the electronic state.

Macroscopic dipolar effects --
Magnetic dipole moment

$$\mathbf{m} = \frac{1}{2}\int d^3r\, \mathbf{r} \times \mathbf{J}(\mathbf{r})$$

Note that the intrinsic spin of elementary particles is associated with a magnetic dipole moment, but we often do not have a detailed knowledge of its **J**(**r**).

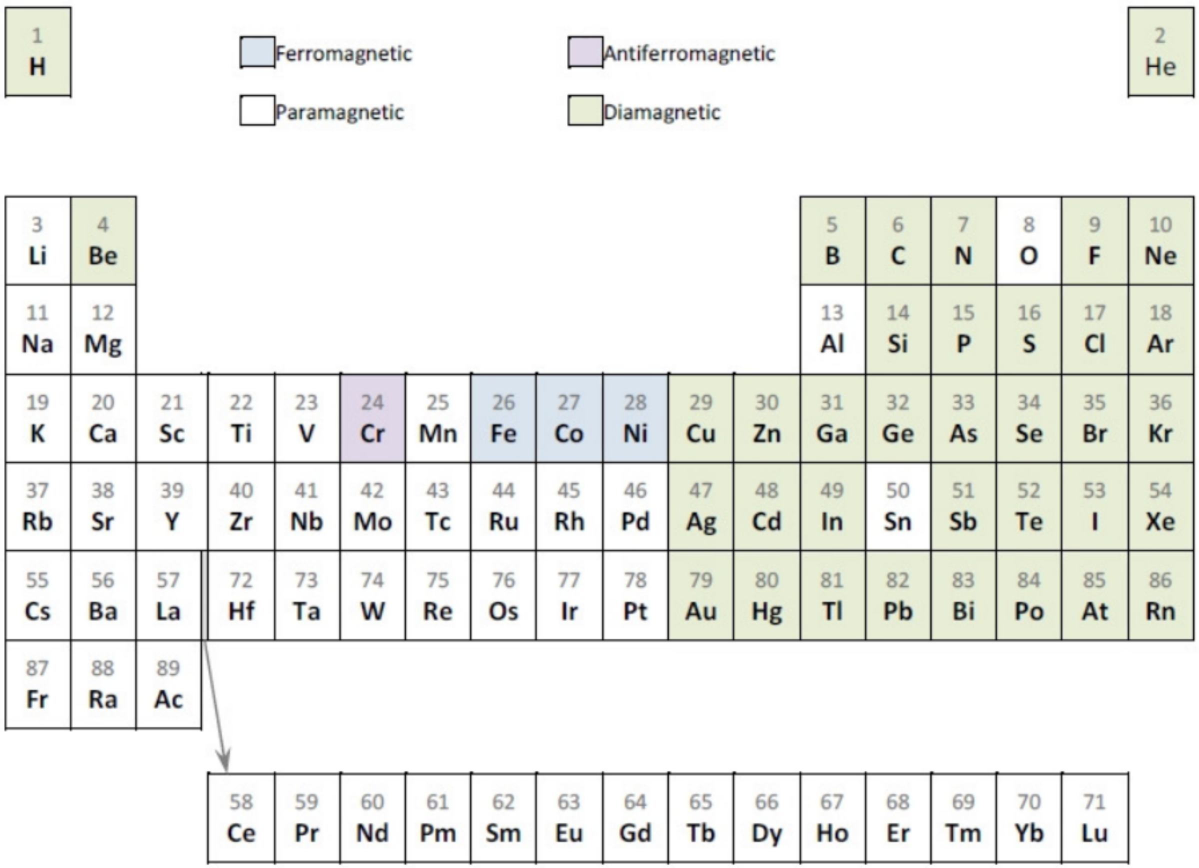
Vector potential for magnetic dipole moment

