PHY 712 Electrodynamics
10-10:50 AM MWF Olin 103

Notes for Lecture 25:

Continue reading Chap. 9

A. Electromagnetic waves due to specific sources

B. Dipole radiation examples

C. Radiation from antennas

An Evening with Professor S. James Gates Jr.

Dr. S. James Gates, Department of Physics and School of Public Policy, University of Maryland.

A pioneering theoretical physicist, Dr. Gates is one of the most distinguished scientists in the nation. He is Past President and Fellow of the National Society of Black Physicists and Fellow of the American Physical Society, the American Association for the Advancement of Science, and the Institute of Physics based in the UK. He is an elected member of the American Academy of Arts and Sciences and the American Philosophical Society. He was elected to the National Academy of Sciences in 2013, becoming the first African American theoretical physicist to be recognized in the 150-year history of the Academy. Dr. Gates served on the U.S. President's Council of Advisors on Science and Technology in the Obama administration and was awarded the National Medal of Science by President Obama. Dr. Gates will reflect on his path through supersymmetry in space-time, advising President Obama on Science and Technology, and working to increase diversity in Science.

Admission is free and open to everyone.

Thursday, March 16 at 7:00pm

Wait Chapel

Physics Colloquium

Friday

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March 17, 2023

A Surprising Connection: Mathematical Coloring Problems & Supersymmetry!

The Four Color Map Conjecture was apparently the first example in the mathematical literature where a theorem was created that depended heavily on the existence of computer algorithms and IT applications. A related problem is the Graph Vertex Coloring Problem and recent study shows evidence that supersymmetry is connected with solutions of the Graph Vertex Coloring Problem.

3:00 pm - Olin 101

Reception at 2:30pm - Olin Entrance

Professor S. James Gates, Jr.

Department of Physics and School of Public Policy

University of Maryland

20	Fri: 02/24/2023	Chap. 1-7	Review		
21	Mon: 02/27/2023	Chap. 8	Short lectures on waveguides	Exam	
22	Wed: 03/01/2023	Chap. 8	Short lectures on waveguides	Exam	
23	Fri: 03/03/2023	Chap. 8	Short lectures on waveguides	Exam	03/03/2023
	Mon: 03/06/2023	No class	*Spring Break*		
	Wed: 03/08/2023	No class	*Spring Break*		
	Fri: 03/10/2023	No class	*Spring Break*		
24	Mon: 03/13/2023	Chap. 9	Radiation from localized oscillating sources	#17	03/17/2023
25	Wed: 03/15/2023	Chap. 9	Radiation from oscillating sources		
26	Fri: 03/17/2023	Chap. 9 & 10	Radiation and scattering		

Important results from last time – EM waves from time harmonic sources – open isotropic boundaries

For scalar potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\rho}(\mathbf{r}',\omega),$$

where $\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{\Phi}_0(\mathbf{r},\omega) = 0$

For vector potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \tilde{\mathbf{A}}_0(\mathbf{r},\omega) + \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega),$$

where $\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{\mathbf{A}}_0(\mathbf{r},\omega) = 0$

Useful expansion :

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik\sum_{lm} j_l(kr_<)h_l(kr_>)Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}')$$

Spherical Bessel function : $j_l(kr)$

Spherical Hankel function : $h_l(kr) = j_l(kr) + in_l(kr)$

$$\widetilde{\Phi}(\mathbf{r},\omega) = \widetilde{\Phi}_0(\mathbf{r},\omega) + \sum_{lm}\widetilde{\phi}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\widetilde{\phi}_{lm}(r,\omega) = \frac{ik}{\varepsilon_0}\int d^3r'\,\widetilde{\rho}(\mathbf{r}',\omega)\,j_l(kr_<)h_l(kr_>)Y^*_{lm}(\hat{\mathbf{r}}')$$

$$\widetilde{\mathbf{A}}(\mathbf{r},\omega) = \widetilde{\mathbf{A}}_0(\mathbf{r},\omega) + \sum_{lm}\widetilde{\mathbf{a}}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\widetilde{\mathbf{a}}_{lm}(r,\omega) = ik\mu_0\int d^3r'\,\widetilde{\mathbf{J}}(\mathbf{r}',\omega)\,j_l(kr_<)h_l(kr_>)Y^*_{lm}(\hat{\mathbf{r}}')$$

