PHY 712 Electrodynamics
10-10:50 AM in Olin 103

Discussion for Lecture 20:

Review of Chapters 1-7

1. Comment on what to expect with the take- home exam
2. Main topics covered
3. Some details of past HW problems

17	Fri: 02/17/2023	Chap. 7	Electromagnetic plane waves	#15	02/20/2023
18	Mon: 02/20/2023	Chap. 7	Electromagnetic plane waves	#16	02/22/2023
19	Wed: 02/22/2023	Chap. 7	Optical effects of refractive indices		
20	Fri: 02/24/2023	Chap. 1-7	Review		
21	Mon: 02/27/2023	Chap. 8	Short lectures on waveguides	Exam	
22	Wed: 03/01/2023	Chap. 8	Short lectures on waveguides	Exam	
23	Fri: 03/03/2023	Chap. 8	Short lectures on waveguides	Exam	03/03/2023
	Mon: 03/06/2023	No class	*Spring Break*		
	Wed: 03/08/2023	No class	*Spring Break*		
	Fri: 03/10/2023	No class	*Spring Break*		
24	Mon: 03/13/2023	Chap. 9	Radiation from localized oscillating sources		

For 2/27/2023-3/03/2023:

- Individual work on take home exam
- Shortened class lectures on Chapter 8 of **Jackson**

Motivation for giving/taking mid-term exam

1. Opportunity to review/solidify knowledge in the topic
2. Opportunity to practice problem solving techniques appropriate to the topic
3. Assessment of performance. Accordingly, the work you turn in must be your own (of course).
   - You are encouraged to consult with your instructor (but no one else!) if any questions arise about the exam questions
   - Extra credit awarded if you report errors/inconsistencies/ambiguities in the exam questions

Instructions on exam:

Note: This is a ``take-home'' exam which can be turned in any time before 4 PM Friday, March 3, 2023. In addition to each worked problem, please attach ALL Maple (or Mathematica, Matlab, Wolfram, etc.), work sheets as well as a full list of resources used to complete these problems. It is assumed that all work on the exam is performed under the guidelines of the honor code. In particular, if you have any questions about the material, you may consult with the instructor ***but no one else***. For grading purposes, each question in multi-part problems are worth equal weight. Credit will be assigned on the basis of both the logical steps of the solution and on the correct answer.

More advice about exam –

- It is important that the instructor is able to read your work and understand your reasoning.
- Since you will be using Maple or Mathematica or ?? to evaluate some of your results, please include the software work (or snips of it) into your exam materials.
- Your exam paper does not need to be a work of art, but it does need to be readable. If you prefer to submit your exam paper electronically, that will be fine. (I may print it myself.)

More advice – accumulated trusted equations/mathematical relationships and know how to use them

Jackson

pg. 783

Table 4 Conversion Table for Given Amounts of a Physical Quantity

The table is arranged so that a given amount of some physical quantity, expressed as so many SI or Gaussian units of that quantity, can be expressed as an equivalent number of units in the other system. Thus the entries in each row stand for the same amount, expressed in different units. All factors of 3 (apart from exponents) should, for accurate work, be replaced by (2.997 924 58), arising from the numerical value of the velocity of light. For example, in the row for displacement (D), the entry ($12\pi \times 10^5$) is actually ($2.997\,924\,58 \times 4\pi \times 10^5$) and "9" is actually $10^{-16}\, c^2 = 8.987\,55 \ldots$. Where a name for a unit has been agreed on or is in common usage, that name is given. Otherwise, one merely reads so many Gaussian units, or SI units.

