PHY 712 Electrodynamics
10-10:50 AM MWF in Olin 103

Notes for Lecture 26:

Complete reading of Chap. 9 & 10

A. Superposition of radiation

B. Scattered radiation

24	Mon: 03/13/2023	Chap. 9	Radiation from localized oscillating sources	#17	03/17/2023
25	Wed: 03/15/2023	Chap. 9	Radiation from oscillating sources		
26	Fri: 03/17/2023	Chap. 9 & 10	Radiation and scattering	#18	03/20/2023
27	Mon: 03/20/2023	Chap. 11	Special Theory of Relativity		
28	Wed: 03/22/2023	Chap. 11	Special Theory of Relativity		
29	Fri: 03/24/2023	Chap. 11	Special Theory of Relativity		

PHY 712 -- Assignment #18

March 17, 2023

Finish reading Chapters 9 and 10 in **Jackson** .

1. Work problem 9.16(a) in **Jackson**. Note that you can use an approach similar to that discussed in Section 9.4 of the textbook, replacing the "center-fed" antenna with the given antenna configuration.

Note that your presentations will be given during the week of April 17th.
Please think about your topic for your ~10 min. presentation.

Physics Colloquium

Friday

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March 17, 2023

A Surprising Connection: Mathematical Coloring Problems & Supersymmetry!

The Four Color Map Conjecture was apparently the first example in the mathematical literature where a theorem was created that depended heavily on the existence of computer algorithms and IT applications. A related problem is the Graph Vertex Coloring Problem and recent study shows evidence that supersymmetry is connected with solutions of the Graph Vertex Coloring Problem.

Today at 3 PM in Olin 101

Professor S. James Gates, Jr.
Department of Physics and School of Public Policy
University of Maryland

Electromagnetic waves from time harmonic sources – review:

For scalar potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

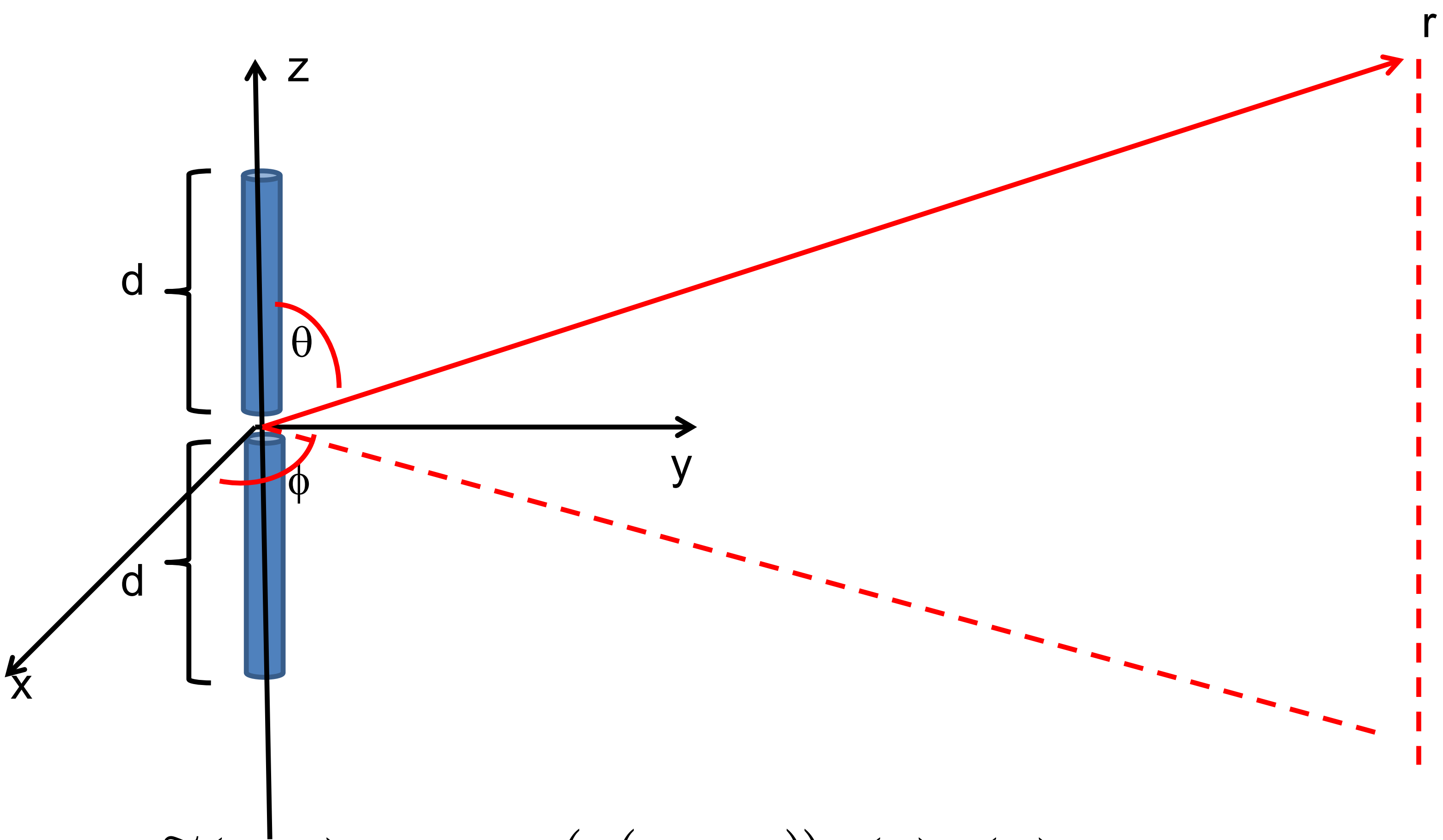
$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\rho}(\mathbf{r}',\omega)$$

For vector potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \tilde{\mathbf{A}}_0(\mathbf{r},\omega) + \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

Consider antenna source (center-fed)

