

PHY 712 Electrodynamics

10-10:50 AM Online

Notes for Lecture 21:

Chap. 8 in Jackson – Wave Guides

(shortened lecture)

1. TEM, TE, and TM modes
2. Justification for boundary conditions; behavior of waves near conducting surfaces

17	Fri: 02/17/2023	Chap. 7	Electromagnetic plane waves	#15	02/20/2023
18	Mon: 02/20/2023	Chap. 7	Electromagnetic plane waves	#16	02/22/2023
19	Wed: 02/22/2023	Chap. 7	Optical effects of refractive indices		
20	Fri: 02/24/2023	Chap. 1-7	Review		
21	Mon: 02/27/2023	Chap. 8	Short lectures on waveguides	Exam	
22	Wed: 03/01/2023	Chap. 8	Short lectures on waveguides	Exam	
23	Fri: 03/03/2023	Chap. 8	Short lectures on waveguides	Exam	03/03/2023
	Mon: 03/06/2023	No class	<i>Spring Break</i>		
	Wed: 03/08/2023	No class	<i>Spring Break</i>		
	Fri: 03/10/2023	No class	<i>Spring Break</i>		
24	Mon: 03/13/2023	Chap. 9	Radiation from localized oscillating sources		

For 2/27/2023-3/03/2023:

- Individual work on take home exam
- Shortened class lectures on Chapter 8 of **Jackson**

Maxwell's equations

For linear isotropic media and no sources: $\mathbf{D} = \epsilon\mathbf{E}$; $\mathbf{B} = \mu\mathbf{H}$

Coulomb's law: $\nabla \cdot \mathbf{E} = 0$

Ampere-Maxwell's law: $\nabla \times \mathbf{B} - \mu\epsilon \frac{\partial \mathbf{E}}{\partial t} = 0$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

Analysis of Maxwell's equations without sources -- continued:

Coulomb's law : $\nabla \cdot \mathbf{E} = 0$

Ampere - Maxwell's law : $\nabla \times \mathbf{B} - \mu\epsilon \frac{\partial \mathbf{E}}{\partial t} = 0$

Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles : $\nabla \cdot \mathbf{B} = 0$

$$\begin{aligned}\nabla \times \left(\nabla \times \mathbf{B} - \mu\epsilon \frac{\partial \mathbf{E}}{\partial t} \right) &= -\nabla^2 \mathbf{B} - \mu\epsilon \frac{\partial (\nabla \times \mathbf{E})}{\partial t} \\ &= -\nabla^2 \mathbf{B} + \mu\epsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0\end{aligned}$$

$$\begin{aligned}\nabla \times \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) &= -\nabla^2 \mathbf{E} + \frac{\partial (\nabla \times \mathbf{B})}{\partial t} \\ &= -\nabla^2 \mathbf{E} + \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0\end{aligned}$$

Analysis of Maxwell's equations without sources -- continued:

Both \mathbf{E} and \mathbf{B} fields are solutions to a wave equation:

$$\nabla^2 \mathbf{B} - \frac{1}{v^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

$$\nabla^2 \mathbf{E} - \frac{1}{v^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\text{where } v^2 \equiv c^2 \frac{\mu_0 \epsilon_0}{\mu \epsilon} \equiv \frac{c^2}{n^2}$$

Plane wave solutions to wave equation :

$$\mathbf{B}(\mathbf{r}, t) = \Re\left(\mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right) \quad \mathbf{E}(\mathbf{r}, t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right)$$

Analysis of Maxwell's equations without sources -- continued:

Plane wave solutions to wave equation :

$$\mathbf{B}(\mathbf{r}, t) = \Re\left(\mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right) \quad \mathbf{E}(\mathbf{r}, t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}$$

Note: ε, μ, n, k can all be complex; for the moment we will assume that they are all real (no dissipation).