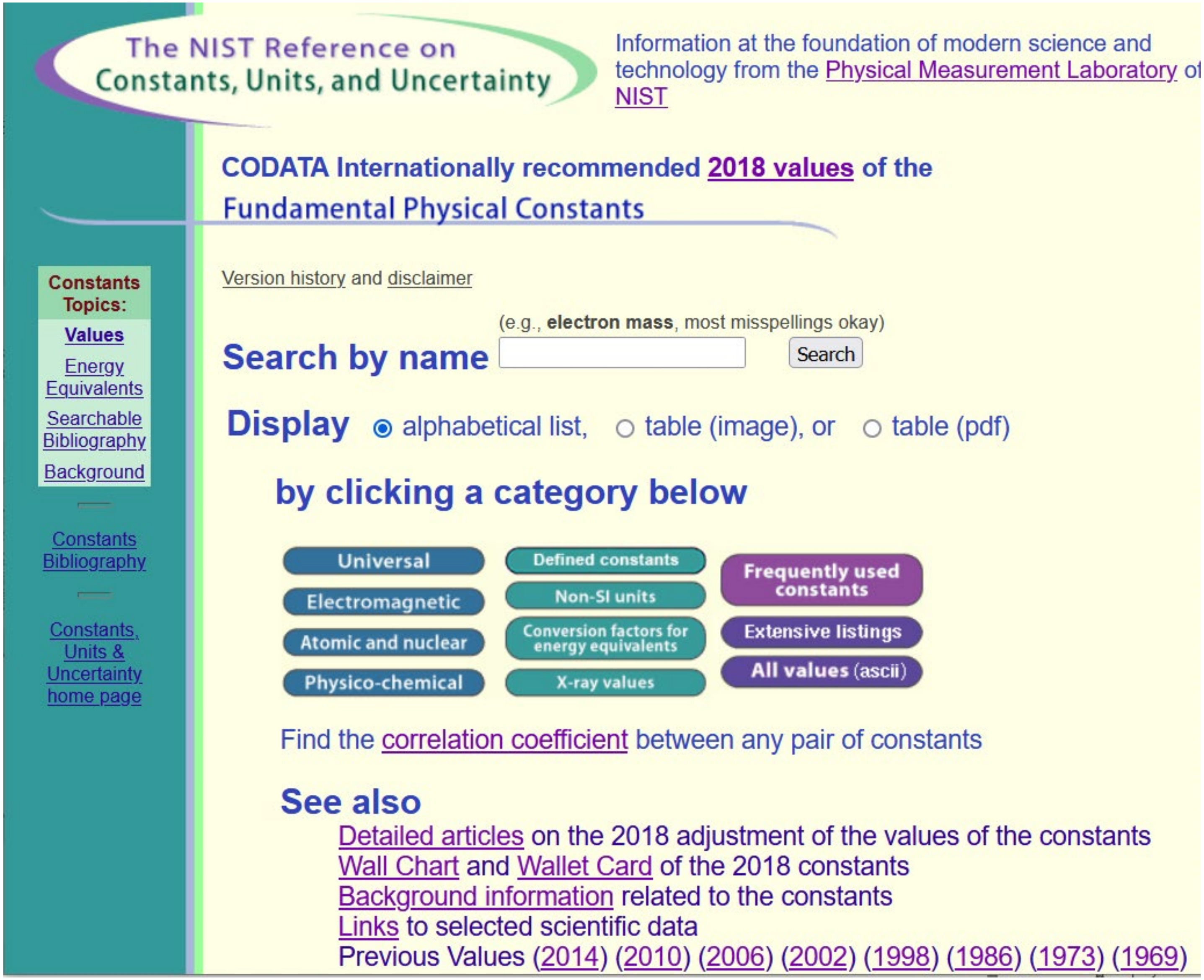
Jackson pg. 783

Physical Quantity	Symbol	SI		Gaussian
Length	l	1 meter (m)	10^2	centimeters (cm)
Mass	m	1 kilogram (kg)	10^3	grams (g)
Time	t	1 second (s)	1	second (s)
Frequency	ν	1 hertz (Hz)	1	hertz (Hz)
Force	F	1 newton (N)	10^5	dynes
Work / Energy	W / U	1 joule (J)	10^7	ergs
Power	P	1 watt (W)	10^7	ergs s^{-1}
Charge	q	1 coulomb (C)	3×10^9	statcoulombs
Charge density	ρ	1 C m^{-3}	3×10^3	statcoul cm^{-3}
Current	I	1 ampere (A)	3×10^9	statamperes
Current density	J	1 A m^{-2}	3×10^5	statamp cm^{-2}
Electric field	E	1 volt m^{-1} (Vm^{-1})	$\frac{1}{3} \times 10^{-4}$	statvolt cm^{-1}
Potential	Φ, V	1 volt (V)	$\frac{1}{300}$	statvolt
Polarization	P	1 C m^{-2}	3×10^5	dipole moment cm^{-3}
Displacement	D	1 C m^{-2}	$12\pi \times 10^5$	statvolt cm^{-1} (statcoul cm^{-2})
Conductivity	σ	1 mho m^{-1}	9×10^9	s^{-1}
Resistance	R	1 ohm (Ω)	$\frac{1}{9} \times 10^{-11}$	s cm^{-1}
Capacitance	C	1 farad (F)	9×10^{11}	cm
Magnetic flux	ϕ, F	1 weber (Wb)	10^8	gauss cm^2 or maxwells
Magnetic induction	B	1 tesla (T)	10^4	gauss (G)
Magnetic field	H	1 A m^{-1}	$4\pi \times 10^{-3}$	oersted (Oe)
Magnetization	M	1 A m^{-1}	10^{-3}	magnetic moment cm^{-3}
Inductance*	L	1 henry (H)	$\frac{1}{9} \times 10^{-11}$	

*There is some confusion about the unit of inductance in Gaussian units. This stems from the use by some authors of a modified system of Gaussian units in which current is measured in electromagnetic units, so that the connection between charge and current is $I_m = (1/c)(dq/dt)$. Since inductance is defined through the induced voltage $V = L(dI/dt)$ or the energy $U = \frac{1}{2}LI^2$, the choice of current defined in Section 2 means that our Gaussian unit of inductance is equal in magnitude and dimensions (t^2l^{-1}) to the electrostatic unit of inductance. The electromagnetic current I_m is related to our Gaussian current I by the relation $I_m = (1/c)I$. From the energy definition of inductance, we see that the electromagnetic inductance L_m is related to our Gaussian inductance L through $L_m = c^2L$. Thus L_m has the dimensions of length. The modified Gaussian system generally uses the electromagnetic unit of inductance, as well as current. Then the voltage relation reads $V = (L_m/c)(dI_m/dt)$. The numerical connection between units of inductance is

$$1 \text{ henry} = \tfrac{1}{9} \times 10^{-11} \text{ Gaussian (es) unit} = 10^9 \text{ emu}$$

Source for standard measurements –
https://physics.nist.gov/cuu/Constants/index.html