Note – these notes differ from previous formulation d/2 ←→ d



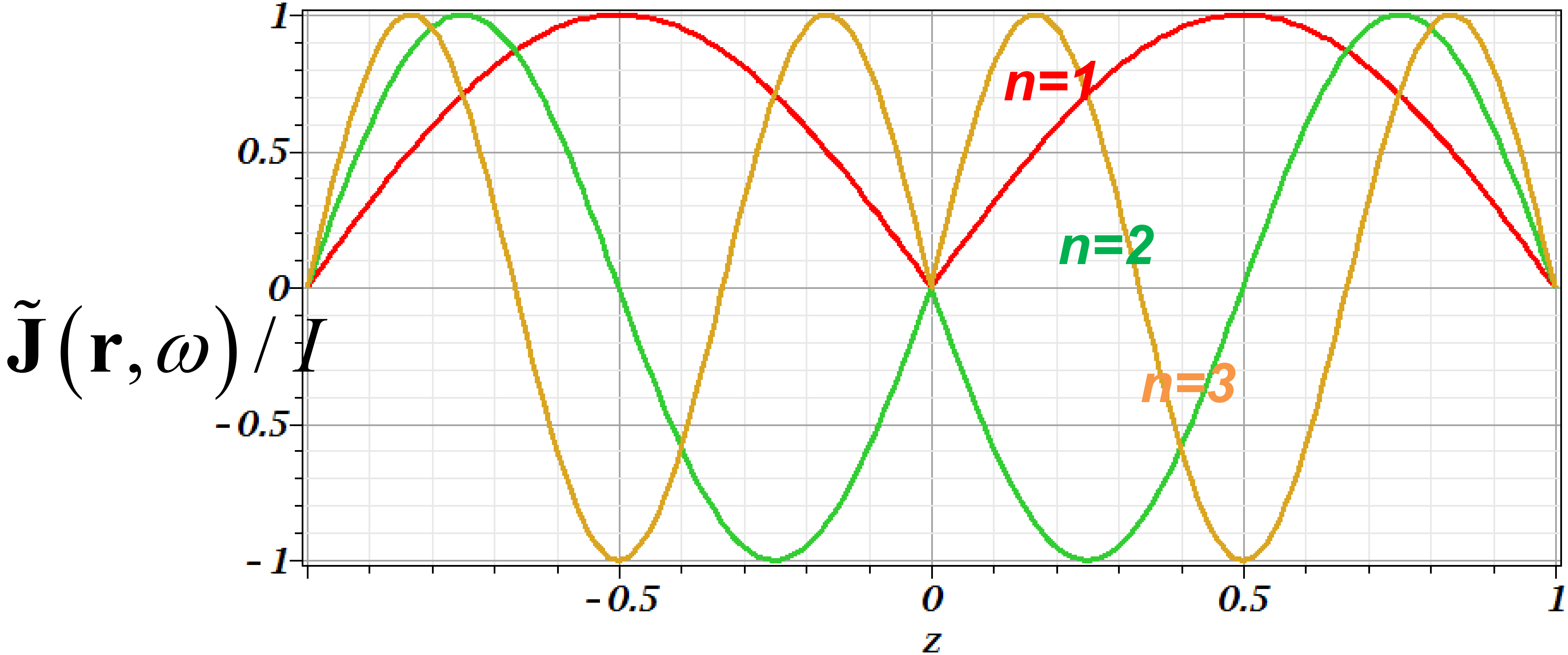
$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} I \sin\left(k\left(d - |z|\right)\right)\delta(x)\delta(y) \quad \text{for } -d \le z \le d$$

$$k \equiv \frac{\omega}{c}$$

Consider antenna source -- continued

$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} I \sin\left(k\left(d-|z|\right)\right)\delta(x)\delta(y) \qquad \text{for} \quad -d \le z \le d$$

$$\text{for} \quad k \equiv \frac{\omega}{c} = \frac{n\pi}{d}; \qquad n = 1, 2, 3....$$



Consider antenna source -- continued

$$\tilde{\mathbf{J}}(\mathbf{r},\omega) = \hat{\mathbf{z}} I \sin\left(k\left(d - |z|\right)\right)\delta(x)\delta(y) \quad \text{for} \quad -d \le z \le d$$

$$k \equiv \frac{\omega}{c}$$

Vector potential from source:

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

For $r >> d$ $$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r' e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r} I \int_{-d}^{d} dz' \, e^{-ikz'\cos\theta} \sin\left(k\left(d - |z'|\right)\right)$$

Consider antenna source -- continued

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \hat{\mathbf{z}} \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} I \int_{-d}^{d} dz\, e^{-ikz\cos\theta} \sin\left(k\left(d - |z|\right)\right)$$

$$= \hat{\mathbf{z}} \frac{\mu_0}{4\pi} \frac{e^{ikr}}{kr} 2I \left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin^2\theta} \right]$$

In the radiation zone:

$$\tilde{\mathbf{B}}(\mathbf{r},\omega) = \nabla \times \tilde{\mathbf{A}}(\mathbf{r},\omega) \approx ik\hat{\mathbf{r}} \times \tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) \approx -ikc\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \tilde{\mathbf{A}}(\mathbf{r},\omega)\right)$$

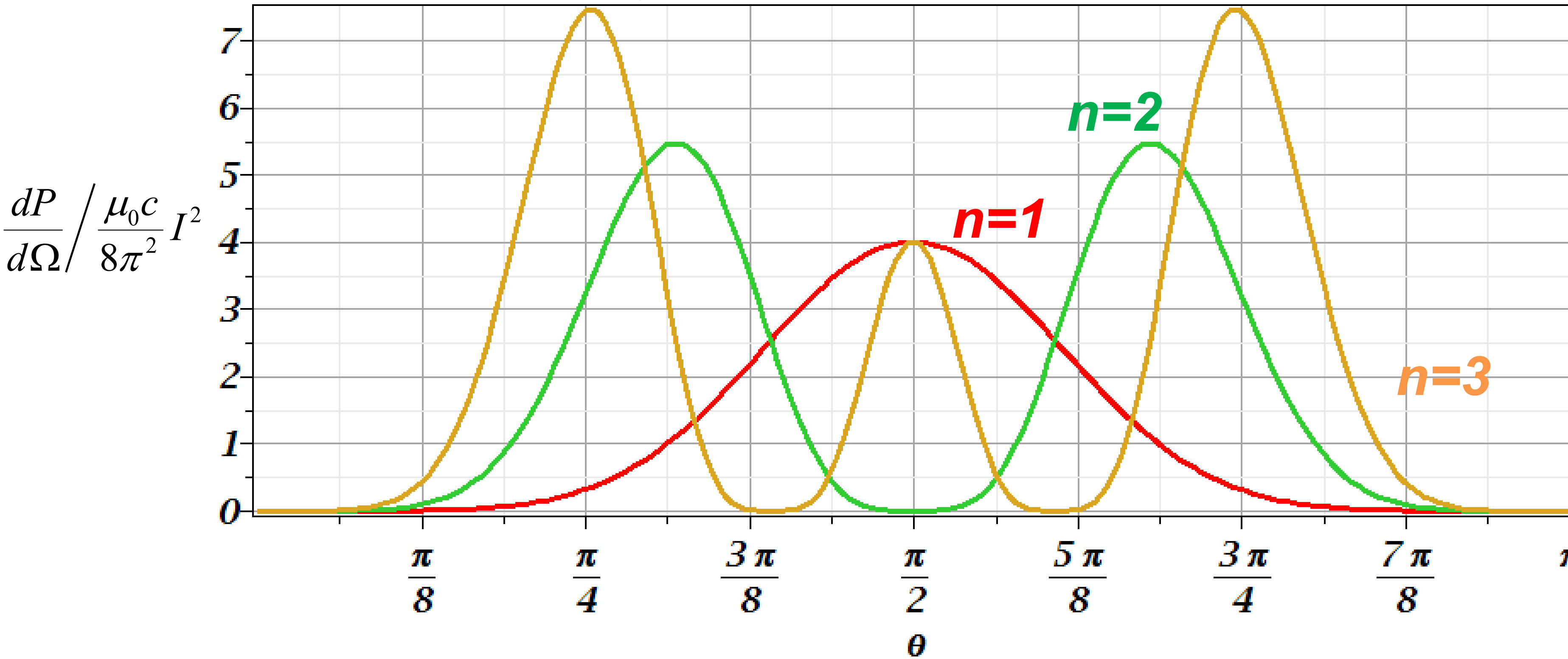
$$\frac{dP}{d\Omega} = \frac{1}{2\mu_0} r^2 \hat{\mathbf{r}} \cdot \Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega) \times \tilde{\mathbf{B}}^*(\mathbf{r},\omega)\right) = \frac{k^2 c}{2\mu_0} r^2 \left(\left|\tilde{\mathbf{A}}(\mathbf{r},\omega)\right|^2 - \left|\hat{\mathbf{r}} \cdot \tilde{\mathbf{A}}(\mathbf{r},\omega)\right|^2 \right)$$