Example of dipole radiation source

$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} J_0 e^{-r/R} \qquad \tilde{\rho}(\mathbf{r},\omega) = \frac{J_0}{-i\omega R}\cos\theta e^{-r/R}$$

Note that the continuity of charge and current must be satisfied. For the Fourier amplitudes, the relations are as bellow:

Recall continuity condition: $\quad -i\omega\,\tilde{\rho}(\mathbf{r},\omega) + \nabla\cdot\tilde{\mathbf{J}}(\mathbf{r},\omega) = 0$

$$-i\omega\mathbf{r}\;\tilde{\rho}(\mathbf{r},\omega) + \mathbf{r}\nabla\cdot\tilde{\mathbf{J}}(\mathbf{r},\omega)$$

Jackson's clever trick!

$$\int d^3r\;\mathbf{r}\;\tilde{\rho}(\mathbf{r},\omega) = \frac{1}{i\omega}\int d^3r\;\mathbf{r}\nabla\cdot\tilde{\mathbf{J}}(\mathbf{r},\omega)$$

$$= -\frac{1}{i\omega}\int d^3r\;\tilde{\mathbf{J}}(\mathbf{r},\omega) = \mathbf{p}(\omega)$$

Example of dipole radiation source

$$\widetilde{\mathbf{J}}(\mathbf{r},\omega)=\hat{\mathbf{z}}J_0e^{-r/R} \qquad \widetilde{\rho}(\mathbf{r},\omega)=\frac{J_0}{-i\omega R}\cos\theta e^{-r/R}$$

$$\mathbf{p}(\omega)=\hat{\mathbf{z}}\frac{J_0}{-i\omega R}\int d^3r\ r\cos(\theta)\left(\cos(\theta)e^{-r/R}\right)$$

$$=\hat{\mathbf{z}}\frac{J_0}{-i\omega R}\frac{4\pi}{3}\int_0^\infty dr\ r^3e^{-r/R}=\hat{\mathbf{z}}J_0\frac{8\pi R^3}{-i\omega}$$

$$=\frac{1}{-i\omega}\int d^3r\ \tilde{\mathbf{J}}(\mathbf{r},\omega)$$

From the analysis valid for *kr>>1* and *kR<<1*:

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)=-\frac{i\mu_0\omega}{4\pi}\mathbf{p}(\omega)\frac{e^{ikr}}{r}=\hat{\mathbf{z}}J_0\mu_0 2R^3\frac{e^{ikr}}{r}$$

$$\tilde{\Phi}(\mathbf{r},\omega)=-\frac{ik}{4\pi\varepsilon_0}\mathbf{p}(\omega)\cdot\hat{\mathbf{r}}\left(1+\frac{i}{kr}\right)\frac{e^{ikr}}{r}=\frac{J_0 2R^3}{\varepsilon_0 c}\left(1+\frac{i}{kr}\right)\frac{e^{ikr}}{r}\cos\theta$$

Example of dipole radiation source -- exact results for r>>R:

$$\tilde{\mathbf{J}}(\mathbf{r},\omega)=\hat{\mathbf{z}}J_0 e^{-r/R} \qquad \tilde{\rho}(\mathbf{r},\omega)=\frac{J_0}{-i\omega R}\cos\theta e^{-r/R}$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)=\hat{\mathbf{z}}J_0\,(ik\mu_0)\int_0^\infty r'^2\,dr'\,e^{-r'/R}h_0(kr_>)j_0(kr_<)$$

$$\tilde{\Phi}(\mathbf{r},\omega)=-\frac{J_0 k}{\varepsilon_0\omega R}\cos\theta\int_0^\infty r'^2\,dr'\,e^{-r'/R}h_1(kr_>)j_1(kr_<)$$

Evaluation for $r>>R$:

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)=\hat{\mathbf{z}}J_0\mu_0\frac{e^{ikr}}{r}\;\frac{2R^3}{(1+k^2R^2)^2}$$

$$\tilde{\Phi}(\mathbf{r},\omega)=\frac{J_0 k}{\varepsilon_0\omega}\cos\theta\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\;\frac{2R^3}{(1+k^2R^2)^2}$$

Agrees with dipole approximation for *kR<<1*.