Vector relations

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\nabla\times\nabla\psi = 0$$

$$\nabla\cdot(\nabla\times\mathbf{a}) = 0$$

$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$$

$$\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$$

$$\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}$$

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$$

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$

If $\mathbf{x}$ is the coordinate of a point with respect to some origin, with magnitude $r = |\mathbf{x}|$, $\mathbf{n} = \mathbf{x}/r$ is a unit radial vector, and $f(r)$ is a well-behaved function of r, then

$$\nabla\cdot\mathbf{x} = 3 \qquad \nabla\times\mathbf{x} = 0$$

$$\nabla\cdot[\mathbf{n}f(r)] = \frac{2}{r}f + \frac{\partial f}{\partial r} \qquad \nabla\times[\mathbf{n}f(r)] = 0$$

$$(\mathbf{a}\cdot\nabla)\mathbf{n}f(r) = \frac{f(r)}{r}[\mathbf{a} - \mathbf{n}(\mathbf{a}\cdot\mathbf{n})] + \mathbf{n}(\mathbf{a}\cdot\mathbf{n})\frac{\partial f}{\partial r}$$

$$\nabla(\mathbf{x}\cdot\mathbf{a}) = \mathbf{a} + \mathbf{x}(\nabla\cdot\mathbf{a}) + i(\mathbf{L}\times\mathbf{a})$$

where $\mathbf{L} = \frac{1}{i}(\mathbf{x}\times\nabla)$ is the angular-momentum operator.

In the following ϕ, ψ, and $\mathbf{A}$ are well-behaved scalar or vector functions, V is a three-dimensional volume with volume element d^3x, S is a closed two-dimensional surface bounding V, with area element da and unit outward normal $\mathbf{n}$ at da.

$$\int_V \nabla \cdot \mathbf{A}\, d^3x = \int_S \mathbf{A} \cdot \mathbf{n}\, da \qquad \text{(Divergence theorem)}$$

$$\int_V \nabla\psi\, d^3x = \int_S \psi\mathbf{n}\, da$$

$$\int_V \nabla \times \mathbf{A}\, d^3x = \int_S \mathbf{n} \times \mathbf{A}\, da$$

$$\int_V (\phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi)\, d^3x = \int_S \phi\mathbf{n} \cdot \nabla\psi\, da \qquad \text{(Green's first identity)}$$

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, d^3x = \int_S (\phi\nabla\psi - \psi\nabla\phi) \cdot \mathbf{n}\, da \qquad \text{(Green's theorem)}$$

In the following S is an open surface and C is the contour bounding it, with line element $d\mathbf{l}$. The normal $\mathbf{n}$ to S is defined by the right-hand-screw rule in relation to the sense of the line integral around C.

$$\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, da = \oint_C \mathbf{A} \cdot d\mathbf{l} \qquad \text{(Stokes's theorem)}$$

$$\int_S \mathbf{n} \times \nabla\psi\, da = \oint_C \psi\, d\mathbf{l}$$

Explicit Forms of Vector Operations

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian
$(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical (ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}\right) + \mathbf{e}_3\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical
(r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1 \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right]$$
$$+ \mathbf{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi).\right]$$

Comment on cartesian unit vectors versus local (cylindrical or spherical) unit vectors

$$\hat{\mathbf{r}} = \sin\theta\ \cos\phi\ \hat{\mathbf{x}} + \sin\theta\ \sin\phi\ \hat{\mathbf{y}} + \cos\theta\ \hat{\mathbf{z}}$$

$$\hat{\boldsymbol{\theta}} = \cos\theta\ \cos\phi\ \hat{\mathbf{x}} + \cos\theta\ \sin\phi\ \hat{\mathbf{y}} - \sin\theta\ \hat{\mathbf{z}}$$

$$\hat{\boldsymbol{\varphi}} = \quad -\sin\phi\ \hat{\mathbf{x}} + \cos\phi\ \hat{\mathbf{y}}$$

$$\text{Note that } \nabla^2\mathbf{A} = \left(\nabla^2 A_x\right)\hat{\mathbf{x}} + \left(\nabla^2 A_y\right)\hat{\mathbf{y}} + \left(\nabla^2 A_z\right)\hat{\mathbf{z}}$$

$$\text{Also note that } \nabla^2 f(r) = \frac{\partial^2 f(r)}{\partial r^2} + \frac{2}{r}\frac{\partial f(r)}{\partial r}$$

Special functions -- many are described in Jackson

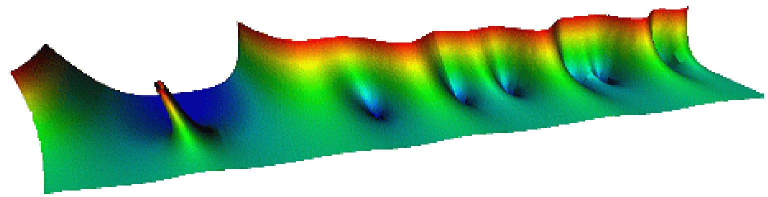
Additional source [-- https://dlmf.nist.gov/](https://dlmf.nist.gov/)