Note that \mathbf{E}_0 and \mathbf{B}_0 are not independent;

from Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

$$\Rightarrow \mathbf{B}_0 = \frac{\mathbf{k} \times \mathbf{E}_0}{\omega} = \frac{n \hat{\mathbf{k}} \times \mathbf{E}_0}{c}$$

} For real
 ε, μ, n, k

also note : $\hat{\mathbf{k}} \cdot \mathbf{E}_0 = 0$ and $\hat{\mathbf{k}} \cdot \mathbf{B}_0 = 0$

Analysis of Maxwell's equations without sources -- continued:

Summary of plane electromagnetic waves:

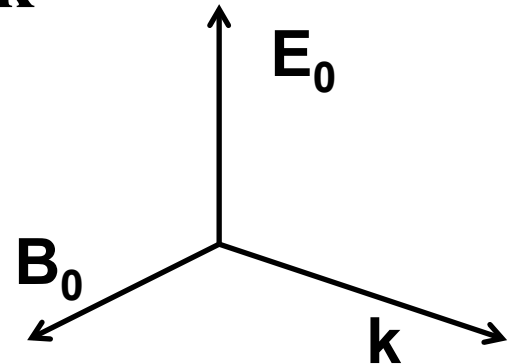
$$\mathbf{B}(\mathbf{r}, t) = \Re\left(\frac{n\hat{\mathbf{k}} \times \mathbf{E}_0}{c} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right) \quad \mathbf{E}(\mathbf{r}, t) = \Re(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t})$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} \quad \text{and } \hat{\mathbf{k}} \cdot \mathbf{E}_0 = 0$$

Poynting vector and energy density:

$$\langle \mathbf{S} \rangle_{avg} = \frac{n|\mathbf{E}_0|^2}{2\mu c} \hat{\mathbf{k}} = \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}} |\mathbf{E}_0|^2 \hat{\mathbf{k}}$$

$$\langle u \rangle_{avg} = \frac{1}{2} \epsilon |\mathbf{E}_0|^2$$



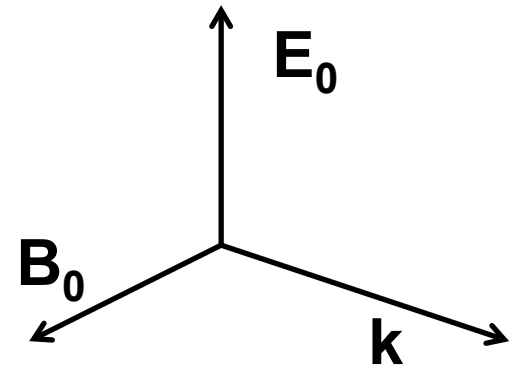
Transverse electric and magnetic waves (TEM)

$$\mathbf{B}(\mathbf{r}, t) = \Re\left(\frac{n\hat{\mathbf{k}} \times \mathbf{E}_0}{c} e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t}\right) \quad \mathbf{E}(\mathbf{r}, t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}} \cdot \mathbf{E}_0 = 0$$

TEM modes describe
electromagnetic waves in lossless
media and vacuum

For real
 ε, μ, n, k



Effects of complex dielectric; fields near the surface on an ideal conductor

Suppose for an isotropic medium : $\mathbf{D} = \epsilon_b \mathbf{E}$ $\mathbf{J} = \sigma \mathbf{E}$

Maxwell's equations in terms of \mathbf{H} and \mathbf{E} :

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{H} = 0$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \qquad \nabla \times \mathbf{H} = \sigma \mathbf{E} + \epsilon_b \frac{\partial \mathbf{E}}{\partial t}$$

$$\left(\nabla^2 - \mu \sigma \frac{\partial}{\partial t} - \mu \epsilon_b \frac{\partial^2}{\partial t^2} \right) \mathbf{F} = 0 \qquad \mathbf{F} = \mathbf{E}, \mathbf{H}$$

Plane wave form for \mathbf{E} :

$$\mathbf{E}(\mathbf{r}, t) = \Re(\mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t}) \qquad \text{where } \mathbf{k} = (n_R + in_I) \frac{\omega}{c} \hat{\mathbf{k}}$$

$$\Rightarrow \mathbf{E}(\mathbf{r}, t) = e^{-\hat{\mathbf{k}} \cdot \mathbf{r} / \delta} \Re(\mathbf{E}_0 e^{in_R (\omega/c) \hat{\mathbf{k}} \cdot \mathbf{r} - i\omega t})$$