More details --

$$\tilde{\mathbf{J}}(\mathbf{r},\omega)=\hat{\mathbf{z}}J_0e^{-r/R} \qquad \tilde{\rho}(\mathbf{r},\omega)=\frac{J_0}{-i\omega R}\cos\theta e^{-r/R}$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)=\hat{\mathbf{z}}J_0\,(ik\mu_0)\int_0^\infty r'^2\,dr'\,e^{-r'/R}h_0(kr_>)j_0(kr_<)$$

$$\tilde{\Phi}(\mathbf{r},\omega)=-\frac{J_0k}{\varepsilon_0\omega R}\cos\theta\int_0^\infty r'^2\,dr'\,e^{-r'/R}h_1(kr_>)j_1(kr_<)$$

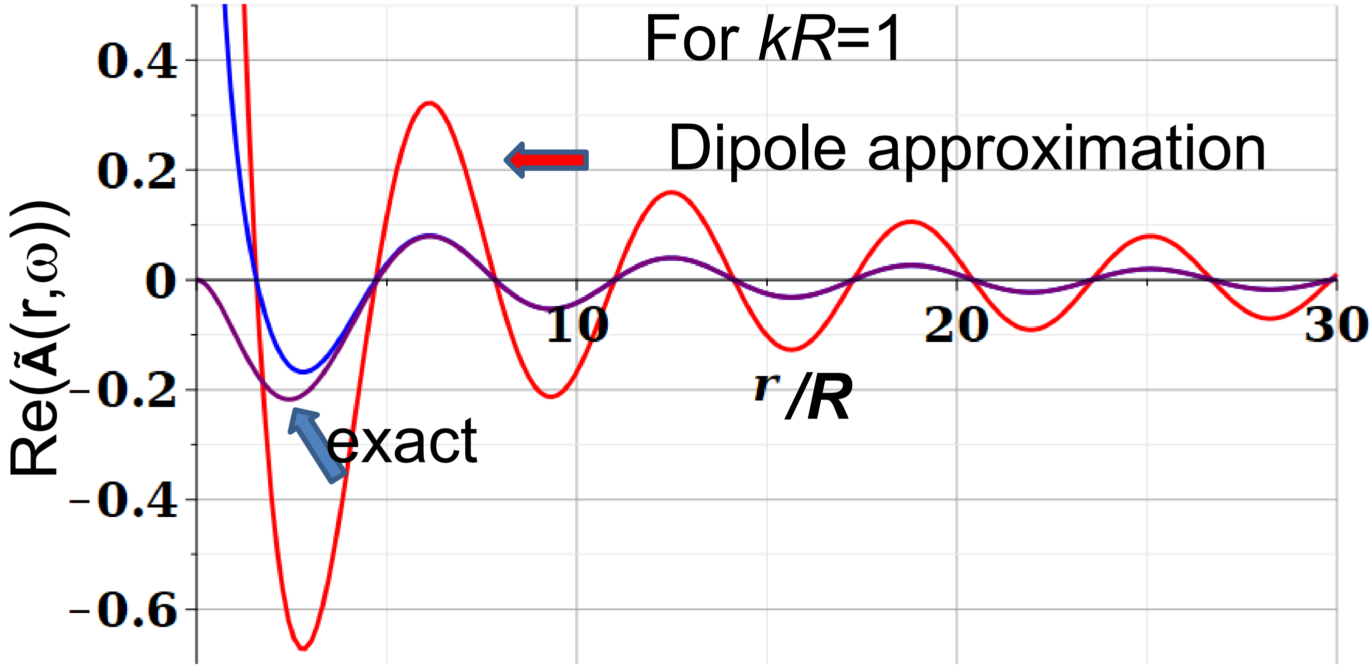
$$\tilde{\mathbf{A}}(r,\omega)=\hat{\mathbf{z}}J_0\mu_0\left(\frac{e^{ikr}}{kr}\int_0^r r'dr'e^{-r'/R}\sin(kr')+\frac{\sin(kr)}{kr}\int_r^\infty r'dr'e^{-r'/R+ikr'}\right)$$