Digital Library of Mathematical Functions

Index
Notations
Search
Help?
Citing
Customize

About the Project

NIST National Institute of Standards and Technology U.S. Department of Commerce



NIST Digital Library of Mathematical Functions

Project News

2022-03-15 *DLMF Update; Version 1.1.5*
2022-01-15 *DLMF Update; Version 1.1.4*
2021-09-15 *DLMF Update; Version 1.1.3*
2021-07-19 *Brian D. Sleeman, Associate Editor of the DLMF, dies at age 81*
More news

Basic equations of electrodynamics

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = \epsilon\mathbf{E}$$

$$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} = \frac{1}{\mu}\mathbf{B}$$

CGS (Gaussian)	SI
$\nabla \cdot \mathbf{D} = 4\pi\rho$	$\nabla \cdot \mathbf{D} = \rho$
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$
$\mathbf{F} = q(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})$	$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$
$u = \frac{1}{8\pi}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$	$u = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$
$\mathbf{S} = \frac{c}{4\pi}(\mathbf{E} \times \mathbf{H})$	$\mathbf{S} = (\mathbf{E} \times \mathbf{H})$

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon\mathbf{E}$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} = \frac{1}{\mu}\mathbf{B}$$

More relationships

CGS (Gaussian)

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = \epsilon\mathbf{E}$$

$$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} = \frac{1}{\mu}\mathbf{B}$$

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

MKS (SI)

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon\mathbf{E}$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} = \frac{1}{\mu}\mathbf{B}$$

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

$$\epsilon \Leftrightarrow \epsilon/\epsilon_0$$

$$\mu \Leftrightarrow \mu/\mu_0$$

More SI relationships:

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} \qquad \mathbf{D} = \varepsilon\mathbf{E}$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}) \qquad \mathbf{B} = \mu\mathbf{H} \qquad \mathbf{B} = F(\mathbf{H})$$

for ferromagnet

More Gaussian relationships:

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} \qquad \mathbf{D} = \varepsilon\mathbf{E}$$

$$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}) \qquad \mathbf{B} = \mu\mathbf{H} \qquad \mathbf{B} = F(\mathbf{H})$$

for ferromagnet

elementary charge: $e = 1.6021766208 \times 10^{-19}$ C
$= 4.80320467299766 \times 10^{-10}$ statC

Energy and power (SI units)

Electromagnetic energy density: $u \equiv \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B}\right)$

Poynting vector: $\mathbf{S} \equiv \mathbf{E}\times\mathbf{H}$

Equations for time harmonic fields:

$$\mathbf{E}(\mathbf{r},t)=\Re\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}\right)\equiv\frac{1}{2}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}+\widetilde{\mathbf{E}}^{*}(\mathbf{r},\omega)e^{i\omega t}\right)$$

$$\left\langle u\left(\mathbf{r},t\right)\right\rangle_{t\text{ avg}}=\frac{1}{4}\Re\left(\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\cdot\tilde{\mathbf{D}}^{*}(\mathbf{r},\omega)+\tilde{\mathbf{B}}(\mathbf{r},\omega)\cdot\tilde{\mathbf{H}}^{*}(\mathbf{r},\omega)\right)\right)$$

$$\left\langle \mathbf{S}\left(\mathbf{r},t\right)\right\rangle_{t\text{ avg}}=\frac{1}{2}\Re\left(\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\times\tilde{\mathbf{H}}^{*}(\mathbf{r},\omega)\right)\right)$$

Solution of Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0 \qquad \nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

Introduction of vector and scalar potentials:

$$\nabla \cdot \mathbf{B} = 0 \qquad \Rightarrow \quad \mathbf{B} = \nabla \times \mathbf{A}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \Rightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi \qquad \text{or} \quad \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$$

Scalar and vector potentials continued:

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0 :$$

$$-\nabla^2 \Phi - \frac{\partial \left(\nabla \cdot \mathbf{A} \right)}{\partial t} = \rho / \varepsilon_0$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

$$\nabla \times \left(\nabla \times \mathbf{A} \right) + \frac{1}{c^2} \left(\frac{\partial \left(\nabla \Phi \right)}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) = \mu_0 \mathbf{J}$$

Analysis of the scalar and vector potential equations:

$$-\nabla^2\Phi - \frac{\partial(\nabla\cdot\mathbf{A})}{\partial t} = \rho/\varepsilon_0$$

$$\nabla\times(\nabla\times\mathbf{A}) + \frac{1}{c^2}\left(\frac{\partial(\nabla\Phi)}{\partial t} + \frac{\partial^2\mathbf{A}}{\partial t^2}\right) = \mu_0\mathbf{J}$$

Lorentz gauge form -- require $\nabla\cdot\mathbf{A}_L + \frac{1}{c^2}\frac{\partial\Phi_L}{\partial t} = 0$

$$-\nabla^2\Phi_L + \frac{1}{c^2}\frac{\partial^2\Phi_L}{\partial t^2} = \rho/\varepsilon_0$$

$$-\nabla^2\mathbf{A}_L + \frac{1}{c^2}\frac{\partial^2\mathbf{A}_L}{\partial t^2} = \mu_0\mathbf{J}$$

Solution methods for scalar and vector potentials and their electrostatic and magnetostatic analogs:

$$-\nabla^2\Phi_L + \frac{1}{c^2}\frac{\partial^2\Phi_L}{\partial t^2} = \rho / \varepsilon_0$$

$$-\nabla^2\mathbf{A}_L + \frac{1}{c^2}\frac{\partial^2\mathbf{A}_L}{\partial t^2} = \mu_0\mathbf{J}$$