Some details:

Plane wave form for \mathbf{E} :

$$\mathbf{E}(\mathbf{r}, t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right) \quad \text{where } \mathbf{k} = (n_R + in_I) \frac{\omega}{c} \hat{\mathbf{k}}$$

$$\left(\nabla^2 - \mu\sigma \frac{\partial}{\partial t} - \mu\epsilon_b \frac{\partial^2}{\partial t^2} \right) \mathbf{E} = 0$$

$$-(n_R + in_I)^2 + i \frac{\mu\sigma c^2}{\omega} + \mu\epsilon_b c^2 = 0$$

Note that in this formulation, we are assuming ϵ_b and σ are real numbers.

Fields near the surface on an ideal conductor -- continued

For our system :

$$\frac{\omega}{c} n_R = \omega \sqrt{\frac{\mu \epsilon_b}{2} \left(\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon_b} \right)^2} + 1 \right)^{1/2}}$$

$$\frac{\omega}{c} n_I = \omega \sqrt{\frac{\mu \epsilon_b}{2} \left(\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon_b} \right)^2} - 1 \right)^{1/2}}$$

For $\frac{\sigma}{\omega} \gg 1$ $\frac{\omega}{c} n_R \approx \frac{\omega}{c} n_I \approx \sqrt{\frac{\mu \sigma \omega}{2}} \equiv \frac{1}{\delta}$ ← “skin depth”

$$\Rightarrow \mathbf{E}(\mathbf{r}, t) = e^{-\hat{\mathbf{k}} \cdot \mathbf{r} / \delta} \Re \left(\mathbf{E}_0 e^{i \hat{\mathbf{k}} \cdot \mathbf{r} / \delta - i \omega t} \right)$$

$$\Rightarrow \mathbf{H}(\mathbf{r}, t) = \frac{n}{c \mu} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r}, t) = \frac{1+i}{\delta \mu \omega} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r}, t)$$

Some representative values of skin depth

Ref: Lorrain² and Corson

$$\frac{\omega}{c} n_R \approx \frac{\omega}{c} n_I \approx \sqrt{\frac{\mu\sigma\omega}{2}} \equiv \frac{1}{\delta}$$

Note that frequency given in
units of Hz $\Rightarrow \frac{\omega}{2\pi}$

	σ (10^7 S/m)	μ/μ_0	δ (0.001m) at 60 Hz	δ (0.001m) at 1 MHz
Al	3.54	1	10.9	84.6
Cu	5.80	1	8.5	66.1
Fe	1.00	100	1.0	10.0
Mumetal	0.16	2000	0.4	3.0
Zn	1.86	1	15.1	117

Relative energies associated with field

Electric energy density: $\epsilon_b |\mathbf{E}|^2$

Magnetic energy density: $\mu |\mathbf{H}|^2$

Ratio inside conducting media: $\frac{\epsilon_b |\mathbf{E}|^2}{\mu |\mathbf{H}|^2} = \frac{\epsilon_b}{\mu \left| \frac{1+i}{\delta \mu \omega} \right|^2} = \frac{\epsilon_b \mu \omega^2 \delta^2}{2}$

$$\lambda = \frac{2\pi c}{\omega} = \frac{c}{f}$$

$$= 2\pi^2 \frac{\epsilon_b}{\epsilon_0} \frac{\mu}{\mu_0} \frac{\delta^2}{\lambda^2}$$

For $\frac{\epsilon_b |\mathbf{E}|^2}{\mu |\mathbf{H}|^2} \ll 1 \Rightarrow$ magnetic energy dominates

Note that in free space, $\frac{\epsilon_0 |\mathbf{E}|^2}{\mu_0 |\mathbf{H}|^2} = 1$