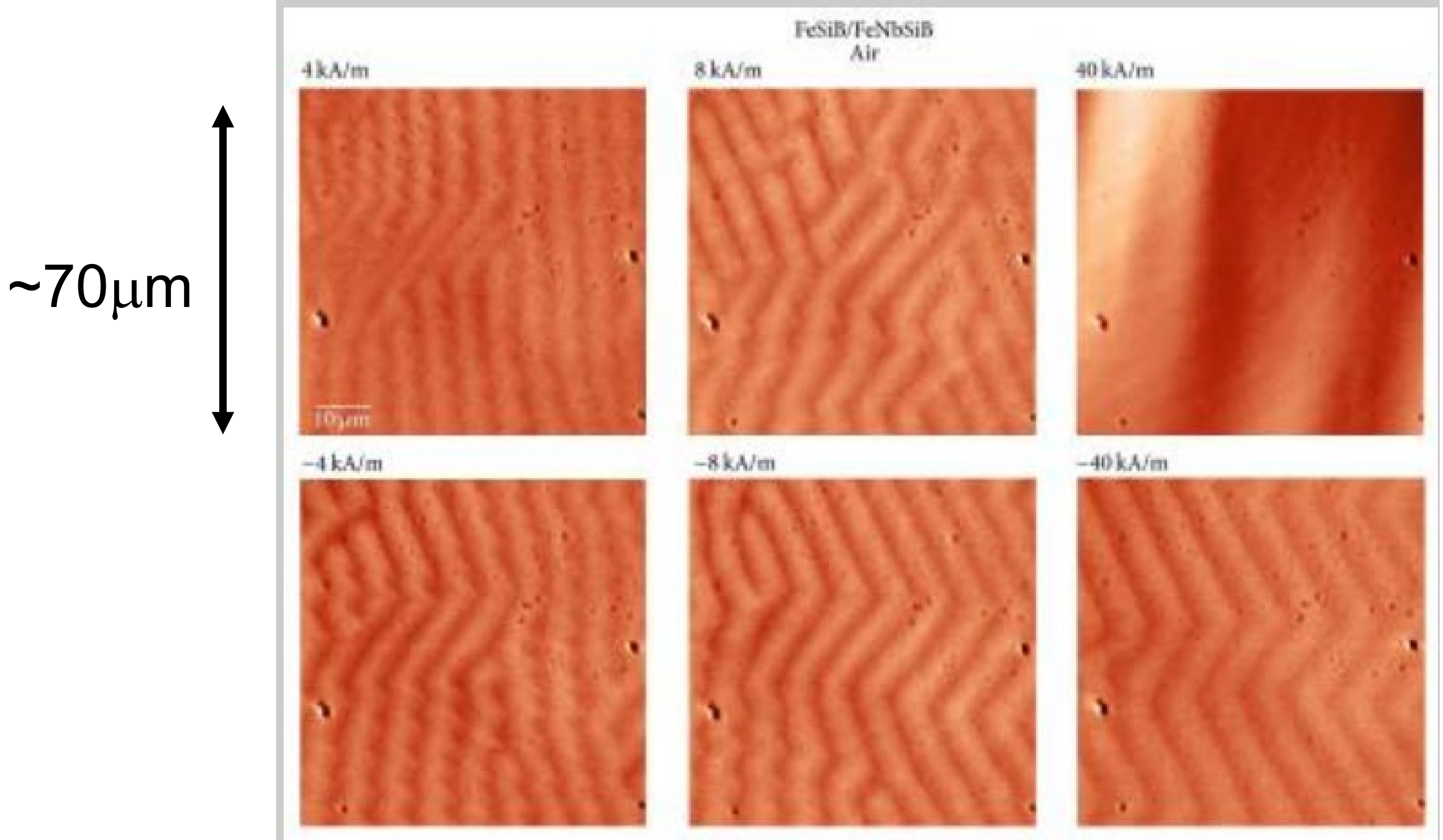
$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \mathbf{r}}{|\mathbf{r}|^3}$$

Valid outside the extent of **J**(**r**)

Classification of Magnetic Materials

All materials can be classified in terms of their magnetic behaviour falling into one of five categories depending on their bulk magnetic susceptibility. The two most common types of magnetism are diamagnetism and paramagnetism, which account for the magnetic properties of most of the periodic table of elements at room temperature (see figure 3).





Scanning. 2018; 2018: 8308460. PMCID: PMC5892230
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Magnetic Domain Patterns in Bilayered Ribbons Studied by Magnetic Force Microscopy and Magneto-Optical Kerr Microscopy

Jana Trojková,[1] Ondřej Životský,[1,2] Aleš Hendrych,[1,3] Dmitry Markov,[1] and Klára Drobíková[4,5]

Macroscopic magnetization

$$\mathbf{M}(\mathbf{r}) = \sum_i \mathbf{m}_i \delta^3(\mathbf{r} - \mathbf{r}_i)$$

Vector potential due to "free" current $\mathbf{J}_{\mathbf{free}}(\mathbf{r})$ and macroscopic magnetization $\mathbf{M}(\mathbf{r})$. Note: the designation $\mathbf{J}_{\mathbf{free}}(\mathbf{r})$ implies that this current does not also contribute to the magnetization density. $\mathbf{m}_i$ may include contributions from "bound" currents as well as from intrinsic spins.

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \left(\frac{\mathbf{J}_{free}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \frac{\mathbf{M}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \right)$$

Vector potential contributions from macroscopic magnetization -- continued

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \left(\frac{\mathbf{J}_{free}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \frac{\mathbf{M}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \right)$$

Note that :

$$\frac{\mathbf{M}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = \mathbf{M}(\mathbf{r}')\times\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}$$

$$= -\nabla'\times\left(\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right) + \frac{\nabla'\times\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$\Rightarrow \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\mathbf{J}_{free}(\mathbf{r}') + \nabla'\times\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Vector potential contributions from macroscopic magnetization -- continued