In your "bag" of tricks:

- **Direct (analytic or numerical) solution of differential equations**
- **Solution by expanding in appropriate orthogonal functions**
- **Green's function techniques**

How to choose most effective solution method --

- In general, Green's functions methods work well when source is contained in a finite region of space

Consider the electrostatic problem:

$$-\nabla^2\Phi_L = \rho / \varepsilon_0$$

Define: $\nabla'^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$

$$\Phi_L(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r' \rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') +$$

$$\frac{1}{4\pi}\int_S d^2r' \left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'.$$

For electrostatic problems where $\rho(\mathbf{r})$ is contained in a small region of space and $S \to \infty$, $\quad G(\mathbf{r},\mathbf{r}') = \dfrac{1}{|\mathbf{r}-\mathbf{r}'|}$

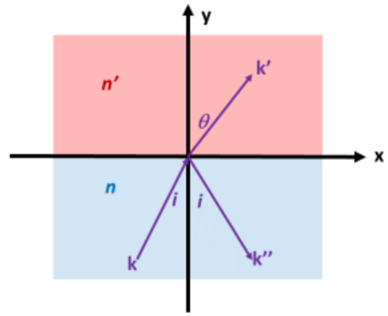
$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_<^{\,l}}{r_>^{\,l+1}} Y_{lm}(\theta,\varphi) Y_{lm}^{*}(\theta',\varphi')$$

Some HW problems

February 20, 2023

PHY 712 – Problem Set #16

Continue reading Chaper 7 in **Jackson**



1. The figure shows the plane of incidence of a plane polarized electromagnetic wave of harmonic frequency ω as it is reflected and refracted at the boundary between two uniform media with real refractive indices n and n', similar to Fig. 7.6 of your textbook. In this case, both media have permeabilities $\mu = \mu' = \mu_0$. The permittivities are ϵ' and ϵ for the upper and lower media, respectively. The wavevectors for the incident, reflected, and refracted plane waves are given by:

$$\mathbf{k} = \frac{n\omega}{c}(\sin i\hat{\mathbf{x}} + \cos i\hat{\mathbf{y}}), \quad \mathbf{k}'' = \frac{n\omega}{c}(\sin i\hat{\mathbf{x}} - \cos i\hat{\mathbf{y}}), \quad \text{and} \quad \mathbf{k}' = \frac{n'\omega}{c}(\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}}),$$

respectively, where c denotes the speed of light in vacuum. In this case, the surface between the media is in the $x - z$ plane with $\hat{\mathbf{y}}$ as the surface normal direction. We can assume that the normal components of the **D** and **B** fields and the tangential components of the **E** and **H** fields are continuous at this surface boundary.

(a) We can express the magnetic field of the incident, reflected, and refracted plane waves as

$$\mathbf{H}(\mathbf{r},t) = \Re\left\{\mathbf{H}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right\}, \quad \mathbf{H}''(\mathbf{r},t) = \Re\left\{\mathbf{H}_0'' e^{i\mathbf{k}''\cdot\mathbf{r}-i\omega t}\right\}, \quad \text{and} \quad \mathbf{H}'(\mathbf{r},t) = \Re\left\{\mathbf{H}_0' e^{i\mathbf{k}'\cdot\mathbf{r}-i\omega t}\right\},$$

respectively. Express the continuity of the **D**, **B**, **E**, and **H** field components at the surface boundary in terms of the $\mathbf{H}_0$, $\mathbf{H}_0''$, and $\mathbf{H}_0'$ amplitudes, analogous to Eq. 7.37 in **Jackson**.

(b) Assuming that the incident magnetic field amplitude is in the plane of incidence,

$$\mathbf{H}_0 = H_0(-\cos i\hat{\mathbf{x}} + \sin i\hat{\mathbf{y}}),$$

determine the corresponding reflected amplitude ratio $\dfrac{H_0''}{H_0}$.

(c) Assuming that the incident magnetic field amplitude is perpendicular to the plane of incidence,

$$\mathbf{H}_0 = H_0\hat{\mathbf{z}},$$

determine the corresponding reflected amplitude ratio $\dfrac{H_0''}{H_0}$.