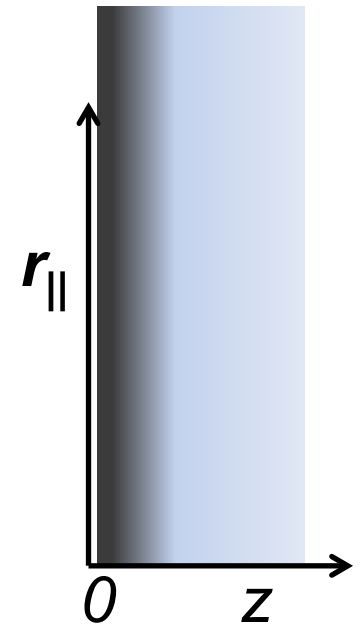
Fields near the surface on an ideal conductor -- continued

$$\text{For } \frac{\sigma}{\omega} \gg 1 \quad \frac{\omega}{c} n_R \approx \frac{\omega}{c} n_I \approx \sqrt{\frac{\mu\sigma\omega}{2}} \equiv \frac{1}{\delta}$$

$$\text{In this limit, } \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} = c\sqrt{\mu\varepsilon} = n_R + in_I = \frac{c}{\omega} \frac{1}{\delta} (1+i)$$

$$\mathbf{E}(\mathbf{r}, t) = e^{-\hat{\mathbf{k}} \cdot \mathbf{r} / \delta} \Re \left(\mathbf{E}_0 e^{i\hat{\mathbf{k}} \cdot \mathbf{r} / \delta - i\omega t} \right)$$

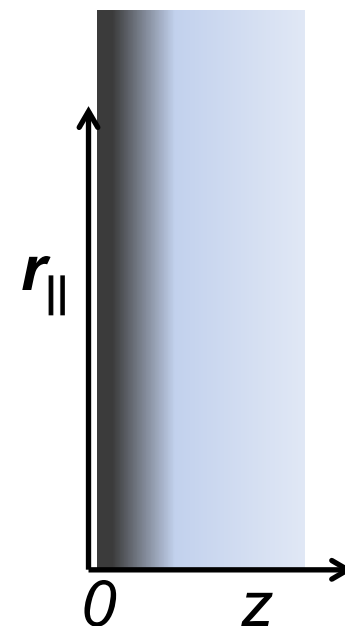
$$\mathbf{H}(\mathbf{r}, t) = \frac{n}{c\mu} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r}, t) = \frac{1+i}{\delta\mu\omega} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r}, t)$$



Fields near the surface on an ideal conductor -- continued

$$\mathbf{E}(\mathbf{r}, t) = e^{-\hat{\mathbf{k}} \cdot \mathbf{r} / \delta} \Re \left(\mathbf{E}_0 e^{i\hat{\mathbf{k}} \cdot \mathbf{r} / \delta - i\omega t} \right)$$

$$\mathbf{H}(\mathbf{r}, t) = \frac{n}{c\mu} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r}, t) = \frac{1+i}{\delta\mu\omega} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r}, t)$$



Note that it is convenient to express the EM fields in terms of the \mathbf{H} amplitude:

$$\mathbf{H}(\mathbf{r}, t) = e^{-\hat{\mathbf{k}} \cdot \mathbf{r} / \delta} \Re \left(\mathbf{H}_0 e^{i\hat{\mathbf{k}} \cdot \mathbf{r} / \delta - i\omega t} \right)$$

$$\mathbf{E}(\mathbf{r}, t) = \delta\mu\omega \frac{1-i}{2} \hat{\mathbf{k}} \times \mathbf{H}(\mathbf{r}, t)$$

Boundary values for ideal conductor

Inside the conductor :

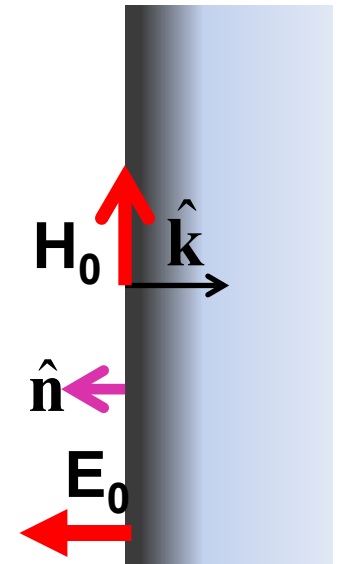
$$\mathbf{H}(\mathbf{r}, t) = e^{-\hat{\mathbf{k}} \cdot \mathbf{r} / \delta} \Re \left(\mathbf{H}_0 e^{i\hat{\mathbf{k}} \cdot \mathbf{r} / \delta - i\omega t} \right)$$

$$\mathbf{E}(\mathbf{r}, t) = \delta \mu \omega \frac{1-i}{2} \hat{\mathbf{k}} \times \mathbf{H}(\mathbf{r}, t)$$

At the boundary of an ideal conductor, the \mathbf{E} and \mathbf{H} fields decay in the direction normal to the interface.