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\mathbf{J}_{free}(\mathbf{r}') + \nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Note that for the case that $\nabla \cdot \mathbf{A} = 0$:

$$\nabla \times \mathbf{B}(\mathbf{r}) = \nabla \times (\nabla \times \mathbf{A}(\mathbf{r})) = -\nabla^2 \mathbf{A}(\mathbf{r})$$

$$= \frac{\mu_0}{4\pi}\int d^3r' \left(4\pi\delta^3(\mathbf{r}-\mathbf{r}')\right)\left(\mathbf{J}_{free}(\mathbf{r}') + \nabla' \times \mathbf{M}(\mathbf{r}')\right)$$

$$= \mu_0\left(\mathbf{J}_{free}(\mathbf{r}) + \nabla \times \mathbf{M}(\mathbf{r})\right)$$

$$\Rightarrow \nabla \times \left(\mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{M}(\mathbf{r})\right) = \mu_0 \mathbf{J}_{free}(\mathbf{r})$$

Magnetic field contributions

$$\nabla \times \left(\mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{M}(\mathbf{r})\right) = \mu_0 \mathbf{J}_{free}(\mathbf{r})$$

Define "new" magnetic field vector:

$$\mu_0 \mathbf{H}(\mathbf{r}) \equiv \mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{M}(\mathbf{r})$$

$$\Rightarrow \nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}_{free}(\mathbf{r})$$

$$\nabla \times \left(\mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{M}(\mathbf{r})\right) = \mu_0 \mathbf{J}_{free}(\mathbf{r})$$

Note that $\mathbf{B}(\mathbf{r}) \equiv$ the magnetic flux density

Define $\mathbf{H}(\mathbf{r}) \equiv$ the magnetic field

$$\mu_0 \mathbf{H}(\mathbf{r}) \equiv \mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{M}(\mathbf{r})$$

$$\Rightarrow \nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}_{free}(\mathbf{r})$$

Energy associated with magnetic fields

Note: We previously used without proof --

the force on a magnetic dipole $\mathbf{m}$ in an external $\mathbf{B}$ field is:

$$\mathbf{F} = \nabla\left(\mathbf{m}\cdot\mathbf{B}\right)$$

This implies that energy associated with aligning a magnetic dipole $\mathbf{m}$ in an external $\mathbf{B}$ field is given by:

$$E_{\text{int}} = -\mathbf{m}\cdot\mathbf{B}$$

Macroscopic energies --

It can be shown that: $$W_B = \frac{1}{2}\int d^3r\ \mathbf{B}(\mathbf{r})\cdot\mathbf{H}(\mathbf{r})$$

In analogy to: $$W_E = \frac{1}{2}\int d^3r\ \mathbf{E}(\mathbf{r})\cdot\mathbf{D}(\mathbf{r})$$

$$\text{Summary of equations of magnetostatics :}$$

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}_{total}(\mathbf{r})$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}_{free}(\mathbf{r})$$

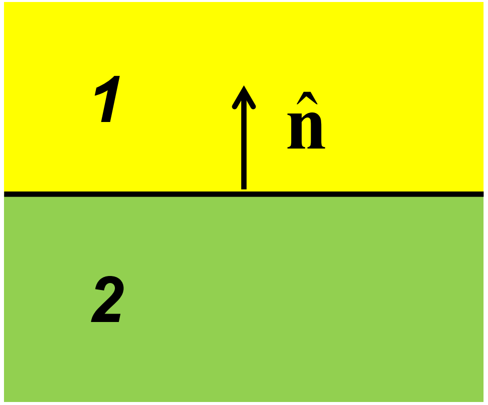
$$\mathbf{B}(\mathbf{r}) = \mu_0 (\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

For the case that $\mathbf{J}_{free}(\mathbf{r}) = 0$:

$$\nabla \times \mathbf{H}(\mathbf{r}) = 0$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$



For the case that $\mathbf{J}_{free}(\mathbf{r}) = 0$:

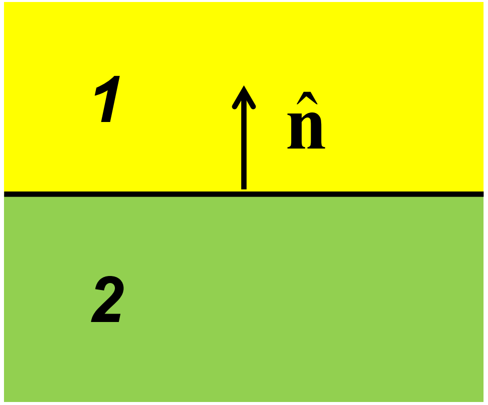
$$\nabla \times \mathbf{H}(\mathbf{r}) = 0$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

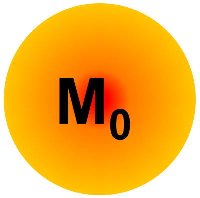
At boundary :

$$\mathbf{H}_1 \times \hat{\mathbf{n}} = \mathbf{H}_2 \times \hat{\mathbf{n}}$$

$$\mathbf{B}_1 \cdot \hat{\mathbf{n}} = \mathbf{B}_2 \cdot \hat{\mathbf{n}}$$



Example magnetostatic boundary value problem



$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0 \hat{\mathbf{z}} & r \leq a \\ 0 & r > a \end{cases}$$