Maxwell's equations

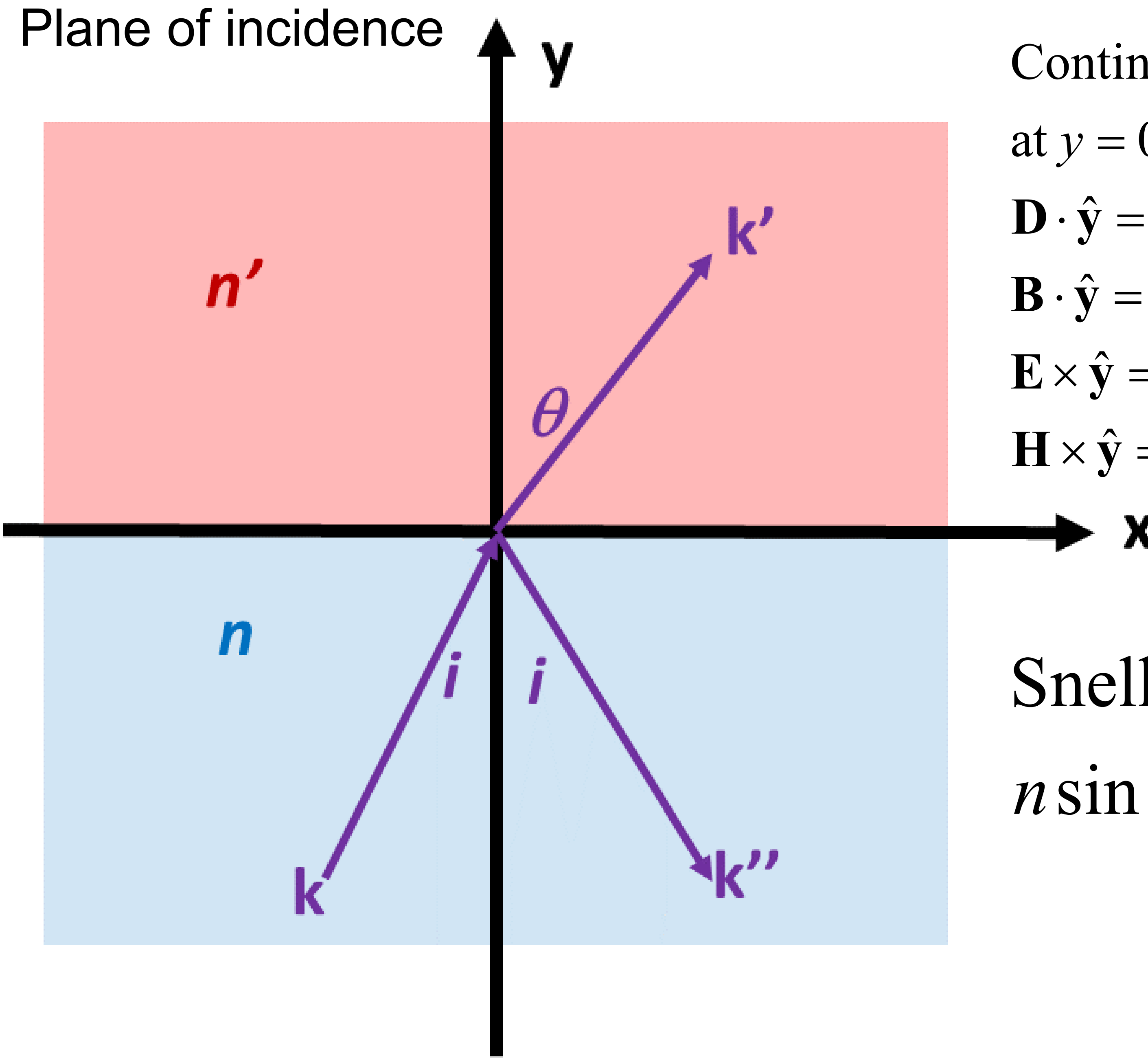
For linear isotropic media and no sources: $\mathbf{D} = \varepsilon\mathbf{E}; \quad \mathbf{B} = \mu\mathbf{H}$

Coulomb's law: $\nabla \cdot \mathbf{E} = 0$

Ampere-Maxwell's law: $\nabla \times \mathbf{B} - \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} = 0$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$



Continuity conditions at $y = 0$ interface:

$\mathbf{D} \cdot \hat{\mathbf{y}} = \text{continuous}$

$\mathbf{B} \cdot \hat{\mathbf{y}} = \text{continuous}$

$\mathbf{E} \times \hat{\mathbf{y}} = \text{continuous}$

$\mathbf{H} \times \hat{\mathbf{y}} = \text{continuous}$

Snell's "law"

$$n \sin i = n' \sin \theta$$

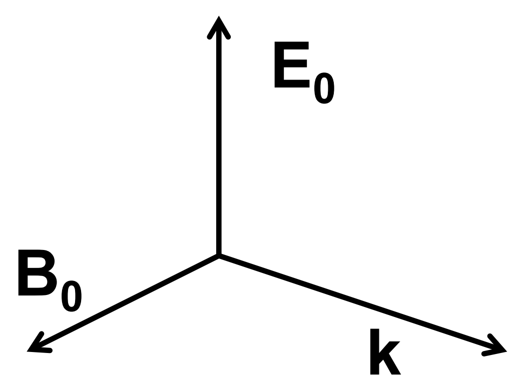
Summary of plane electromagnetic waves:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{B}(\mathbf{r},t) = \Re\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