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta} \right]^2$$

Consider antenna source -- continued

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta} \right]^2$$

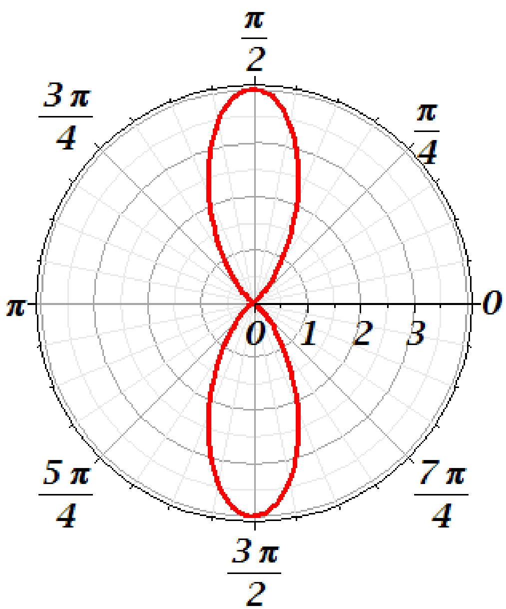
$$\text{for } k \equiv \frac{\omega}{c} = \frac{n\pi}{d}; \qquad n = 1, 2, 3....$$

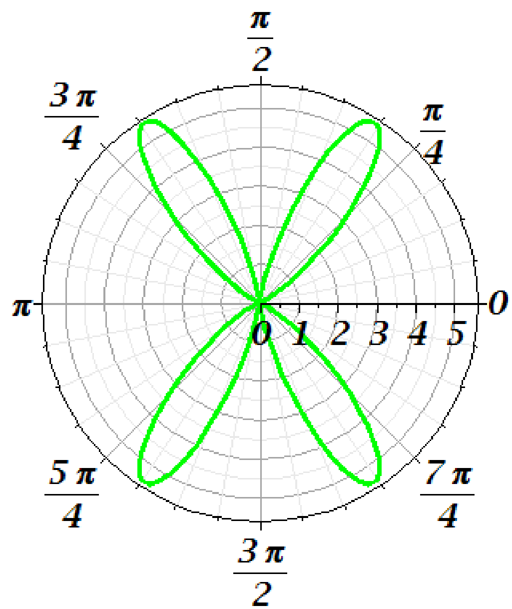


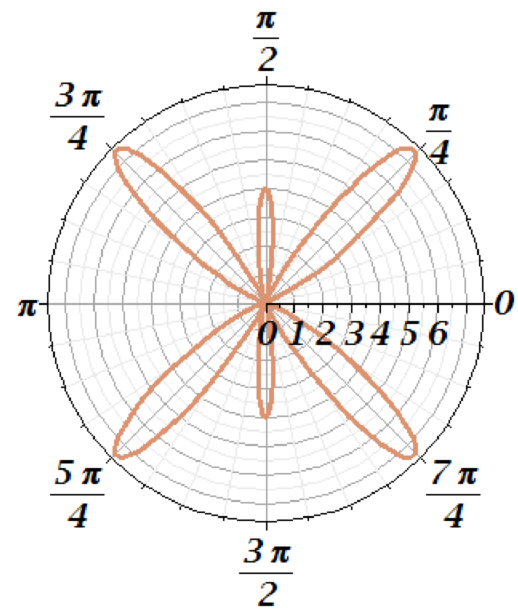
Consider antenna source -- continued

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta} \right]^2$$

For $kd = n\pi$:

