PHY 712 Electrodynamics
10-10:50 AM MWF in Olin 103

Notes for Lecture 16:

Finish reading Chapter 6

1. Some details of Liénard-Wiechert results
2. Energy density and flux associated with electromagnetic fields
3. Time harmonic fields

Physics Colloquium

Thursday

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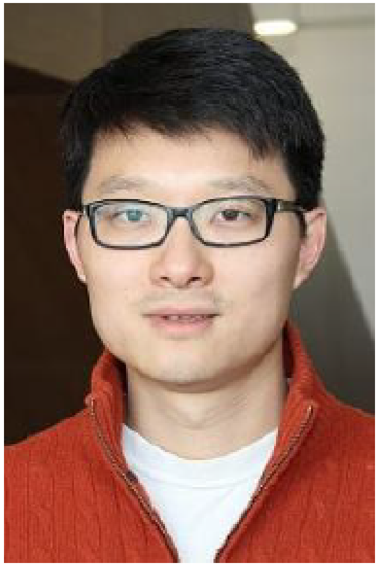
February 16, 2023

Tunneling Probe of 2D Kitaev and Moiré Magnetism

I will discuss our recent results using tunneling magnetoresistance and spectroscopy to probe the underlying magnetic order and/or excitations in two "complex" atomically thin spin systems: the Kitaev material alpha-RuCl3 in monolayer form and twisted bilayers of the layer antiferromagnet CrI3. We find a reversal of the magnetic anisotropy together with an enhancement of the Kitaev interaction in the former and multiple skyrmion states in the latter that can be tuned by magnetic field.



Adam Wei Tsen, Ph.D.

Assistant Professor,

University of Waterloo

4:00 pm - Olin 101*

Note: For additional information on the seminar, contact wfuphys@wfu.edu

Reception at 3:30pm - Olin Entrance

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Mon: 01/9/2023	Chap. 1 & Appen.	Introduction, units and Poisson equation	#1	01/13/2023
2	Wed: 01/11/2023	Chap. 1	Electrostatic energy calculations	#2	01/18/2023
3	Fri: 01/13/2023	Chap. 1	Electrostatic energy calculations thanks to Ewald	#3	01/18/2023
	Mon: 01/16/2023		MLK Holiday -- no class		
4	Wed: 01/18/2023	Chap. 1 & 2	Electrostatic potentials and fields	#4	01/20/2023
5	Fri: 01/20/2023	Chap. 1 - 3	Poisson's equation in 2 and 3 dimensions	#5	01/23/2023
6	Mon: 01/23/2023	Chap. 1 - 3	Brief introduction to numerical methods	#6	01/25/2023
7	Wed: 01/25/2023	Chap. 2 & 3	Image charge constructions	#7	01/30/2023
8	Fri: 01/27/2023	Chap. 2 & 3	Cylindrical and spherical geometries		
9	Mon: 01/30/2023	Chap. 3 & 4	Spherical geometry and multipole moments	#8	02/01/2023
10	Wed: 02/01/2023	Chap. 4	Dipoles and Dielectrics	#9	02/03/2023
11	Fri: 02/03/2023	Chap. 4	Dipoles and Dielectrics	#10	02/06/2023
12	Mon: 02/06/2023	Chap. 5	Magnetostatics	#11	02/08/2023
13	Wed: 02/08/2023	Chap. 5	Magnetic dipoles and hyperfine interaction	#12	02/10/2023
14	Fri: 02/10/2023	Chap. 5	Magnetic dipoles and dipolar fields	#13	02/13/2023
15	Mon: 02/13/2023	Chap. 6	Maxwell's Equations	#14	02/17/2023
16	Wed: 02/15/2023	Chap. 6	Electromagnetic energy and forces		
17	Fri: 02/17/2023	Chap. 7	Electromagnetic plane waves		
18	Mon: 02/20/2023	Chap. 7	Electromagnetic plane waves		
19	Wed: 02/22/2023	Chap. 7	Optical effects of refractive indices		
20	Fri: 02/24/2023	Chap. 1-7	Review		

Solution of Maxwell's equations in the Lorentz gauge – Review from previous lecture --

Liénard-Wiechert potentials and fields --
Determination of the scalar and vector potentials for a moving point particle (also see Landau and Lifshitz ***The Classical Theory of Fields***, Chapter 8.)