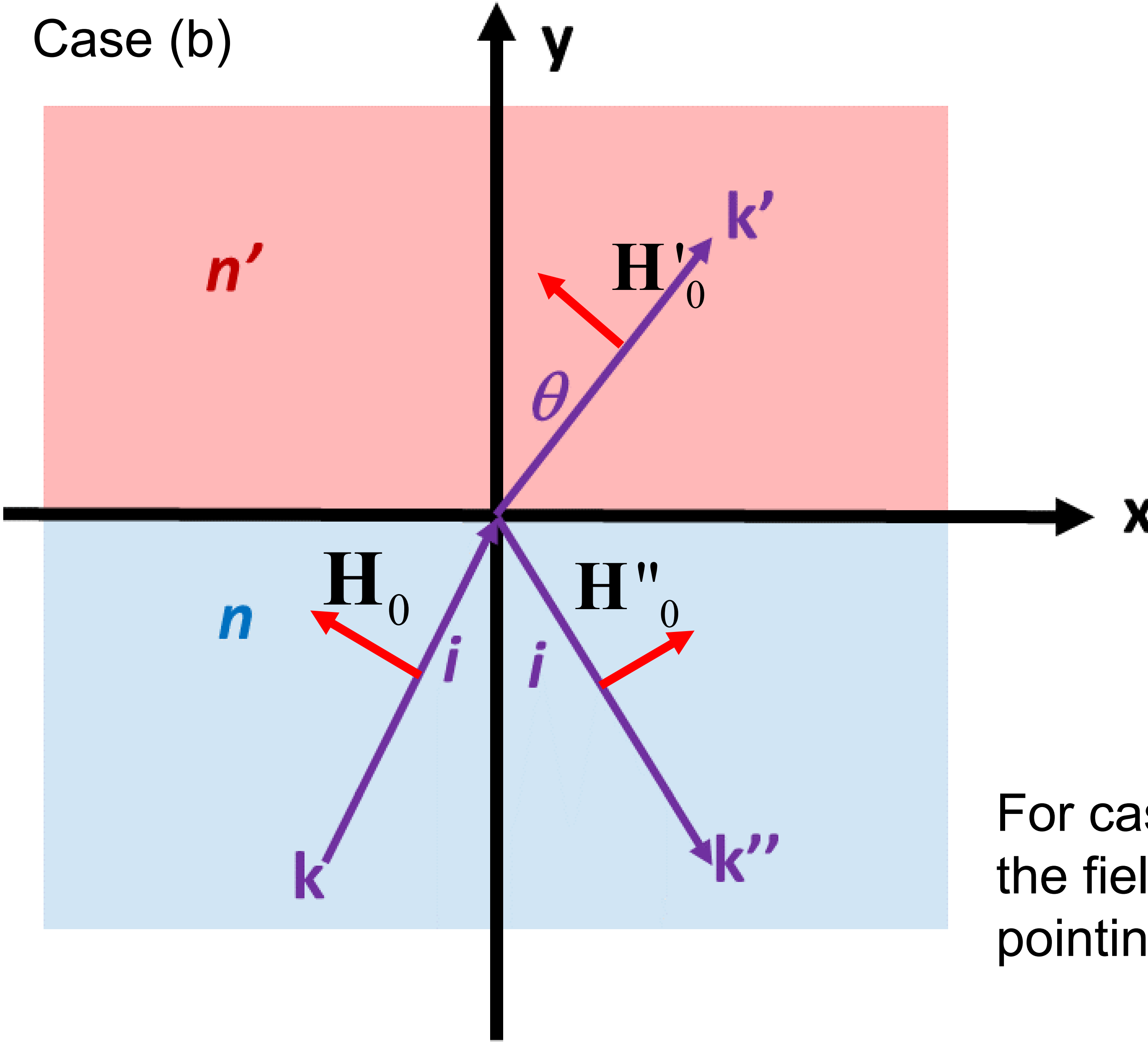
$$\mathbf{D}(\mathbf{r},t) = \varepsilon\mathbf{E}(\mathbf{r},t) \qquad \mathbf{H}(\mathbf{r},t) = \frac{1}{\mu}\mathbf{B}(\mathbf{r},t)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}}\cdot\mathbf{E}_0 = 0$$

$$\mathbf{B}_0 = \frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c} \quad \text{and} \quad \hat{\mathbf{k}}\cdot\mathbf{B}_0 = 0$$



Case (b)



For case (c),
the fields are all
pointing out of plane.

Find four equations for H_0, H'_0, and H''_0

Continuity conditions

at $y = 0$ interface:

$\mathbf{D} \cdot \hat{\mathbf{y}} = \text{continuous}$

$\mathbf{B} \cdot \hat{\mathbf{y}} = \text{continuous}$

$\mathbf{E} \times \hat{\mathbf{y}} = \text{continuous}$

$\mathbf{H} \times \hat{\mathbf{y}} = \text{continuous}$

Solution strategy

1 equation will be trivial because of geometry

3 equations will be non-trivial, but one will be redundant

Find $\dfrac{H'_0}{H_0}$ and $\dfrac{H''_0}{H_0}$

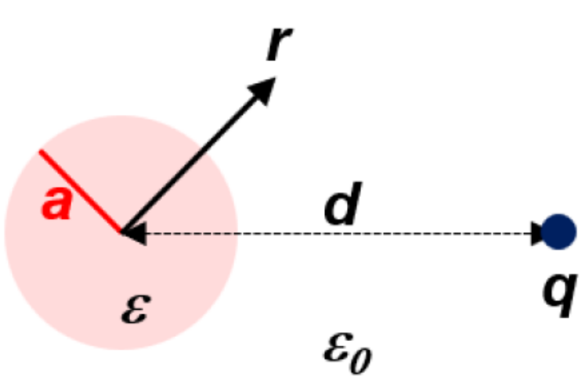
Do you expect that your answer will be related to the forms of the Fresnel equations derived in Jackson?

HW10

February 3, 2023

Finish reading Chapter 4 in **Jackson** .