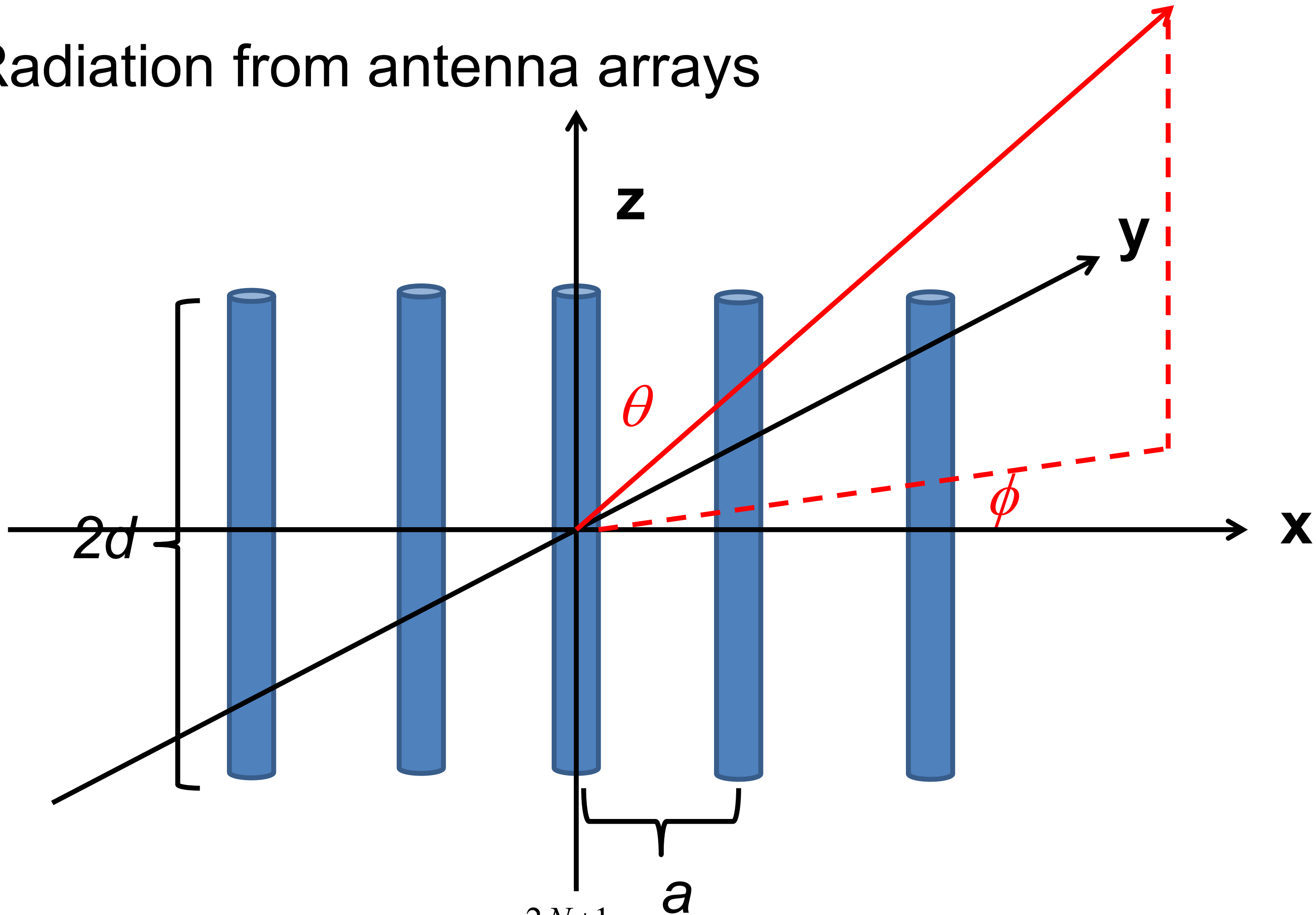






n=1 ***n=2*** ***n=3***

Radiation from antenna arrays



$$\tilde{\mathbf{J}}(\mathbf{r},\omega)=\hat{\mathbf{z}}I\sin\left(k\left(d-|z|\right)\right)\sum_{j=1}^{2N+1}\delta\left(x-\left(N+1-j\right)a\right)\delta(y)\quad \text{for } -d\leq z\leq d$$

$$k\equiv\frac{\omega}{c}=\frac{n\pi}{d};\qquad n=1,2,3....$$

Note that these antennas are all “in phase”.

Radiation from antenna arrays -- continued

Vector potential from array source :

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)=\frac{\mu_0}{4\pi}\int d^3r'\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)\approx\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r' e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{J}}(\mathbf{r},\omega)=\hat{\mathbf{z}}I\sin\left(k\left(d-|z|\right)\right)\sum_{j=1}^{2N+1}\delta\left(x-\left(N+1-j\right)a\right)\delta(y) \quad \text{for } -d\leq z\leq d$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)\approx\hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\sum_{j=-N}^{N}e^{-ikaj\sin\theta\cos\phi}\right)I\int_{-d}^{d}dz\, e^{-ikz\cos\theta}\ \sin\left(k\left(d-|z|\right)\right)$$

$$\sum_{j=-N}^{N}e^{-ikaj\sin\theta\cos\phi}=\frac{\sin\left(\frac{1}{2}ka(2N+1)\sin\theta\cos\phi\right)}{\sin\left(\frac{1}{2}ka\sin\theta\cos\phi\right)}$$

Digression – summation of a geometric series

$$\sum_{j=-N}^{N} e^{-iAj} = e^{-iA}\sum_{j=-N}^{N} e^{-iAj} + e^{iAN} - e^{-iA(N+1)}$$

$$\sum_{j=-N}^{N} e^{-iAj} = \frac{e^{iAN} - e^{-iA(N+1)}}{1-e^{-iA}} = \frac{e^{iA/2}}{e^{iA/2}}\frac{e^{iAN} - e^{-iA(N+1)}}{1-e^{-iA}}$$

$$= \frac{2i\sin\left(A\left(N+1/2\right)\right)}{2i\sin(A/2)}$$

$$= \frac{\sin\left(A\left(N+1/2\right)\right)}{\sin(A/2)}$$

$$\sum_{j=-N}^{N} e^{-ikaj\sin\theta\cos\varphi} = \frac{\sin\left(\frac{1}{2}ka\left(2N+1\right)\sin\theta\cos\varphi\right)}{\sin\left(\frac{1}{2}ka\sin\theta\cos\varphi\right)}$$

Radiation from antenna arrays -- continued

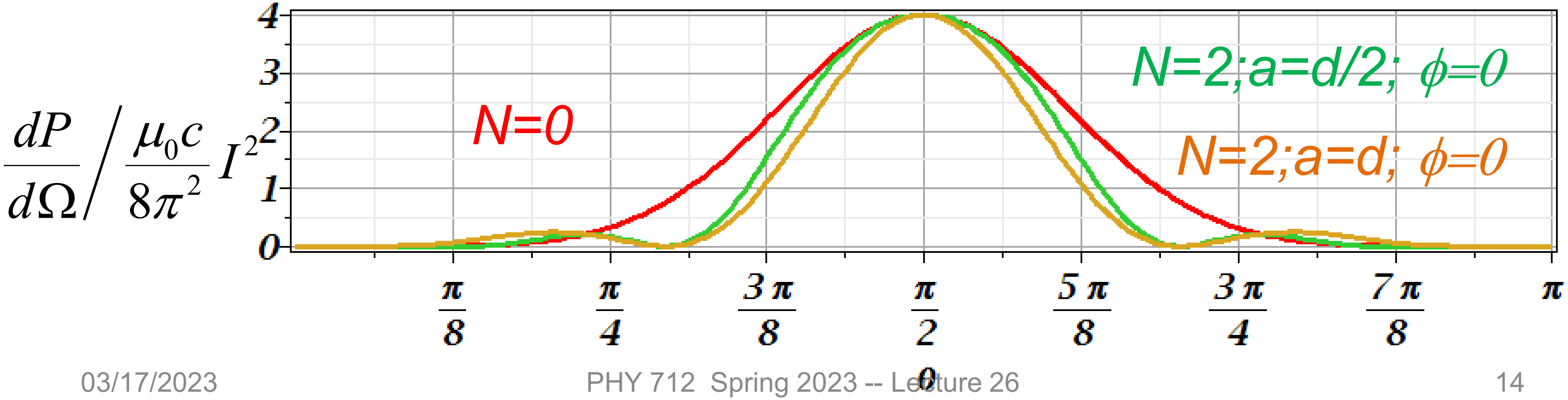
In the radiation zone:

$$\tilde{\mathbf{B}}(\mathbf{r},\omega) = \nabla \times \tilde{\mathbf{A}}(\mathbf{r},\omega) \approx ik\hat{\mathbf{r}} \times \tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) \approx -ikc\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \tilde{\mathbf{A}}(\mathbf{r},\omega)\right)$$

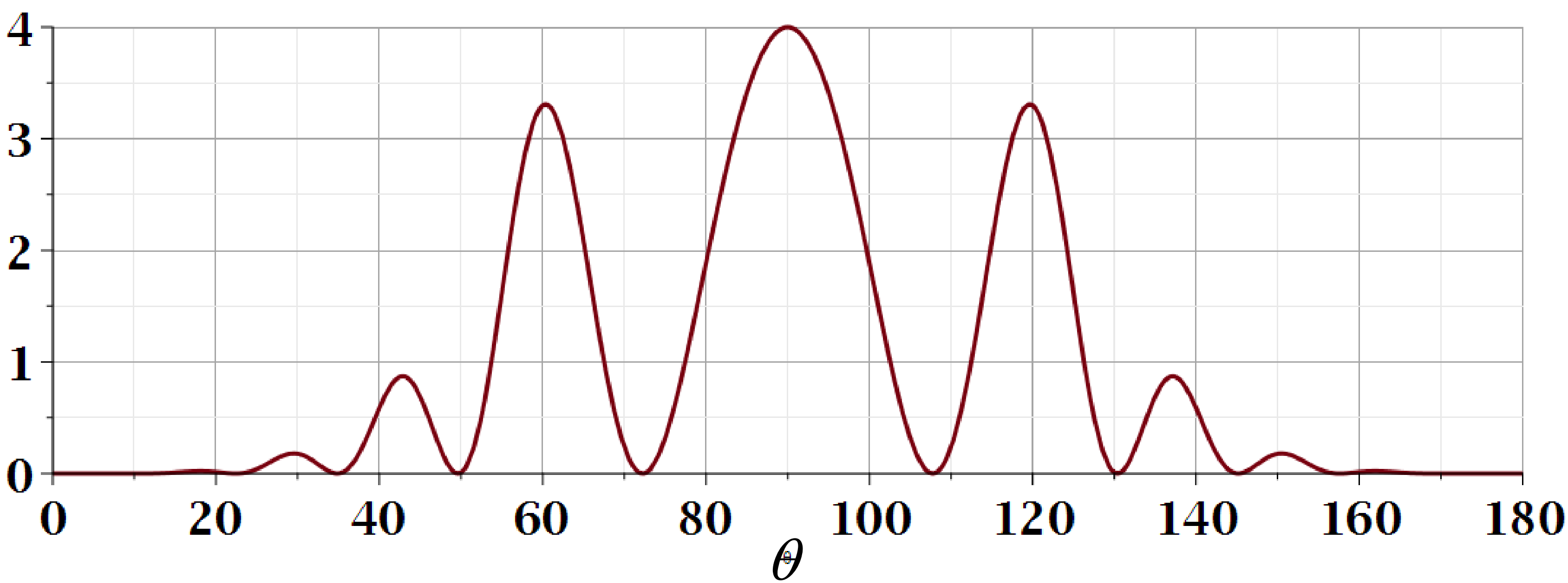
$$\frac{dP}{d\Omega} = \frac{1}{2\mu_0} r^2 \hat{\mathbf{r}} \cdot \Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega) \times \tilde{\mathbf{B}}^*(\mathbf{r},\omega)\right) = \frac{k^2 c r^2}{2\mu_0}\left(\left|\tilde{\mathbf{A}}(\mathbf{r},\omega)\right|^2 - \left|\hat{\mathbf{r}} \cdot \tilde{\mathbf{A}}(\mathbf{r},\omega)\right|^2\right)$$

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta}\right]^2 \left[\frac{\sin\left(\frac{1}{2}ka(2N+1)\sin\theta\cos\phi\right)}{\sin\left(\frac{1}{2}ka\sin\theta\cos\phi\right)}\right]^2$$



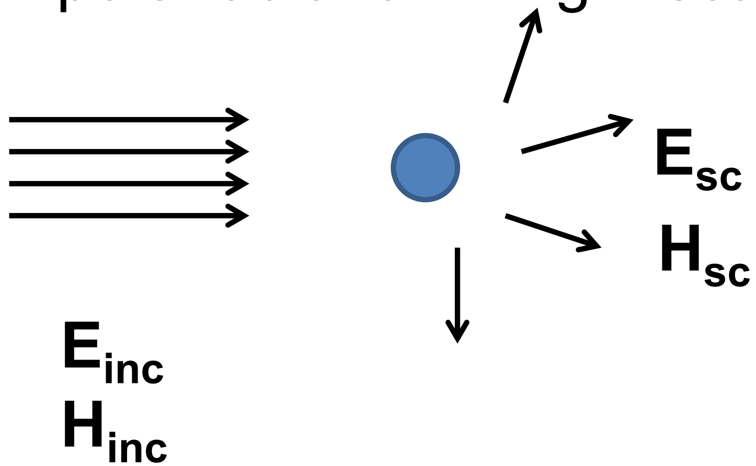
$$\frac{dP}{d\Omega}=\frac{\mu_0 c}{8\pi^2}I^2\left[\frac{\cos(kd\cos\theta)-\cos(kd)}{\sin\theta}\right]^2\left[\frac{\sin\left(\frac{1}{2}ka(2N+1)\sin\theta\cos\varphi\right)}{\sin\left(\frac{1}{2}ka\sin\theta\cos\varphi\right)}\right]^2$$

Example for $\phi=0, N=10, \quad kd=\pi=2ka$



Additional amplitude patterns can be obtained by controlling relative phases of antennas.