Ideal conductor boundary conditions:

$$\hat{\mathbf{n}} \times \mathbf{E} \Big|_S = 0 \qquad \hat{\mathbf{n}} \cdot \mathbf{H} \Big|_S = 0$$

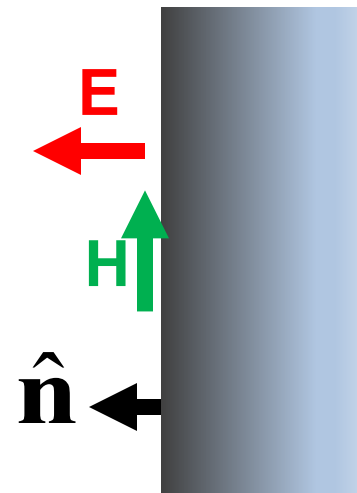


Wave guides – dielectric media with one or more metal boundary

Continuity conditions for fields near metal boundaries --

Ideal conductor boundary conditions:

$$\hat{\mathbf{n}} \times \mathbf{E} \Big|_S = 0 \qquad \hat{\mathbf{n}} \cdot \mathbf{H} \Big|_S = 0$$



Waveguide terminology

- TEM: transverse electric and magnetic (both \mathbf{E} and \mathbf{H} fields are perpendicular to wave propagation direction)
- TM: transverse magnetic (\mathbf{H} field is perpendicular to wave propagation direction)
- TE: transverse electric (\mathbf{E} field is perpendicular to wave propagation direction)

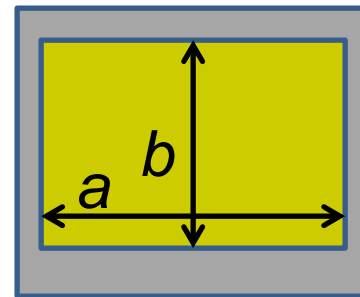
Analysis of rectangular waveguide

Boundary conditions at surface of waveguide:

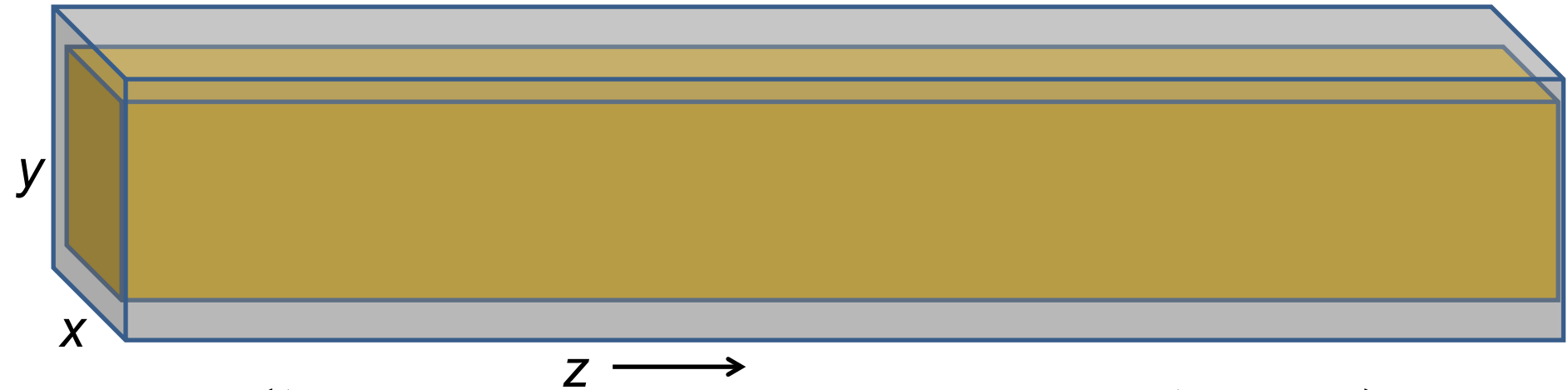
$$\mathbf{E}_{\text{tangential}}=0, \quad \mathbf{B}_{\text{normal}}=0$$