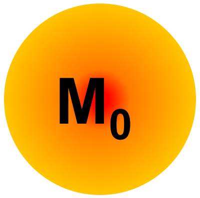
$$\nabla \times \mathbf{H}(\mathbf{r}) = 0 \qquad \Rightarrow \mathbf{H}(\mathbf{r}) = -\nabla \Phi_H(\mathbf{r})$$

$$\mathbf{B}(\mathbf{r}) = \mu_0 \left(\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}) \right)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0 = \mu_0 \nabla \cdot \left(\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}) \right)$$

$$\Rightarrow \nabla^2 \Phi_H(\mathbf{r}) = \nabla \cdot \mathbf{M}(\mathbf{r})$$

Example magnetostatic boundary value problem -- continued



$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0 \hat{\mathbf{z}} & r \leq a \\ 0 & r > a \end{cases}$$

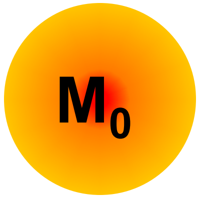
$$\nabla^2 \Phi_H(\mathbf{r}) = \nabla \cdot \mathbf{M}(\mathbf{r})$$

$$\Rightarrow \Phi_H(\mathbf{r}) = -\frac{1}{4\pi} \int d^3r' \frac{\nabla' \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

$$= -\frac{1}{4\pi} \int d^3r' \left[\nabla' \cdot \left(\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) - \mathbf{M}(\mathbf{r}') \cdot \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right]$$

$$= -\frac{1}{4\pi} \nabla \cdot \int d^3r' \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

Example magnetostatic boundary value problem -- continued



$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0\hat{\mathbf{z}} & r \le a \\ 0 & r > a \end{cases} \qquad \Phi_H(\mathbf{r}) = -\frac{1}{4\pi}\nabla\cdot\int d^3r' \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

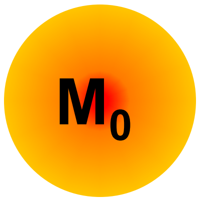
For this example:

$$\Phi_H(\mathbf{r}) = -\frac{M_0}{4\pi}\frac{\partial}{\partial z}\left(4\pi\int_0^a r'^2\, dr' \frac{1}{r_>}\right)$$

For $r \le a$: $$\Phi_H(\mathbf{r}) = -M_0\frac{\partial}{\partial z}\left(\frac{a^2}{2}-\frac{r^2}{6}\right) = \frac{M_0 z}{3}$$

For $r > a$: $$\Phi_H(\mathbf{r}) = -M_0\frac{\partial}{\partial z}\left(\frac{a^3}{3r}\right) = \frac{M_0 a^3 z}{3r^3}$$

Example magnetostatic boundary value problem -- continued



$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0\hat{\mathbf{z}} & r \le a \\ 0 & r > a \end{cases}$$

For $r \le a$: $\quad \Phi_H(\mathbf{r}) = \dfrac{M_0 z}{3} \qquad \mathbf{H}(\mathbf{r}) = -\nabla\Phi_H(\mathbf{r}) = -\dfrac{M_0}{3}\hat{\mathbf{z}}$

For $r > a$: $\quad \Phi_H(\mathbf{r}) = \dfrac{M_0 a^3 z}{3r^3} \qquad \mathbf{H}(\mathbf{r}) = -\nabla\Phi_H(\mathbf{r}) = -\dfrac{M_0 a^3}{3}\left(\dfrac{\hat{\mathbf{z}}}{r^3} - \dfrac{3z\mathbf{r}}{r^5}\right)$

$$\mathbf{B}(\mathbf{r}) = \mu_0\left(\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r})\right)$$

For $r \le a$: $\quad \mathbf{H}(\mathbf{r}) = -\dfrac{M_0\hat{\mathbf{z}}}{3} \qquad \mathbf{B}(\mathbf{r}) = \mu_0\dfrac{2M_0\hat{\mathbf{z}}}{3}$

For $r > a$: $\quad \mathbf{H}(\mathbf{r}) = -\dfrac{M_0 a^3}{3}\left(\dfrac{\hat{\mathbf{z}}}{r^3} - \dfrac{3z\mathbf{r}}{r^5}\right)$

$$\mathbf{B}(\mathbf{r}) = -\mu_0\frac{M_0 a^3}{3}\left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5}\right)$$

Check boundary values:

For $r \le a$: $\mathbf{H}(\mathbf{r}) = -\frac{M_0 \hat{\mathbf{z}}}{3}$ $\quad \mathbf{H}(a\hat{\mathbf{r}}) \times \hat{\mathbf{r}} = -\frac{M_0}{3} \hat{\mathbf{z}} \times \hat{\mathbf{r}}$

For $r > a$: $\mathbf{H}(\mathbf{r}) = -\frac{M_0 a^3}{3} \left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5} \right)$