Consider the fields produced by the following source: a point charge q moving on a trajectory $R_q(t)$.

$$\text{Charge density: } \rho(\mathbf{r},t) = q\delta^3(\mathbf{r}-\mathbf{R}_q(t))$$

$$\text{Current density: } \mathbf{J}(\mathbf{r},t) = q\dot{\mathbf{R}}_q(t)\delta^3(\mathbf{r}-\mathbf{R}_q(t)), \quad \text{where} \quad \dot{\mathbf{R}}_q(t) \equiv \frac{d\mathbf{R}_q(t)}{dt}.$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

$$\Phi(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\int\int d^3r'dt'\frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}']|}\delta\left(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\right)$$

$$\mathbf{A}(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0 c^2}\int\int d^3r'dt'\frac{\mathbf{J}(\mathbf{r}',\mathrm{t}')}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\right).$$

We performing the integrations over first *d^3r'* and then *dt'* making use of the fact that for any function of *t'*,

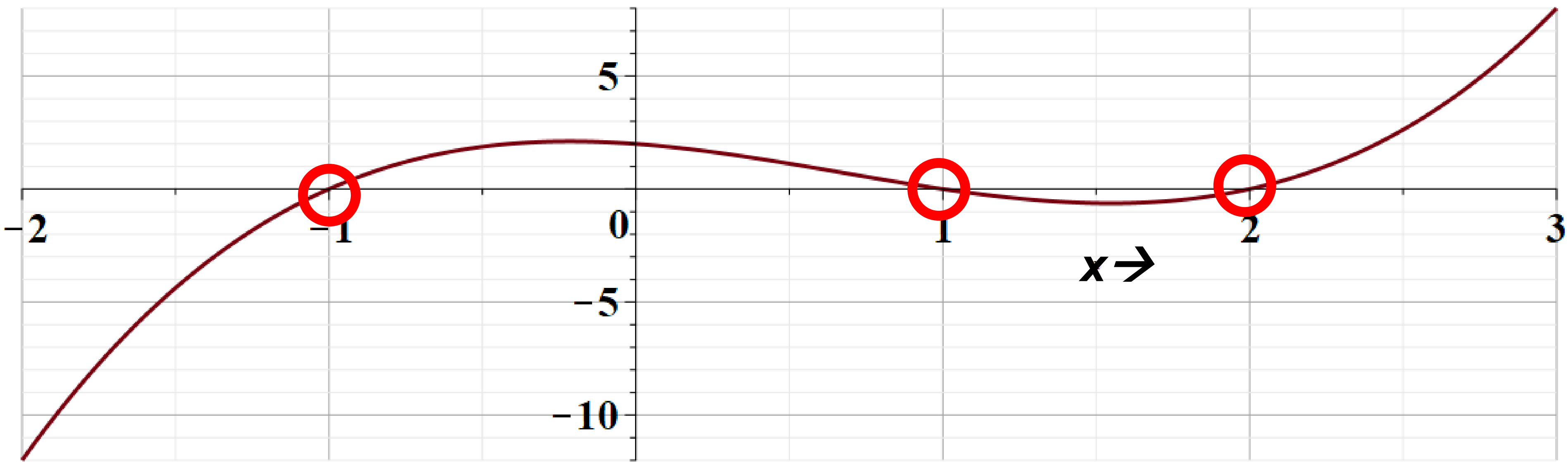
$$\int_{-\infty}^{\infty} dt'\, f(t')\delta\left(t'-(t-|\mathbf{r}-\mathbf{R}_q(t')|/c)\right)=\frac{f(t_r)}{1-\dfrac{\dot{\mathbf{R}}_q(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c\,|\mathbf{r}-\mathbf{R}_q(t_r)|}},$$

where the ``retarded time'' is defined to be

$$t_r \equiv t-\frac{|\mathbf{r}-\ \mathbf{R}_q(t_r)|}{c}.$$

Comment about delta functions -- See Pg. 26 in **Jackson**

$$\int_{-\infty}^{\infty} dx\ \Psi(x)\delta(f(x)) = \sum_i \frac{\Psi(x_i)}{\left|\frac{df(x)}{dx}\right|_{x=x_i}}$$