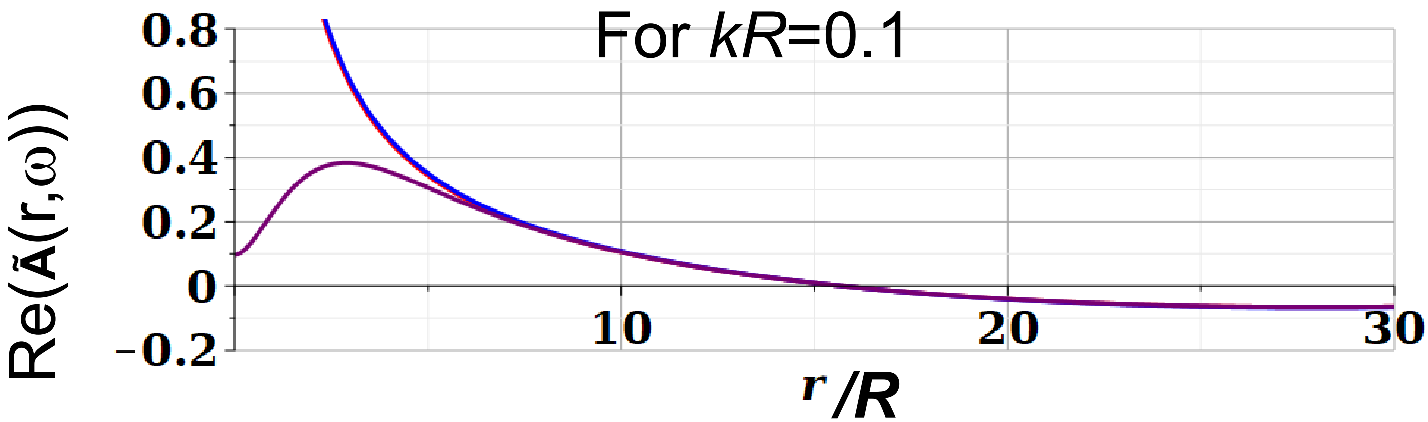
$$\underset{r\gg R}{\approx}\hat{\mathbf{z}}J_0\mu_0\frac{e^{ikr}}{r}\frac{2R^3}{\left(k^2R^2+1\right)^2}$$

Correct when this term is negligible.

$$\tilde{\mathbf{A}}_{\text{dipole approx}}(r,\omega)=\hat{\mathbf{z}}J_0\mu_0\left(\frac{e^{ikr}}{kr}\int_0^\infty r'dr'e^{-r'/R}(kr')\right)=\hat{\mathbf{z}}J_0\mu_0\frac{e^{ikr}}{r}2R^3$$

Example continued





Continued review of dipole results --

Power in the dipole approximation; Section 9.2 of Jackson

Here we use our notation with $\mathbf{n} \rightarrow \hat{\mathbf{r}}$ and $Z_0 \equiv \sqrt{\frac{\mu_0}{\epsilon_0}}$

$$\frac{dP}{d\Omega} = \frac{r^2}{2} \Re \left| \left(\hat{\mathbf{r}} \cdot \left(\mathbf{E} \times \mathbf{H}^* \right) \right) \right|^2$$

Using the expressions for the dipole fields far from the source:

$$\mathbf{H} = \frac{ck^2}{4\pi} \left(\hat{\mathbf{r}} \times \mathbf{p} \right) \frac{e^{ikr}}{r} \qquad \mathbf{E} = Z_0 \mathbf{H} \times \hat{\mathbf{r}}$$