Cross section view



Analysis of rectangular waveguide



$$\mathbf{B} = \Re \left\{ \left(B_x(x, y) \hat{\mathbf{x}} + B_y(x, y) \hat{\mathbf{y}} + B_z(x, y) \hat{\mathbf{z}} \right) e^{ikz - i\omega t} \right\}$$
$$\mathbf{E} = \Re \left\{ \left(E_x(x, y) \hat{\mathbf{x}} + E_y(x, y) \hat{\mathbf{y}} + E_z(x, y) \hat{\mathbf{z}} \right) e^{ikz - i\omega t} \right\}$$

Inside the dielectric medium: (assume ε to be real)

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \times \mathbf{B} - \varepsilon \mu \frac{\partial \mathbf{E}}{\partial t} = 0$$

Solution of Maxwell's equations within the pipe:

Combining Faraday's Law and Ampere's Law, we find that each field component must satisfy a two-dimensional Helmholtz equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k^2 + \mu\epsilon\omega^2 \right) \mathbf{F}(x, y) = 0. \quad \mathbf{F} = \mathbf{E} \text{ or } \mathbf{H}$$

propagation along z.

For the rectangular wave guide discussed in Section 8.4 of your text a solution for a TE mode can have:

$$E_z(x, y) \equiv 0 \quad \text{and} \quad B_z(x, y) = B_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right),$$

$$\text{with } k^2 \equiv k_{mn}^2 = \mu\epsilon\omega^2 - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right]$$

Maxwell's equations within the pipe in terms of all 6 components:

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + ikB_z = 0.$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + ikE_z = 0.$$

$$\frac{\partial E_z}{\partial y} - ikE_y = i\omega B_x.$$

$$ikE_x - \frac{\partial E_z}{\partial x} = i\omega B_y.$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z.$$

For TE mode with $E_z \equiv 0$

$$B_x = -\frac{k}{\omega} E_y$$

$$B_y = \frac{k}{\omega} E_x$$

$$\frac{\partial B_z}{\partial y} - ikB_y = -i\mu\epsilon\omega E_x.$$

$$ikB_x - \frac{\partial B_z}{\partial x} = -i\mu\epsilon\omega E_y.$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -i\mu\epsilon\omega E_z.$$

TE modes for rectangular wave guide continued:

$$E_z(x, y) \equiv 0 \quad \text{and} \quad B_z(x, y) = B_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right),$$

$$E_x = \frac{\omega}{k} B_y = \frac{-i\omega}{k^2 - \mu\epsilon\omega^2} \frac{\partial B_z}{\partial y} = \frac{-i\omega}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]} \frac{n\pi}{b} B_0 \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right),$$

$$E_y = -\frac{\omega}{k} B_x = \frac{i\omega}{k^2 - \mu\epsilon\omega^2} \frac{\partial B_z}{\partial x} = \frac{i\omega}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]} \frac{m\pi}{a} B_0 \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right).$$

Check boundary conditions:

$$\mathbf{E}_{\text{tangential}} = 0 \quad \text{because:} \quad E_z(x, y) \equiv 0, \quad E_x(x, 0) = E_x(x, b) = 0$$
$$\text{and} \quad E_y(0, y) = E_y(a, y) = 0.$$