Solution of Maxwell's equations in the Lorentz gauge -- continued

Resulting scalar and vector potentials:

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0} \frac{1}{R - \dfrac{\mathbf{v}\cdot\mathbf{R}}{c}},$$

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2} \frac{\mathbf{v}}{R - \dfrac{\mathbf{v}\cdot\mathbf{R}}{c}},$$

Notation: $\mathbf{R} \equiv \mathbf{r} - \mathbf{R}_q(t_r)$ $\quad t_r \equiv t - \dfrac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}.$

$$\mathbf{v} \equiv \dot{\mathbf{R}}_q(t_r),$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

In order to find the electric and magnetic fields, we need to evaluate

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}$$

$$\mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t)$$

The trick of evaluating these derivatives is that the retarded time t_r depends on position **r** and on itself. We can show the following results using the shorthand notation:

$$\nabla t_r = -\frac{\mathbf{R}}{c\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)} \quad \text{and} \quad \frac{\partial t_r}{\partial t} = \frac{R}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)}.$$

Some details --

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}$$

$$\frac{\partial t_r}{\partial t} = 1 + \frac{\left(\mathbf{r} - \mathbf{R}_q(t_r)\right) \cdot \frac{d\mathbf{R}_q(t_r)}{dt_r}}{c\,|\mathbf{r} - \mathbf{R}_q(t_r)|} \frac{\partial t_r}{\partial t}$$

Using notation: $\mathbf{R} \equiv \mathbf{r} - \mathbf{R}_q(t_r) \quad \mathbf{v} \equiv \dot{\mathbf{R}}_q(t_r),$

$$\rightarrow \frac{\partial t_r}{\partial t} = \frac{R}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)}.$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

$$-\nabla\Phi(\mathbf{r},t)=\frac{q}{4\pi\epsilon_0}\frac{1}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\mathbf{R}\left(1-\frac{v^2}{c^2}\right)-\frac{v}{c}\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)+\mathbf{R}\frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right],$$

$$-\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}=\frac{q}{4\pi\epsilon_0}\frac{1}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\frac{\mathbf{v}R}{c}\left(\frac{v^2}{c^2}-\frac{\mathbf{v}\cdot\mathbf{R}}{Rc}-\frac{\dot{\mathbf{v}}\cdot R}{c^2}\right)-\frac{\dot{\mathbf{v}}R}{c^2}\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)\right].$$

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{4\pi\epsilon_0}\frac{1}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R}-\frac{\mathbf{v}R}{c}\right)\left(1-\frac{v^2}{c^2}\right)+\left(\mathbf{R}\times\left\{\left(\mathbf{R}-\frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right].$$

$$\mathbf{B}(\mathbf{r},t)=\frac{q}{4\pi\epsilon_0c^2}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1-\frac{v^2}{c^2}+\frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right)-\frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right]=\frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{cR}$$

Back to general case --

Maxwell's equations

$$\text{Coulomb's law:} \qquad \nabla \cdot \mathbf{D} = \rho_{free}$$

$$\text{Ampere-Maxwell's law:} \qquad \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$$

$$\text{Faraday's law:} \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\text{No magnetic monopoles:} \qquad \nabla \cdot \mathbf{B} = 0$$

Energy analysis of electromagnetic fields and sources

Rate of work done on source $\mathbf{J}(\mathbf{r},t)$ by electromagnetic field:

$$\frac{dW_{mech}}{dt} \equiv \frac{dE_{mech}}{dt} = \int d^3r \;\; \mathbf{E} \cdot \mathbf{J}_{free}$$