The power can be written $\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 \left| \left(\left(\hat{\mathbf{r}} \times \mathbf{p} \right) \times \hat{\mathbf{r}} \right) \right|^2$

Defining the angle θ by $\mathbf{p} \cdot \hat{\mathbf{r}} = |\mathbf{p}| \cos\theta$,

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 |\mathbf{p}|^2 \sin^2\theta$$ integrating over solid angles $P = \frac{c^2 Z_0}{12\pi} k^4 |\mathbf{p}|^2$

Review:

Electromagnetic waves from time harmonic sources – continued:

Dipole radiation case:

Define dipole moment at frequency ω:

$$\mathbf{p}(\omega) \equiv \int d^3r \; \mathbf{r}\tilde{\rho}(\mathbf{r},\omega) = -\frac{1}{i\omega}\int d^3r \; \tilde{\mathbf{J}}(\mathbf{r},\omega)$$

For fields outside extent of source and $kr' << 1$ within the source:

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = -\frac{i\mu_0\omega}{4\pi}\mathbf{p}(\omega)\frac{e^{ikr}}{r}$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{ik}{4\pi\varepsilon_0}\mathbf{p}(\omega)\cdot\hat{\mathbf{r}}\left(1+\frac{i}{kr}\right)\frac{e^{ikr}}{r}$$

Review:

Electromagnetic waves from time harmonic sources – continued:

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = -\nabla\tilde{\Phi}(\mathbf{r},\omega) + i\omega\tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left(k^2\left(\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\times\hat{\mathbf{r}}\right) + \left(\frac{3\hat{\mathbf{r}}\left(\hat{\mathbf{r}}\cdot\mathbf{p}(\omega)\right)-\mathbf{p}(\omega)}{r^2}\right)(1-ikr)\right)$$

$$\tilde{\mathbf{B}}(\mathbf{r},\omega) = \nabla\times\tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$= \frac{1}{4\pi\varepsilon_0 c}\frac{e^{ikr}}{r}k^2\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\left(1-\frac{1}{ikr}\right)$$

Power radiated for $kr >> 1$:

$$\frac{dP}{d\Omega} = r^2\hat{\mathbf{r}}\cdot\langle\mathbf{S}\rangle_{avg} = \frac{r^2}{2\mu_0}\hat{\mathbf{r}}\cdot\Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\times\tilde{\mathbf{B}}^*(\mathbf{r},\omega)\right)$$

$$= \frac{c^2k^4}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left|\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\times\hat{\mathbf{r}}\right|^2$$

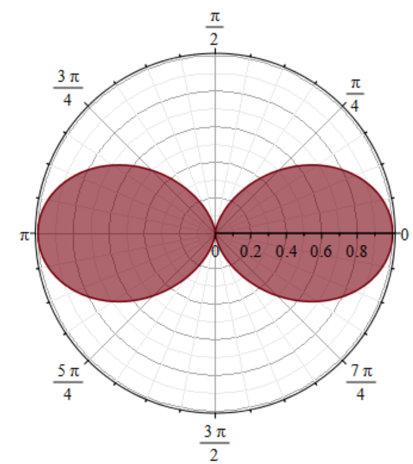
Properties of dipole radiation field for *kr* >>1:

$$\widetilde{\mathbf{E}}(\mathbf{r},\omega)=\frac{1}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left(k^2\left(\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\times\hat{\mathbf{r}}\right)\right)$$

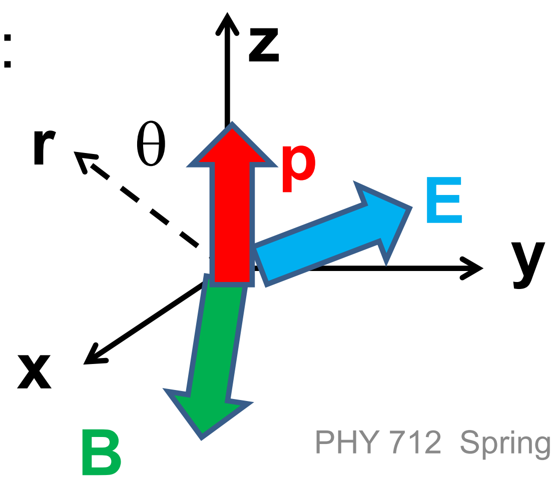
$$\widetilde{\mathbf{B}}(\mathbf{r},\omega)=\frac{1}{4\pi\varepsilon_0 c^2}\frac{e^{ikr}}{r}k^2\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)$$