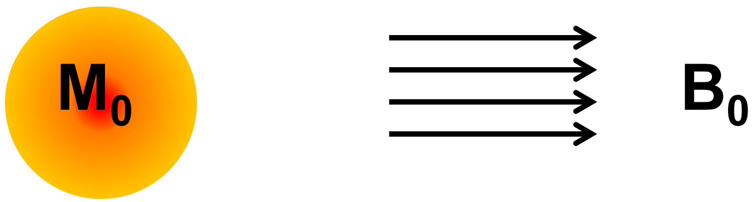
$$\mathbf{H}(a\hat{\mathbf{r}}) \times \hat{\mathbf{r}} = -\frac{M_0 a^3}{3} \frac{\hat{\mathbf{z}} \times \hat{\mathbf{r}}}{a^3}$$

For $r \le a$: $\mathbf{B}(\mathbf{r}) = \mu_0 \frac{2M_0 \hat{\mathbf{z}}}{3}$ $\quad \mathbf{B}(a\hat{\mathbf{r}}) \cdot \hat{\mathbf{r}} = \mu_0 \frac{2M_0}{3} \hat{\mathbf{z}} \cdot \hat{\mathbf{r}}$

For $r > a$: $\mathbf{B}(\mathbf{r}) = -\mu_0 \frac{M_0 a^3}{3} \left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5} \right)$

$$\mathbf{B}(a\hat{\mathbf{r}}) \cdot \hat{\mathbf{r}} = -\mu_0 \frac{M_0 a^3}{3} \hat{\mathbf{z}} \cdot \hat{\mathbf{r}} \left(\frac{1}{a^3} - \frac{3a^2}{a^5} \right)$$

Variation; magnetic sphere plus external field $\mathbf{B}_0$



$$\mathbf{M}(\mathbf{r}) = \begin{cases} \mathbf{M}_0 & r \leq a \\ 0 & r > a \end{cases}$$

By superposition:

For $r \leq a$:

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}_0 + \mu_0 \frac{2}{3}\mathbf{M}_0$$

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\mu_0}\mathbf{B}_0 - \frac{1}{3}\mathbf{M}_0$$

$$\mathbf{B}(\mathbf{r}) + 2\mu_0 \mathbf{H}(\mathbf{r}) = 3\mathbf{B}_0$$

For an isotropic "paramagnetic" material, $\mathbf{B}(\mathbf{r}) = \mu \mathbf{H}(\mathbf{r})$

$$\mathbf{M}_0 = \frac{3}{\mu_0}\left(\frac{\mu - \mu_0}{\mu + 2\mu_0}\right)\mathbf{B}_0$$

Summary of equations of magnetostatics :

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}_{total}(\mathbf{r})$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}_{free}(\mathbf{r})$$

$$\mathbf{B}(\mathbf{r}) = \mu_0 \left(\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}) \right)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

For the case that $\mathbf{J}_{free}(\mathbf{r})$:

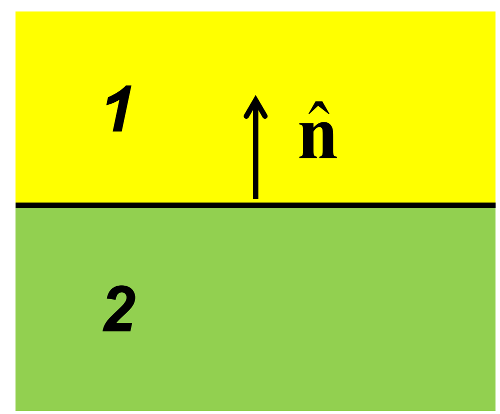
$$\nabla \times \mathbf{H}(\mathbf{r}) = 0$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

At boundary :

$$\mathbf{H}_1 \times \hat{\mathbf{n}} = \mathbf{H}_2 \times \hat{\mathbf{n}}$$

$$\mathbf{B}_1 \cdot \hat{\mathbf{n}} = \mathbf{B}_2 \cdot \hat{\mathbf{n}}$$



Magnetism in materials

$$\mathbf{B}(\mathbf{r}) = \mu_0(\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$

For materials with linear magnetism:

$$\mathbf{B} = \mu\mathbf{H}$$

$$\mu > \mu_0 \quad \Rightarrow \text{paramagnetic material}$$

$$\mu < \mu_0 \quad \Rightarrow \text{diamagnetic material}$$

For ferromagnetic, antiferromagnetic materials

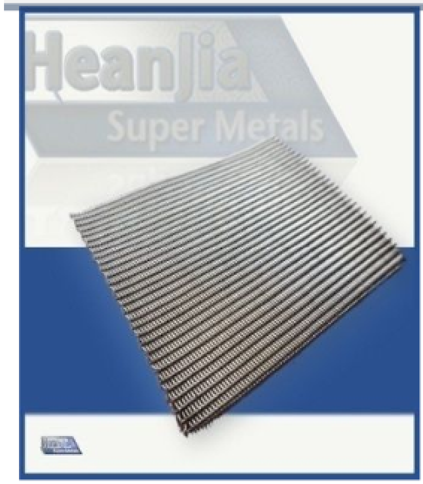
$$\mathbf{B} = f(\mathbf{H}) \quad \text{(with hysteresis)}$$