Dipole radiation in light scattering by small (dielectric) particles

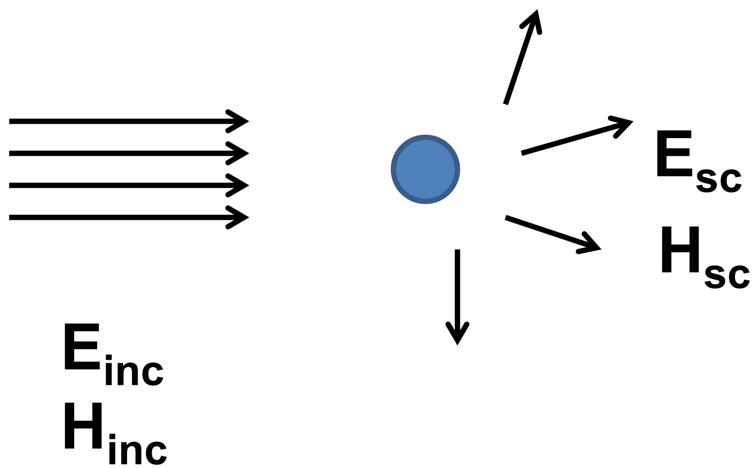


$$\mathbf{E}_{\text{inc}} = \hat{\boldsymbol{\varepsilon}}_0 E_0 e^{ik\hat{\mathbf{k}}_\mathbf{0}\cdot\mathbf{r}} \qquad \mathbf{H}_{\text{inc}} = \frac{1}{\mu_0 c}\hat{\mathbf{k}}_\mathbf{0} \times \mathbf{E}_{\text{inc}}$$

In electric dipole approximation :

$$\mathbf{E}_{\text{sc}} = \frac{1}{4\pi\varepsilon_0}k^2\frac{e^{ikr}}{r}\left(\left(\hat{\mathbf{r}}\times\mathbf{p}\right)\times\hat{\mathbf{r}}\right) \qquad \mathbf{H}_{\text{sc}} = \frac{1}{\mu_0 c}\hat{\mathbf{r}}\times\mathbf{E}_{\text{sc}}$$

Dipole radiation in light scattering by small (dielectric) particles



$$\mathbf{E}_{\text{inc}} = \hat{\mathbf{v}}_0 E_0 e^{ik\hat{\mathbf{k}}_0\cdot\mathbf{r}}$$

$$\mathbf{H}_{\text{inc}} = \frac{1}{\mu_0 c}\hat{\mathbf{k}}_0 \times \mathbf{E}_{\text{inc}}$$

In electric dipole approximation:

$$\mathbf{E}_{\text{sc}} = \frac{1}{4\pi\varepsilon_0}k^2\frac{e^{ikr}}{r}\left(\left(\hat{\mathbf{r}}\times\mathbf{p}\right)\times\hat{\mathbf{r}}\right)$$

$$\mathbf{H}_{\text{sc}} = \frac{1}{\mu_0 c}\hat{\mathbf{r}}\times\mathbf{E}_{\text{sc}}$$

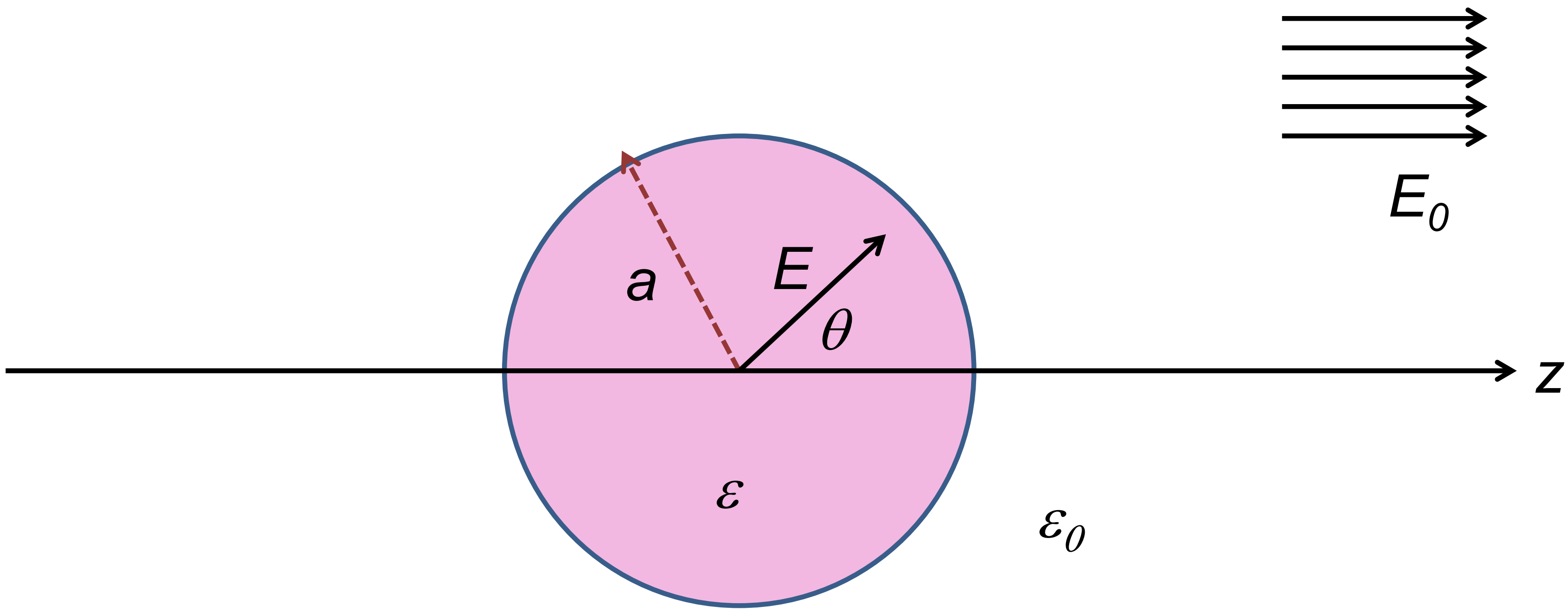
Scattering cross section:

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\mathbf{v}};\hat{\mathbf{k}}_0,\hat{\mathbf{v}}_0\right) = \frac{r^2\hat{\mathbf{r}}\cdot\left\langle\mathbf{S}_{sc}\right\rangle_{avg}}{\hat{\mathbf{k}}_0\cdot\left\langle\mathbf{S}_{inc}\right\rangle_{avg}}$$

$$= \frac{r^2\left|\hat{\mathbf{v}}\cdot\mathbf{E}_{sc}\right|^2}{\left|\hat{\boldsymbol{\varepsilon}}_0\cdot\mathbf{E}_{inc}\right|^2} = \frac{k^4}{\left(4\pi\varepsilon_0 E_0\right)^2}\left|\hat{\mathbf{v}}\cdot\mathbf{p}\right|^2$$

Recall previous analysis for electrostatic case:

Boundary value problems in the presence of dielectrics – example:



$$\text{At } r=a: \quad \varepsilon \frac{\partial \Phi_<(\mathbf{r})}{\partial r} = \varepsilon_0 \frac{\partial \Phi_>(\mathbf{r})}{\partial r}$$

$$\frac{\partial \Phi_<(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_>(\mathbf{r})}{\partial \theta}$$

Boundary value problems in the presence of dielectrics – example -- continued:

$$\Phi_<(\mathbf{r}) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$\Phi_>(\mathbf{r}) = \sum_{l=0}^{\infty} \left(B_l r^l + \frac{C_l}{r^{l+1}} \right) P_l(\cos\theta)$$

At $r = a$: $\quad \varepsilon \frac{\partial \Phi_<(\mathbf{r})}{\partial r} = \varepsilon_0 \frac{\partial \Phi_>(\mathbf{r})}{\partial r}$

$$\frac{\partial \Phi_<(\mathbf{r})}{\partial \theta} = \frac{\partial \Phi_>(\mathbf{r})}{\partial \theta}$$

For $r \to \infty \quad \Phi_>(\mathbf{r}) = -E_0 r \cos\theta$

Solution -- only $l = 1$ contributes

$$B_1 = -E_0$$

$$A_1 = -\left(\frac{3}{2 + \varepsilon/\varepsilon_0} \right) E_0 \qquad C_1 = \left(\frac{\varepsilon/\varepsilon_0 - 1}{2 + \varepsilon/\varepsilon_0} \right) a^3 E_0$$

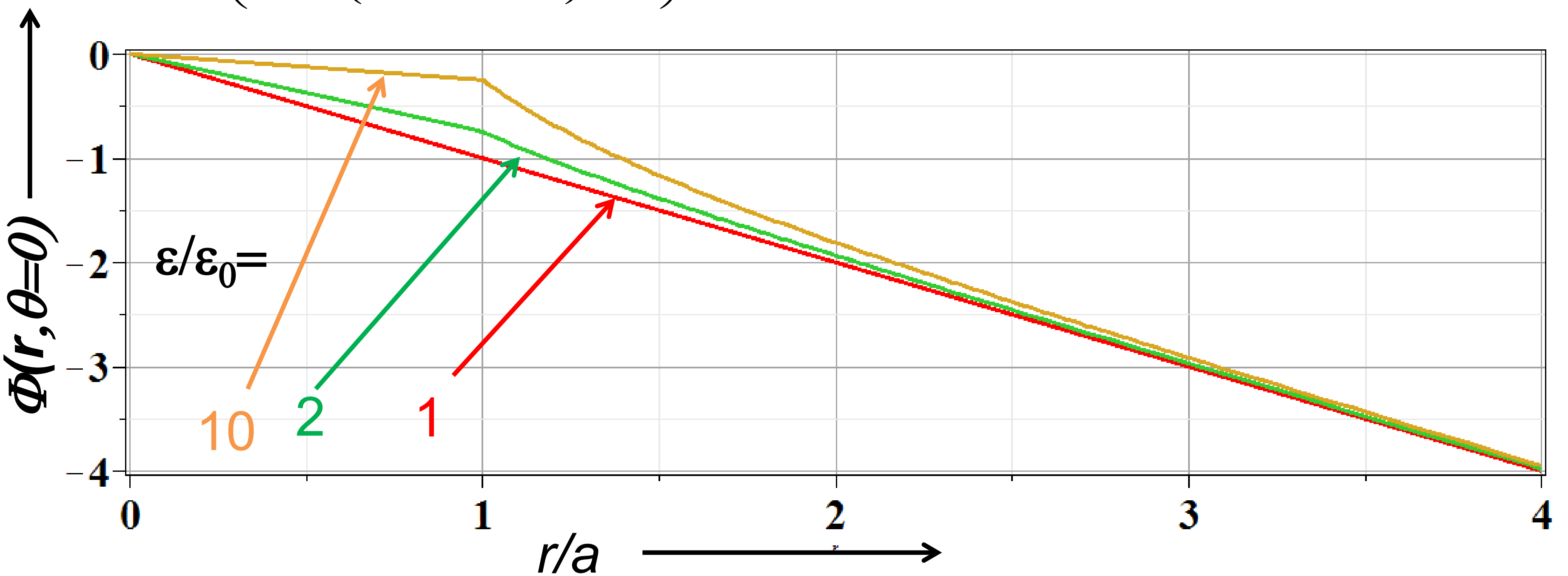
Boundary value problems in the presence of dielectrics – example -- continued:

$$\Phi_<(\mathbf{r}) = -\left(\frac{3}{2+\varepsilon/\varepsilon_0}\right)E_0\, r\cos\theta$$

$$\Phi_>(\mathbf{r}) = -\left(r - \left(\frac{\varepsilon/\varepsilon_0 - 1}{2+\varepsilon/\varepsilon_0}\right)\frac{a^3}{r^2}\right)E_0\cos\theta$$

Induced dipole moment:

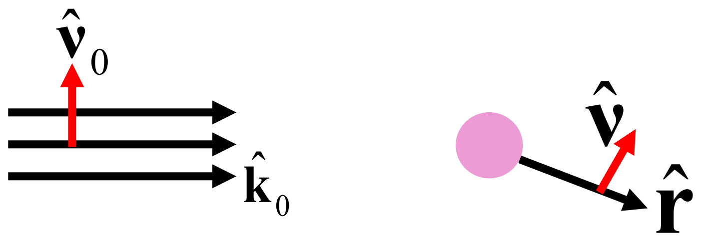
$$\mathbf{p} = 4\pi a^3\varepsilon_0\left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right)\mathbf{E}_0$$



Estimation of scattering dipole moment:

Suppose the scattering particle is a dielectric sphere with permittivity ε and radius *a*:

Note polarization notation change for clarity.