Power radiated for $kr >> 1$:

$$\frac{dP}{d\Omega}=r^2\hat{\mathbf{r}}\cdot\langle\mathbf{S}\rangle_{avg}=\frac{c^2k^4}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left|\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\times\hat{\mathbf{r}}\right|^2$$



Example:



Note that vectors **r, E, B** are mutually orthogonal

Alternative approach

Fields from time harmonic source:

$$\tilde{\Phi}(\mathbf{r},\omega) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\rho}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

For $r >> r'$: $\quad |\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}' + \ldots$

$$\tilde{\Phi}(\mathbf{r},\omega) \approx \frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\int d^3r'\, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\,\tilde{\rho}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r'\, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\,\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

For our example:

$$\tilde{\mathbf{J}}(\mathbf{r},\omega)=\hat{\mathbf{z}}J_0 e^{-r/R} \qquad \tilde{\rho}(\mathbf{r},\omega)=\frac{J_0}{-i\omega R}\cos\theta e^{-r/R}$$

For $r >> r'$: $\quad |\mathbf{r}-\mathbf{r}'| \approx r-\hat{\mathbf{r}}\cdot\mathbf{r}'+\ldots$

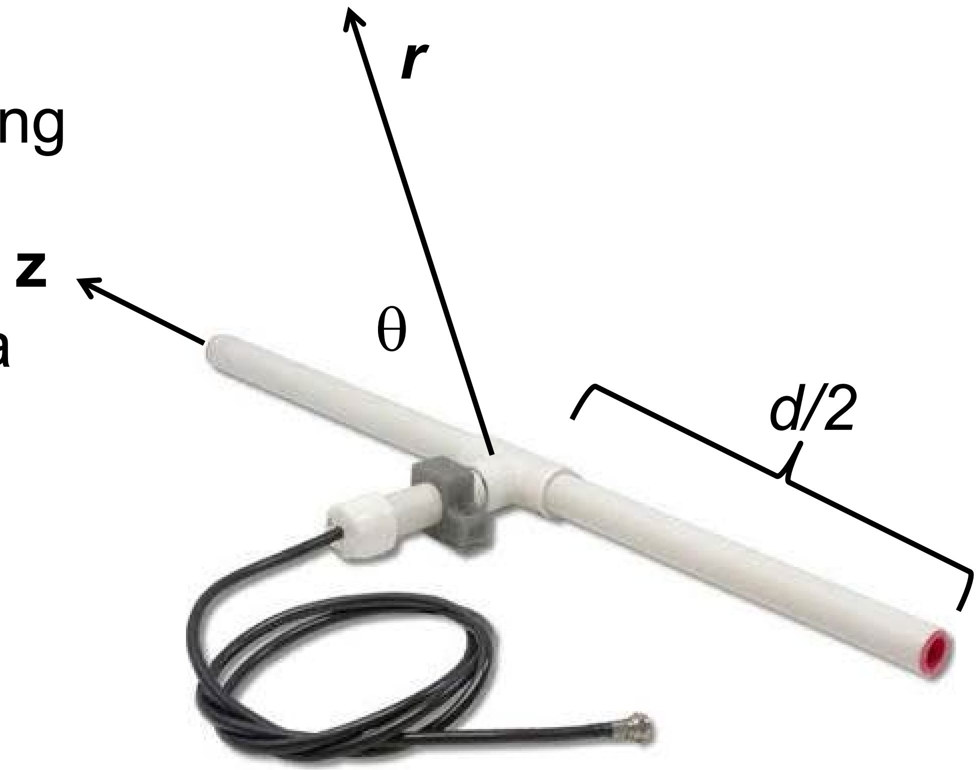
$$\tilde{\Phi}(\mathbf{r},\omega)\approx\frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\int d^3r'\, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\,\tilde{\rho}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)\approx\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r'\, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\,\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

➔Results equivalent to Bessel function expansion in the limit $kr \to \infty$.