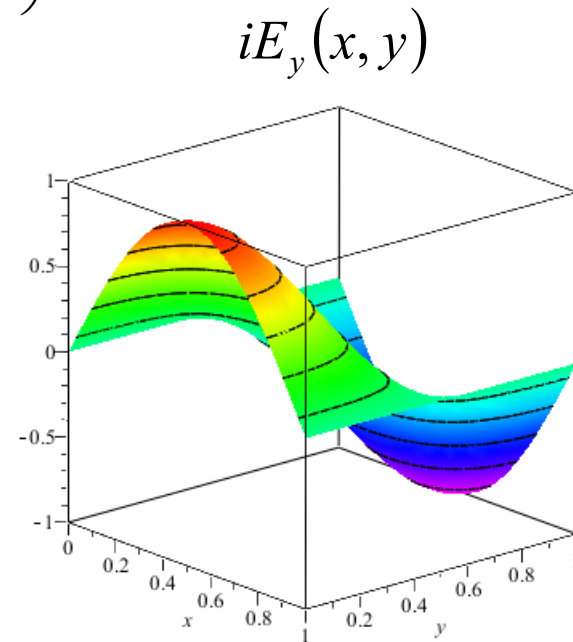
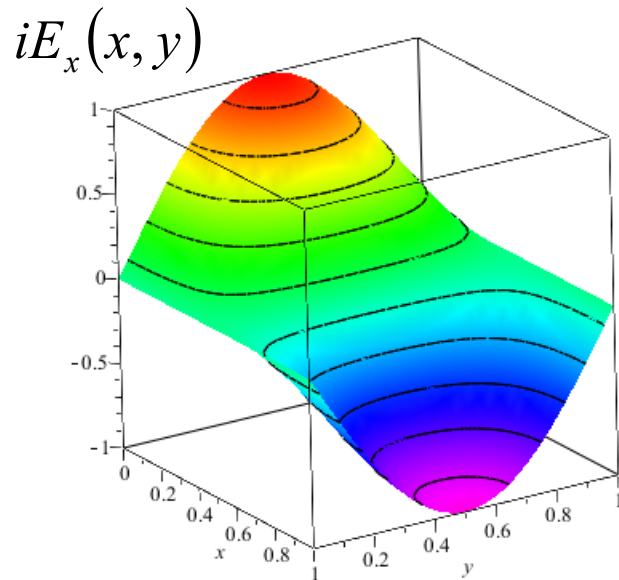
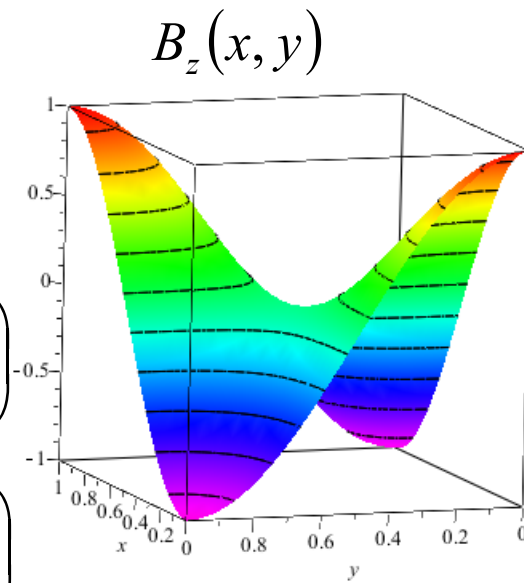
$$\mathbf{B}_{\text{normal}} = 0 \quad \text{because:} \quad B_y(x, 0) = B_y(x, b) = 0$$
$$\text{and} \quad B_x(0, y) = B_x(a, y) = 0.$$