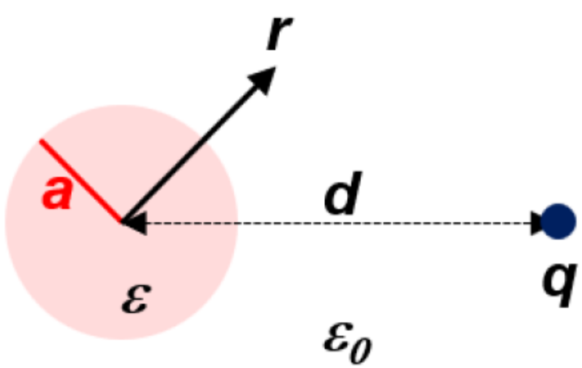
1. Work problem 4.9(a) in **Jackson**. It is probably most convenient to use a coordinate system with the origin at the center of the dielectric sphere.



Note that

$$\frac{1}{|\mathbf{r}-\mathbf{d}|}=\sum_{\ell=0}^{\infty}\frac{r^{\ell}}{d^{\ell+1}}\,P_{\ell}(\hat{\mathbf{r}}\cdot\hat{\mathbf{d}})$$

There are several ways of approaching this problem. One convenient way is to consider the effects of the dielectric sphere and point charge separately



$$\Phi(r,\theta) = \Phi_{\text{sphere}}(r,\theta) + \frac{q}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{d}|}$$

$$\Phi_{\text{sphere}}(r,\theta) = \begin{cases} \sum_{\ell=0} A_\ell r^\ell P_\ell(\cos\theta) & \text{for } r \le a \\ \sum_{\ell=0} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) & \text{for } r \ge a \end{cases}$$

Boundary values at $r = 0$:

$$\varepsilon \left.\frac{\partial\Phi(r,\theta)}{\partial r}\right|_{r=a_-} = \varepsilon_0 \left.\frac{\partial\Phi(r,\theta)}{\partial r}\right|_{r=a_+}$$

$$\left.\frac{\partial\Phi(r,\theta)}{\partial\theta}\right|_{r=a_-} = \left.\frac{\partial\Phi(r,\theta)}{\partial\theta}\right|_{r=a_+}$$

Also: $$\left.\frac{q}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{d}|}\right|_{r=a} = \frac{q}{4\pi\varepsilon_0}\sum_{\ell=0}\frac{a^\ell}{d^{\ell+1}}P_\ell(\cos\theta)$$

$$\ell a^{\ell-1}\left(\varepsilon\left(A_\ell + \frac{q}{4\pi\varepsilon_0 d^{\ell+1}}\right)\right) = -(\ell+1)\varepsilon_0\frac{B_\ell}{a^{\ell+2}} + \ell a^{\ell-1}\frac{q}{4\pi d^{\ell+1}}$$

$$a^\ell\left(\left(A_\ell + \frac{q}{4\pi\varepsilon_0 d^{\ell+1}}\right)\right) = \frac{B_\ell}{a^{\ell+1}} + a^\ell\frac{q}{4\pi\varepsilon_0 d^{\ell+1}} \qquad \Rightarrow A_\ell = \frac{B_\ell}{a^{2\ell+1}}$$

$$\frac{\varepsilon}{\varepsilon_0}\left(A_\ell + \frac{q}{4\pi\varepsilon_0 d^{\ell+1}}\right) = -\frac{(\ell+1)}{\ell}\frac{B_\ell}{a^{2\ell+1}} + \frac{q}{4\pi\varepsilon_0 d^{\ell+1}}$$

$$A_\ell = \frac{q}{4\pi\varepsilon_0 d^{\ell+1}}\left(\frac{1-\dfrac{\varepsilon}{\varepsilon_0}}{\dfrac{\varepsilon}{\varepsilon_0}+\dfrac{(\ell+1)}{\ell}}\right)$$

PHY 712 -- Assignment #11

February 06, 2023

Start reading Chapter 5 in **Jackson** .

1. Consider an infinitely long cylindrical wire with radius a, oriented along the **z** axis. There is a steady uniform current inside the wire. Specifically, in terms of *r* the radial parameter of the cylindrical coordinates of the system the current density is **J**(*r*)=$\mathbf{J_0}$, where $\mathbf{J_0}$ is a constant vector pointing along the z-axis, for $r \leq a$ and zero otherwise.
   a. Find the vector potential (**A**) for all *r*.
   b. Find the magnetic flux field (**B**) for all *r*.

Simple solution using Ampere's law

Know that magnetic field is uniform and pointing in the ϕ direction

$$\text{For } r < a \qquad 2\pi r B = \mu_0 \pi r^2 J_0 \qquad \Rightarrow B = \frac{\mu_0 r J_0}{2}$$

$$\text{For } r > a \qquad 2\pi r B = \mu_0 \pi a^2 J_0 \qquad \Rightarrow B = \frac{\mu_0 a^2 J_0}{2r}$$

$$\mathbf{A} = A_z(r)\hat{\mathbf{z}}$$

$$\text{For } r < a \qquad A_z(r) = -\frac{\mu_0 r^2 J_0}{4}$$

$$\text{For } r > a \qquad A_z(r) = -\frac{\mu_0 a^2 J_0}{4}\left(1 + 2\ln\left(\frac{r}{a}\right)\right)$$

Is this answer unique?