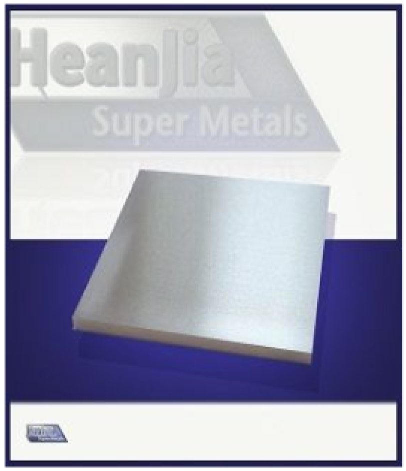
https://en.wikipedia.org/wiki/Permeability_(electromagnetism)

Magnetic susceptibility and permeability data for selected materials

Medium	Susceptibility, volumetric, SI, χ_m	Permeability, μ (H/m)	Relative permeability, max., μ/μ_0	Magnetic field
Metglas 2714A (annealed)		1.26×10^{0}	1 000 000[10]	At 0.5 T
Iron (99.95% pure Fe annealed in H)		2.5×10^{-1}	200 000[11]	
NANOPERM®		1.0×10^{-1}	80 000[12]	At 0.5 T
Mu-metal		2.5×10^{-2}	20 000[13]	At 0.002 T
Mu-metal		6.3×10^{-2}	50 000[14]	
Cobalt-iron (high permeability strip material)		2.3×10^{-2}	18 000[15]	
Permalloy	8000	1.0×10^{-2}	8000[13]	At 0.002 T

Mumetal Magnetic Shielding

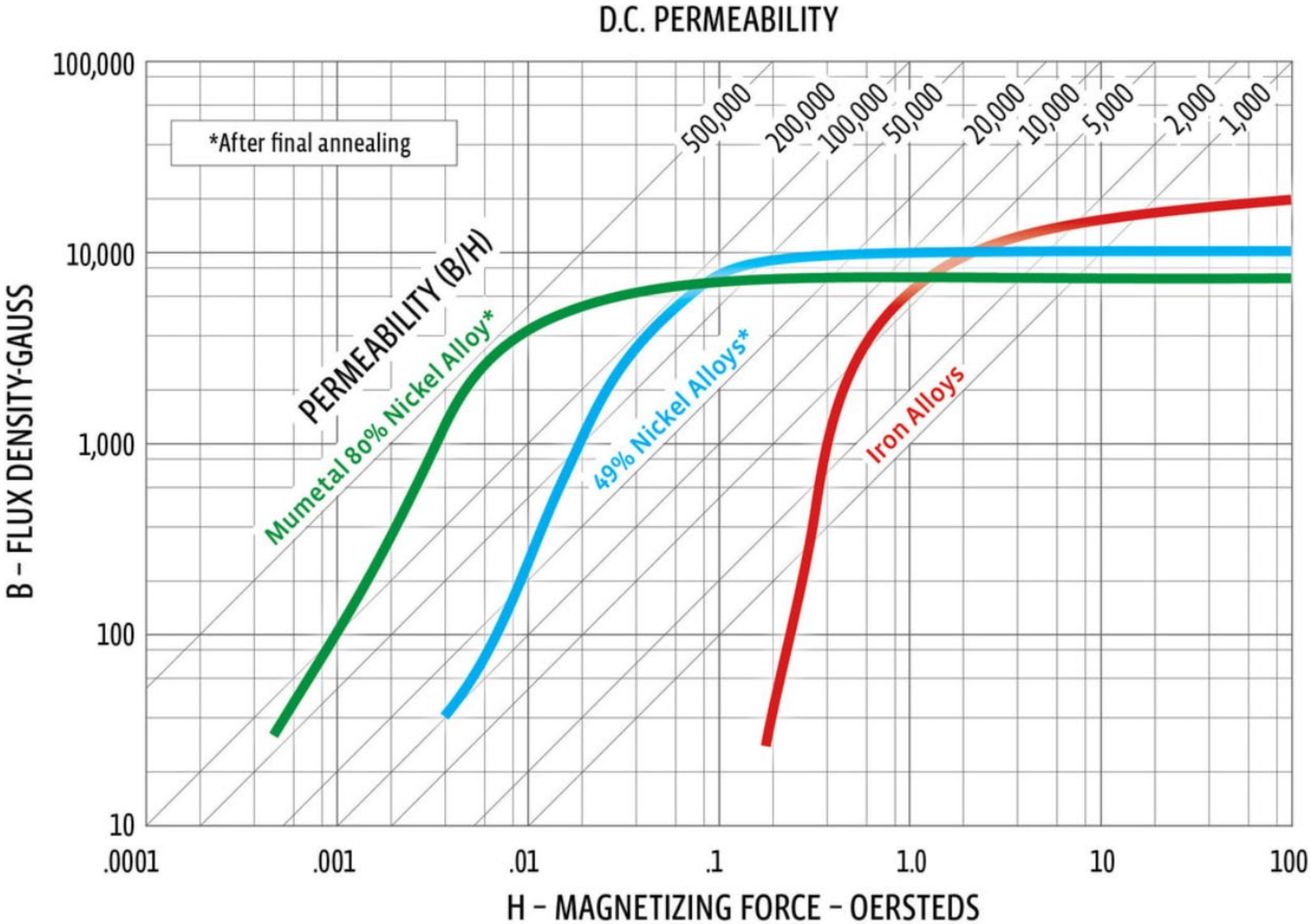





Mu metal is a soft ferromagnetic alloy that has extremely high initial and maximum magnetic permeability. It is used in electric transformer, storage disks, magnetic phonographs, resonance devices and superconducting circuits.