Solution for $m=n=1$

$$B_z(x, y) = B_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

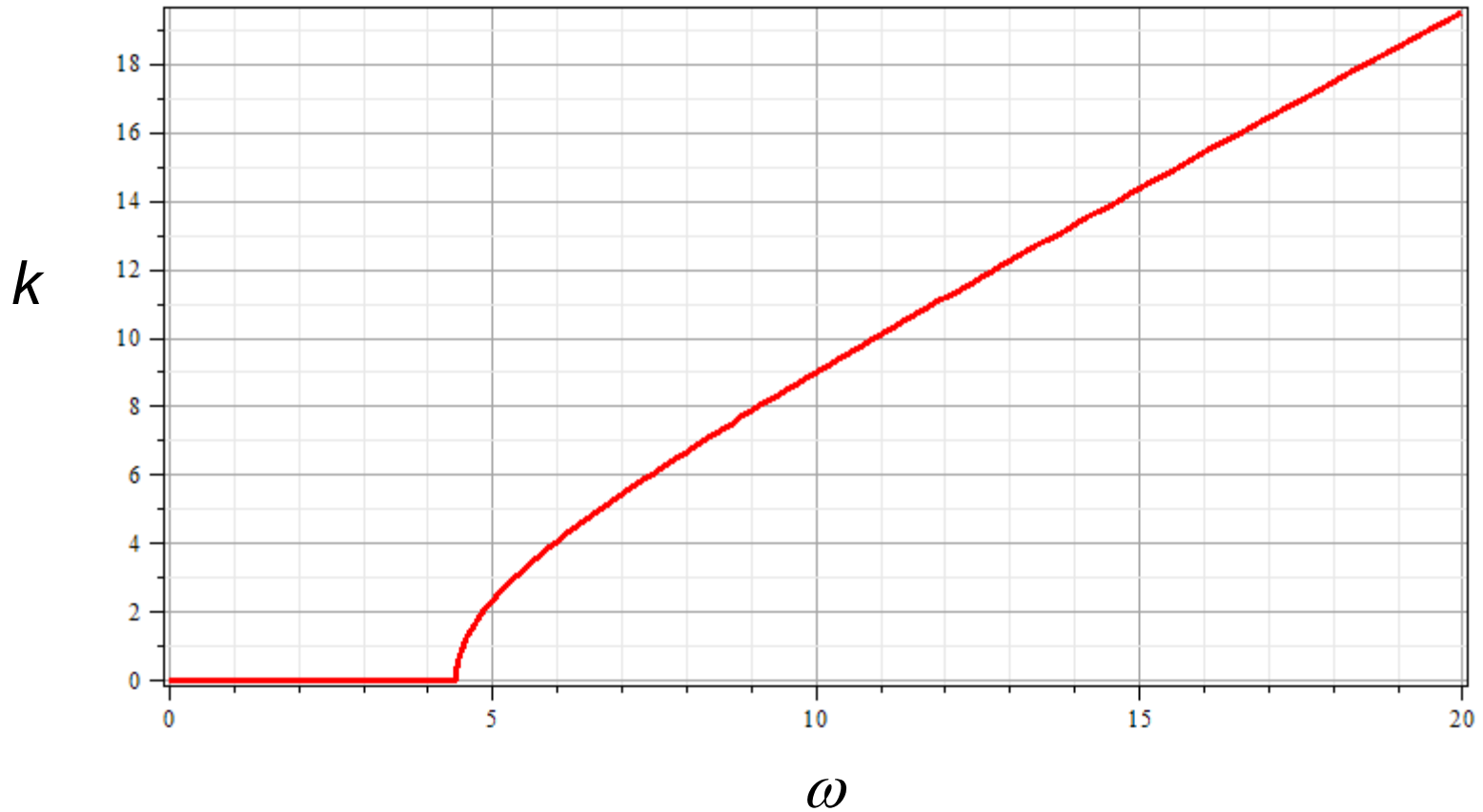
$$iE_x(x, y) = B_0 \left(\frac{\omega n \pi / b}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \right) \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$iE_y(x, y) = B_0 \left(\frac{-\omega m \pi / a}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \right) \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$



Solution for m=n=1

$$k^2 \equiv k_{mn}^2 = \mu\epsilon\omega^2 - \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right]$$



Solution for $m=n=1$ -- more details

$$B_z(x, y) = B_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

$$iE_x(x, y) = B_0 \left(\frac{\omega n\pi / b}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \right) \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) = i \frac{\omega}{k} B_y(x, y)$$

$$iE_y(x, y) = B_0 \left(\frac{-\omega m\pi / a}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \right) \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) = -i \frac{\omega}{k} B_x(x, y)$$

Poynting vector for this case:

$$\langle \mathbf{S} \rangle_{avg} = \frac{1}{2} \Re(\mathbf{E} \times \mathbf{H}^*) = \frac{1}{2} \Re \left(\begin{array}{ccc} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ E_x & E_y & 0 \\ H_x^* & H_y^* & H_z^* \end{array} \right)$$

$$= \frac{\hat{\mathbf{z}}}{2} \Re(E_x H_y^* + E_y H_x^*)$$

(direction along the wave guide)