Expressing source current in terms of fields it produces:

$$\frac{dW_{mech}}{dt} = \int d^3r \;\; \mathbf{E} \cdot \left(\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} \right)$$

Energy analysis of electromagnetic fields and sources - continued

$$\frac{dW_{mech}}{dt} = \int d^3r \;\; \mathbf{E}\cdot\mathbf{J}_{free} = \int d^3r \;\; \mathbf{E}\cdot\left(\nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t}\right)$$

$$= -\int d^3r \;\; \left(\nabla\cdot\left(\mathbf{E}\times\mathbf{H}\right) + \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\right)$$

Let $\mathbf{S} \equiv \mathbf{E}\times\mathbf{H}$ "Poynting vector"

$u \equiv \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}\right)$ energy density

$$\Rightarrow \frac{\partial u}{\partial t} + \nabla\cdot\mathbf{S} = -\mathbf{E}\cdot\mathbf{J}_{free}$$

Assuming that $\mathbf{D}=\epsilon\mathbf{E}$ and that $\mathbf{B}=\mu\mathbf{H}$

Energy analysis of electromagnetic fields and sources - continued

$$\frac{dE_{mech}}{dt} \equiv \int d^3r \;\; \mathbf{E} \cdot \mathbf{J}_{free}$$

Electromagnetic energy density: $u \equiv \frac{1}{2}\left(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}\right)$

$$E_{field} \equiv \int d^3r \;\; u(\mathbf{r},t)$$

Poynting vector: $\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}$

From the previous energy analysis: $\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{J}_{free}$

$$\Rightarrow \frac{dE_{mech}}{dt} + \frac{dE_{field}}{dt} = -\int d^3r \;\; \nabla \cdot \mathbf{S}(\mathbf{r},t) = -\oint d^2r \; \hat{\mathbf{r}} \cdot \mathbf{S}(\mathbf{r},t)$$

Momentum analysis of electromagnetic fields and sources

$$\frac{d\mathbf{P}_{mech}}{dt} \equiv \int d^3r \quad \left(\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}\right)$$

Follows by analogy with Lorentz force:

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

$$\mathbf{P}_{field} = \varepsilon_0 \int d^3r \quad \left(\mathbf{E} \times \mathbf{B}\right)$$

Expression for vacuum fields:

$$\left(\frac{d\mathbf{P}_{mech}}{dt} + \frac{d\mathbf{P}_{field}}{dt}\right)_i = \sum_j \int d^3r \, \frac{\partial T_{ij}}{\partial r_j}$$

Maxwell stress tensor:

$$T_{ij} \equiv \varepsilon_0 \left(E_i E_j + c^2 B_i B_j - \delta_{ij} \frac{1}{2}\left(\mathbf{E} \cdot \mathbf{E} + c^2 \mathbf{B} \cdot \mathbf{B}\right)\right)$$

Summary -- By considering a complete system involving self-contained sources and fields, we examined the energy and force relationships and introduce energy and force equivalents of the electromagnetic fields

Electromagnetic energy density: $u \equiv \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B}\right)$

Poynting vector: $\mathbf{S} \equiv \mathbf{E}\times\mathbf{H}$

Differential relationship: $\frac{\partial u}{\partial t}+\nabla\cdot\mathbf{S}=-\mathbf{E}\cdot\mathbf{J}_{free}$

Maxwell stress tensor (for vacuum case):

$$T_{ij} \equiv \varepsilon_0\left(E_iE_j+c^2B_iB_j-\delta_{ij}\frac{1}{2}\left(\mathbf{E}\cdot\mathbf{E}+c^2\mathbf{B}\cdot\mathbf{B}\right)\right)$$