Other radiation sources using ``alternative approach"

Linear center-fed antenna



$$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx$$

$$\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r'\, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\;\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{J}}(\mathbf{r}',\omega) = I_0 \sin\left(\frac{kd}{2} - k\,|z|\right)\delta(x)\delta(y)\hat{\mathbf{z}}$$

Alternative approach – linear center-fed antenna continued

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \hat{\mathbf{z}} \frac{\mu_0 I_0}{4\pi} \frac{e^{ikr}}{r} \int_{-d/2}^{d/2} dz'\, e^{-ik\cos(\theta)z'} \sin\left(\frac{kd}{2} - k|z'|\right)$$

$$= \hat{\mathbf{z}} \frac{\mu_0 I_0}{2\pi} \frac{e^{ikr}}{kr} \left(\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin^2\theta} \right)$$

Time averaged power:

$$\frac{dP}{d\Omega} = I_0^{\,2} \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{1}{8\pi^2} \left| \frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta} \right|^2$$

Alternative approach – linear center-fed antenna continued

Time averaged power:

$$\frac{dP}{d\Omega} = I_0^{\,2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{1}{8\pi^2}\left|\frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\left(\frac{kd}{2}\right)}{\sin\theta}\right|^2$$

for $kd = \pi:$

$$\frac{dP}{d\Omega} = I_0^{\,2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{1}{8\pi^2}\frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

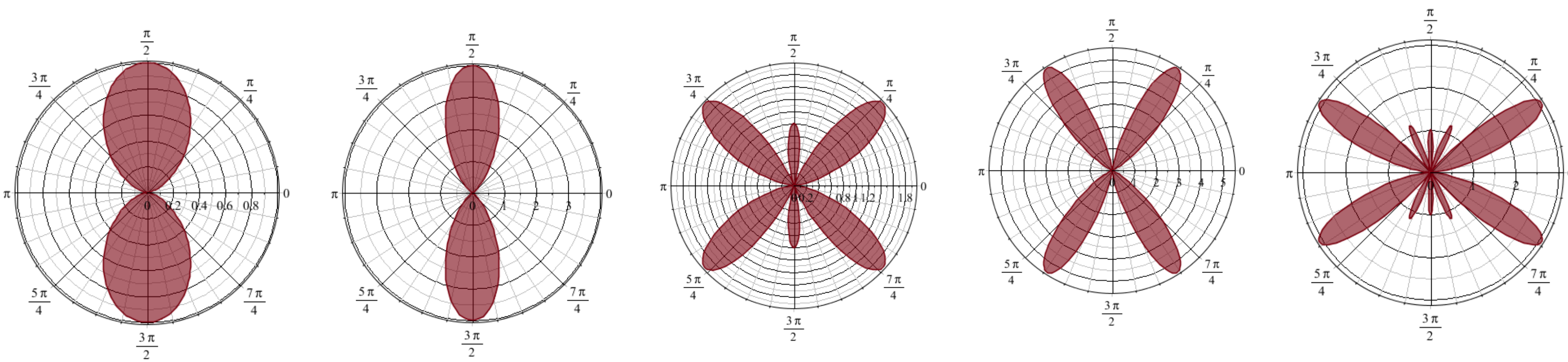
for $kd = 2\pi:$

$$\frac{dP}{d\Omega} = I_0^{\,2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{4}{8\pi^2}\frac{\cos^4\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