Mumetal alloy generally attributes relative permeability about 80,000 to 100,000 than the normal steel alloy. It is also called as soft magnetic alloy and offers low magnetic anisotropy and magnetostriction providing low corceivity to saturate the low magnetic fields. It provides nominal hysteresis loses when the alloy is employed in the AC magnetic circuits.

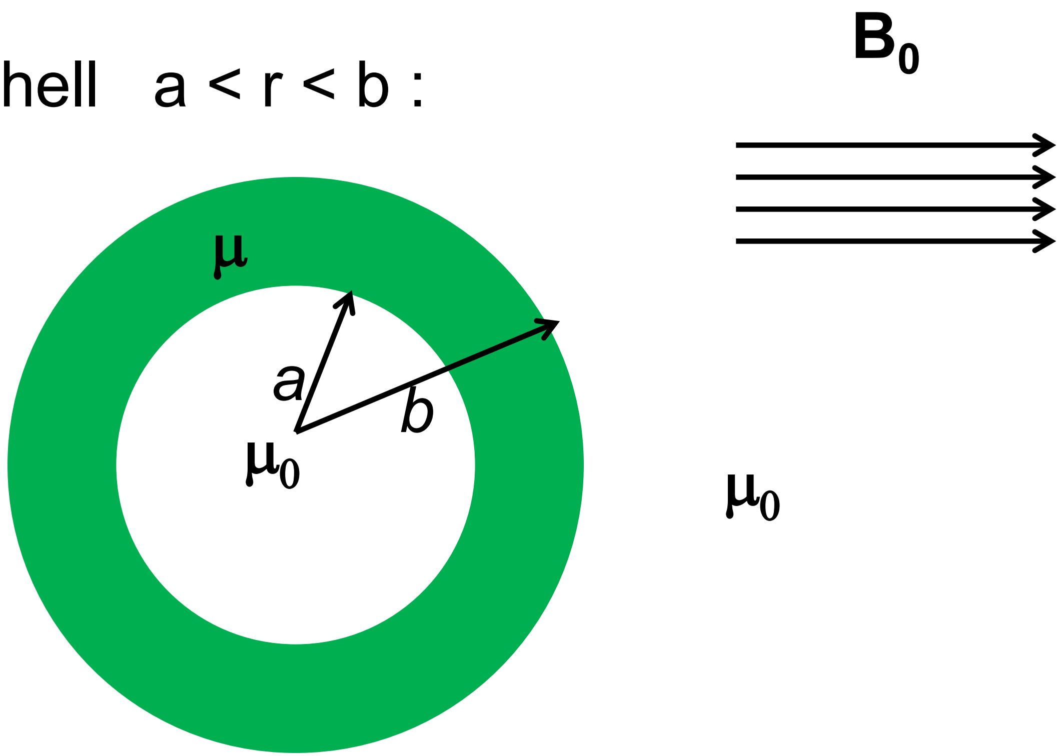
Composed of 80% Ni, 15% Fe, 5% Mo+other materials



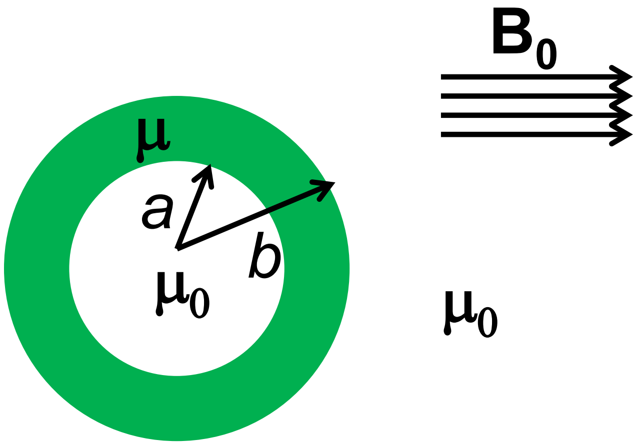
From: http://www.mu-metal.com/shielding-fundamentals.html

Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$

Spherical shell $a < r < b$:



Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$ -- continued



For this case:

$$\nabla \times \mathbf{H}(\mathbf{r}) = 0$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