Integral relationships:

$$\frac{dE_{mech}}{dt} \equiv \int d^3r \ \ \mathbf{E} \cdot \mathbf{J}_{free}$$

$$E_{field} \equiv \int d^3r \ \ u(\mathbf{r},t)$$

$$\Rightarrow \frac{dE_{mech}}{dt} + \frac{dE_{field}}{dt} = -\int d^3r \ \ \nabla \cdot \mathbf{S}(\mathbf{r},t) = -\oint d^2r \ \hat{\mathbf{r}} \cdot \mathbf{S}(\mathbf{r},t)$$

$$\left(\frac{d\mathbf{P}_{mech}}{dt} + \frac{d\mathbf{P}_{field}}{dt} \right)_i = \sum_j \int d^3r \, \frac{\partial T_{ij}}{\partial r_j}$$

Comment on treatment of time-harmonic fields

Fourier transformation in time domain :

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, \widetilde{\mathbf{E}}(\mathbf{r},\omega) \, e^{-i\omega t}$$

$$\widetilde{\mathbf{E}}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} dt \, \mathbf{E}(\mathbf{r},t) \, e^{i\omega t}$$

Note that $\mathbf{E}(\mathbf{r},t)$ is real $\Rightarrow \widetilde{\mathbf{E}}(\mathbf{r},\omega) = \widetilde{\mathbf{E}}^*(\mathbf{r},-\omega)$

These relations and the notion of the superposition principle, lead to the common treatment:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} \right) \equiv \frac{1}{2}\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t} \right)$$

Comment on treatment of time-harmonic fields -- continued

Equations for time harmonic fields:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}\right) \equiv \frac{1}{2}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \widetilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$

Equations	in time domain	in frequency domain
Coulomb's law:	$\nabla \cdot \mathbf{D} = \rho_{free}$	$\nabla \cdot \widetilde{\mathbf{D}} = \widetilde{\rho}_{free}$
Ampere-Maxwell's law:	$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$	$\nabla \times \widetilde{\mathbf{H}} + i\omega\widetilde{\mathbf{D}} = \widetilde{\mathbf{J}}_{free}$
Faraday's law:	$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$	$\nabla \times \widetilde{\mathbf{E}} - i\omega\widetilde{\mathbf{B}} = 0$
No magnetic monopoles:	$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \widetilde{\mathbf{B}} = 0$

Note -- in all of these, the real part is taken at the end of the calculation.

Comment on treatment of time-harmonic fields -- continued

Equations for time harmonic fields :

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}\right) \equiv \frac{1}{2}\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$

Poynting vector: $\mathbf{S}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t)$

$$\mathbf{S}(\mathbf{r},t) = \frac{1}{4}\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right)\times\left(\tilde{\mathbf{H}}(\mathbf{r},\omega)e^{-i\omega t} + \tilde{\mathbf{H}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$

$$= \frac{1}{4}\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\times\tilde{\mathbf{H}}^*(\mathbf{r},\omega) + \tilde{\mathbf{E}}^*(\mathbf{r},\omega)\times\tilde{\mathbf{H}}(\mathbf{r},\omega)\right)$$

$$+ \frac{1}{4}\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\times\tilde{\mathbf{H}}(\mathbf{r},\omega)e^{-2i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r},\omega)\times\tilde{\mathbf{H}}^*(\mathbf{r},\omega)e^{2i\omega t}\right)$$

$$\left\langle \mathbf{S}(\mathbf{r},t)\right\rangle_{t\,\mathrm{avg}} = \Re\left(\frac{1}{2}\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\times\tilde{\mathbf{H}}^*(\mathbf{r},\omega)\right)\right)$$

Maxwell's equations

Coulomb's law : $\nabla \cdot \mathbf{D} = \rho_{free}$

Ampere-Maxwell's law : $\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$

Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles : $\nabla \cdot \mathbf{B} = 0$

Maxwell's equations

For linear isotropic media -- $\mathbf{D} = \varepsilon \mathbf{E}; \quad \mathbf{B} = \mu \mathbf{H}$

and no sources :

Coulomb's law : $\nabla \cdot \mathbf{E} = 0$

Ampere - Maxwell's law : $\nabla \times \mathbf{B} - \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} = 0$

Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles : $\nabla \cdot \mathbf{B} = 0$

Analysis of Maxwell's equations without sources -- continued:

$$\text{Coulomb's law:} \qquad \nabla \cdot \mathbf{E} = 0$$

$$\text{Ampere-Maxwell's law:} \qquad \nabla \times \mathbf{B} - \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} = 0$$

$$\text{Faraday's law:} \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\text{No magnetic monopoles:} \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \left(\nabla \times \mathbf{B} - \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} \right) = -\nabla^2 \mathbf{B} - \mu\varepsilon \frac{\partial (\nabla \times \mathbf{E})}{\partial t}$$

$$= -\nabla^2 \mathbf{B} + \mu\varepsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

$$\nabla \times \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) = -\nabla^2 \mathbf{E} + \frac{\partial (\nabla \times \mathbf{B})}{\partial t}$$

$$= -\nabla^2 \mathbf{E} + \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

Analysis of Maxwell's equations without sources -- continued:

Both E and B fields are solutions to a wave equation:

$$\nabla^2 \mathbf{B} - \frac{1}{v^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

$$\nabla^2 \mathbf{E} - \frac{1}{v^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\text{where} \quad v^2 \equiv c^2 \frac{\mu_0 \varepsilon_0}{\mu \varepsilon} \equiv \frac{c^2}{n^2}$$

Plane wave solutions to wave equation:

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right)$$

Analysis of Maxwell's equations without sources -- continued:

Plane wave solutions to wave equation :

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \qquad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}$$

Note: ε, μ, n, k can all be complex; for the moment we will assume that they are all real (no dissipation).

Note that $\mathbf{E}_0$ and $\mathbf{B}_0$ are not independent;

from Faraday's law : $\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0$

$$\Rightarrow \mathbf{B}_0 = \frac{\mathbf{k}\times\mathbf{E}_0}{\omega} = \frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}$$

also note : $\hat{\mathbf{k}}\cdot\mathbf{E}_0 = 0$ and $\hat{\mathbf{k}}\cdot\mathbf{B}_0 = 0$

Analysis of Maxwell's equations without sources -- continued:

Summary of plane electromagnetic waves:

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\frac{n\hat{\mathbf{k}} \times \mathbf{E}_0}{c} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}} \cdot \mathbf{E}_0 = 0$$

Poynting vector for plane electromagnetic waves:

$$\langle \mathbf{S} \rangle_{avg} = \frac{1}{2} \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \times \frac{1}{\mu} \left(\frac{n\hat{\mathbf{k}} \times \mathbf{E}_0}{c} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \right)^* \right)$$

$$= \frac{n|\mathbf{E}_0|^2}{2\mu c} \hat{\mathbf{k}} = \frac{1}{2} \sqrt{\frac{\varepsilon}{\mu}} |\mathbf{E}_0|^2 \hat{\mathbf{k}}$$

Note that:

$$\mathbf{E}_0 \times \left(\hat{\mathbf{k}} \times \mathbf{E}_0 \right) = \hat{\mathbf{k}} \left(\mathbf{E}_0 \cdot \mathbf{E}_0 \right) - \mathbf{E}_0 \left(\hat{\mathbf{k}} \cdot \mathbf{E}_0 \right.$$

$$= \hat{\mathbf{k}} |\mathbf{E}_0|^2$$

Analysis of Maxwell's equations without sources -- continued:

Transverse Electric and Magnetic (TEM) waves

Summary of plane electromagnetic waves:

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}}\cdot\mathbf{E}_0 = 0$$

Energy density for plane electromagnetic waves:

$$\langle u\rangle_{avg} = \frac{1}{4}\Re\left(\varepsilon\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\cdot\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)^*\right) +$$

$$\frac{1}{4}\Re\left(\frac{1}{\mu}\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\cdot\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)^*\right)$$

$$= \frac{1}{2}\varepsilon|\mathbf{E}_0|^2$$