Alternative approach – linear center-fed antenna continued

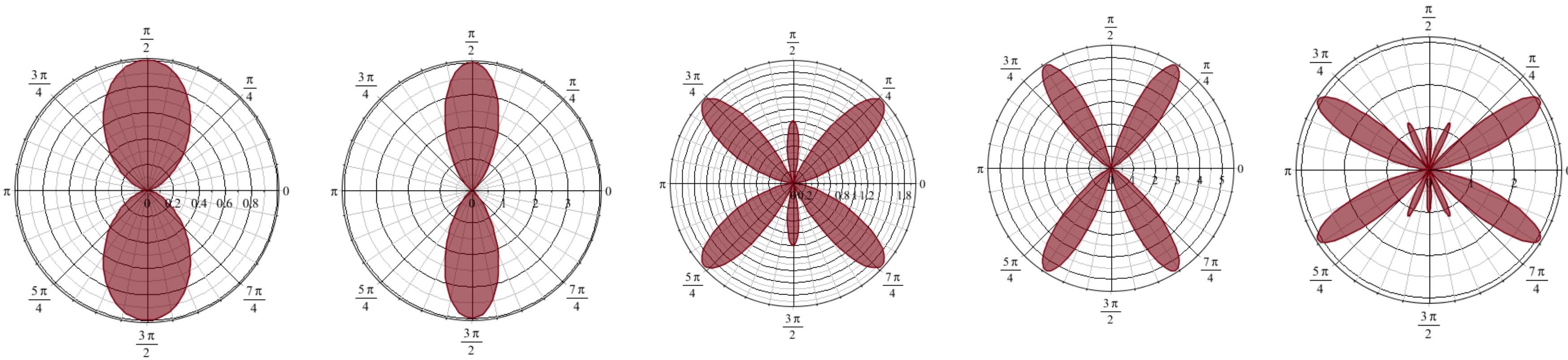
Time averaged power:

$$\frac{dP}{d\Omega} = I_0^{\,2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\,\frac{1}{8\pi^2}\left|\frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\left(\frac{kd}{2}\right)}{\sin\theta}\right|^2$$



kd=π $\quad$ 2π $\quad$ 3π $\quad$ 4π $\quad$ 5π

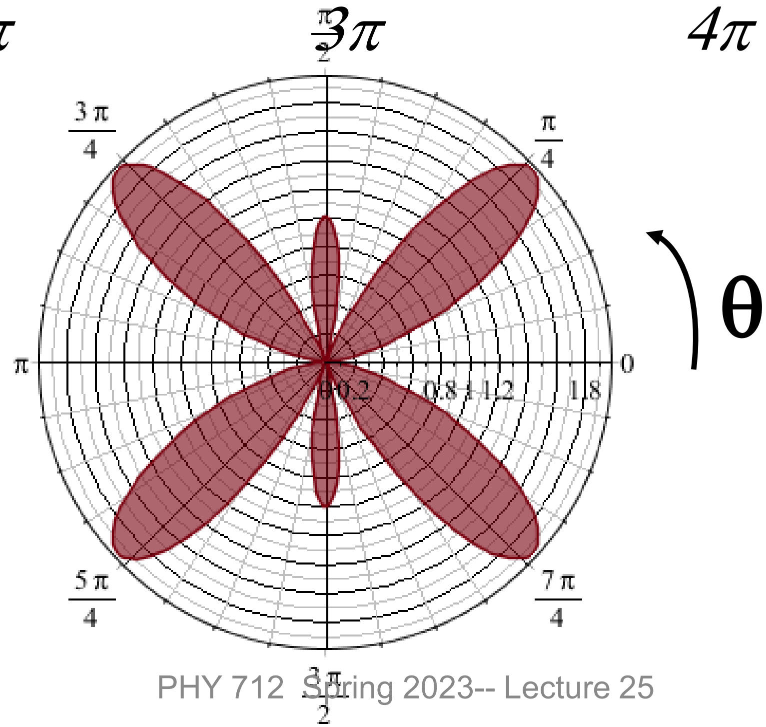
Some details --



kd=π 2π 3π 4π 5π

Polar plot:
Angle indicates values of theta

Radius indicates value scaled to 1.



Next time – we will consider the effects of multiple antennas (antenna arrays including interference effects) and radiation due to light scattering (from Chapter 10 of your textbook).