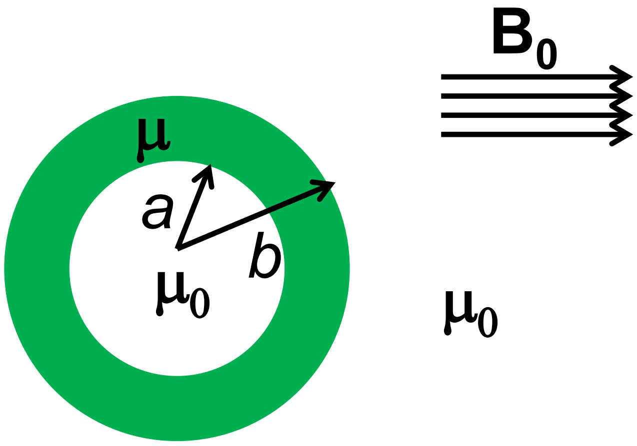
$$\mathbf{B}(\mathbf{r}) = \mu \mathbf{H}(\mathbf{r})$$

Continuity at boundaries:

$$\mathbf{H} \times \hat{\mathbf{n}} = \text{continuous}$$

$$\mathbf{B} \cdot \hat{\mathbf{n}} = \text{continuous}$$

Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$ -- continued



$$\text{Let}: \quad \mathbf{H}(\mathbf{r}) = -\nabla\Phi_H(\mathbf{r})$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0 \qquad \Rightarrow \nabla^2\Phi_H(\mathbf{r}) = 0$$

$$\text{For} \quad 0 \le r \le a \qquad \Phi_H(\mathbf{r}) = \sum_l \delta_l r^l P_l(\cos\theta)$$

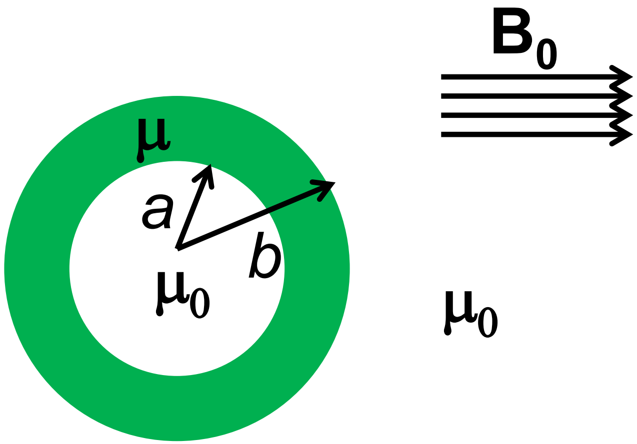
$$\text{For} \quad a \le r \le b \qquad \Phi_H(\mathbf{r}) = \sum_l \left(\beta_l r^l + \frac{\gamma_l}{r^{l+1}}\right) P_l(\cos\theta)$$

$$\text{For} \quad r \ge b \qquad \Phi_H(\mathbf{r}) = -\frac{B_0}{\mu_0} r\cos\theta + \sum_l \frac{\alpha_l}{r^{l+1}} P_l(\cos\theta)$$

Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$ -- continued

Applying boundary conditions

(only $l = 1$ terms contribute) :



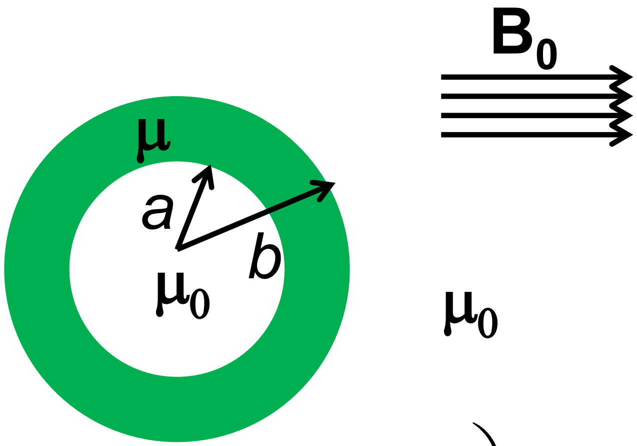
At $r = a$ $$\delta_1 = \frac{\mu}{\mu_0}\left(\beta_1 - 2\frac{\gamma_1}{a^3}\right)$$

$$a\delta_1 = a\beta_1 + \frac{\gamma_1}{a^2}$$

At $r = b$ $$\frac{\mu}{\mu_0}\left(\beta_1 - 2\frac{\gamma_1}{b^3}\right) = -\frac{B_0}{\mu_0} - 2\frac{\alpha_1}{b^3}$$

$$b\beta_1 + \frac{\gamma_1}{b^2} = -b\frac{B_0}{\mu_0} + \frac{\alpha_1}{b^2}$$

Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$ -- continued



When the dust clears:

$$\delta_1 = \left(\frac{-9\mu/\mu_0}{(2\mu/\mu_0+1)(\mu/\mu_0+2)-2(a/b)^3(\mu/\mu_0-1)^2} \right) \frac{B_0}{\mu_0}$$

$$\approx \frac{1}{\mu/\mu_0} \left(\frac{-9/2}{\left(1-(a/b)^3\right)} \frac{B_0}{\mu_0} \right)$$