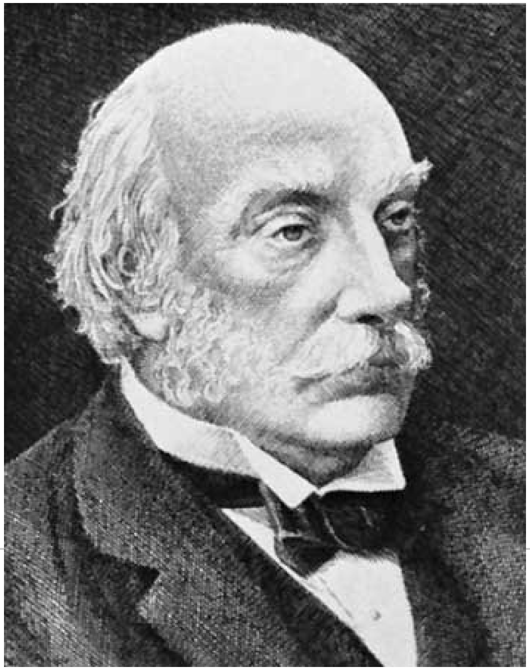
$$\mathbf{p} = 4\pi a^3 \varepsilon_0 \left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right) \mathbf{E}_{inc} \qquad \mathbf{E}_{\text{inc}} = \hat{\mathbf{v}}_0 E_0 e^{ik\hat{\mathbf{k}}_0 \cdot \mathbf{r}}$$

Scattering cross section:

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}}, \hat{\mathbf{v}}; \hat{\mathbf{k}}_0, \hat{\mathbf{v}}_0\right) = \frac{r^2 \left|\hat{\mathbf{v}} \cdot \mathbf{E}_{sc}\right|^2}{\left|\hat{\mathbf{v}}_0 \cdot \mathbf{E}_{inc}\right|^2} = \frac{k^4}{\left(4\pi\varepsilon_0 E_0\right)^2} \left|\hat{\mathbf{v}} \cdot \mathbf{p}\right|^2$$

$$= k^4 a^6 \left| \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right|^2 \left|\hat{\mathbf{v}} \cdot \hat{\mathbf{v}}_0\right|^2$$

https://www.britannica.com/biography/John-William-Strutt-3rd-Baron-Rayleigh



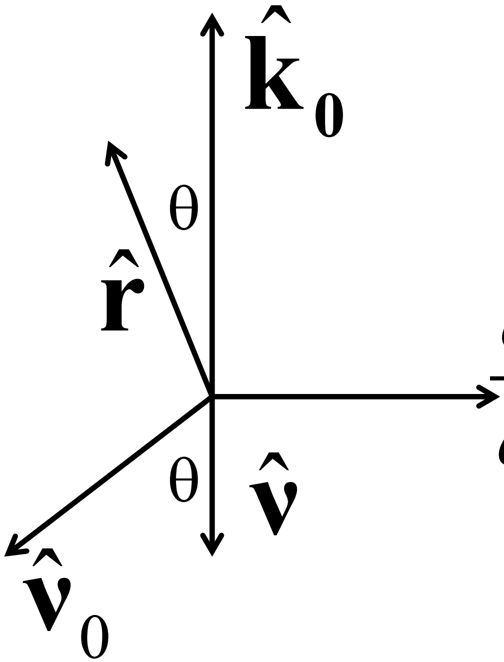
WRITTEN BY: R. Bruce Lindsay
See Article History

Alternative Titles: John William Strutt, 3rd Baron Rayleigh of Terling Place

Lord Rayleigh, in full **John William Strutt, 3rd Baron Rayleigh of Terling Place**, (born November 12, 1842, Langford Grove, Maldon, Essex, England—died June 30, 1919, Terling Place, Witham, Essex), English physical scientist who made fundamental discoveries in the fields of acoustics and optics that are basic to the theory of wave propagation in fluids. He received the Nobel Prize for Physics in 1904 for his successful isolation of argon, an inert atmospheric gas.

Scattering by dielectric sphere with permittivity ε and radius *a*:

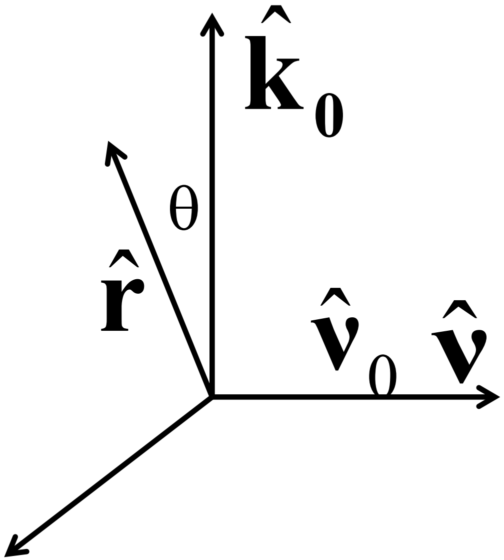
For $\mathbf{E}_{\text{inc}}$ polarized in scattering plane:



$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\mathbf{v}};\hat{\mathbf{k}}_0,\hat{\mathbf{v}}_0\right) == k^4a^6\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2\left|\hat{\mathbf{v}}\cdot\hat{\mathbf{v}}_0\right|^2$$

$$= k^4a^6\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2\cos^2\theta$$

Scattering by dielectric sphere with permittivity ε and radius *a*:



For $\mathbf{E}_{inc}$ polarized perpendicular to scattering plane:

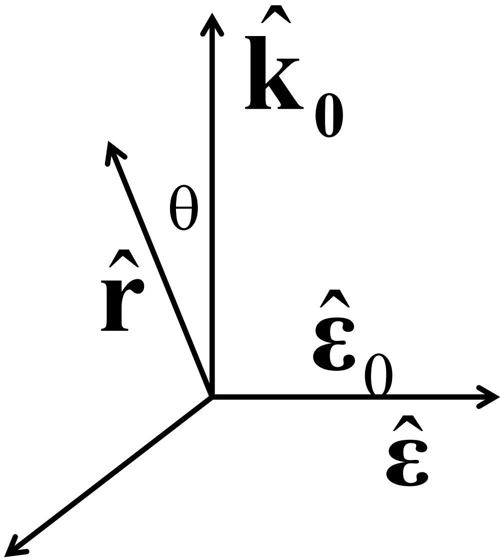
$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\mathbf{v}};\hat{\mathbf{k}}_0,\hat{\mathbf{v}}_0\right)=k^4a^6\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2\left|\hat{\mathbf{v}}\cdot\hat{\mathbf{v}}_0\right|^2$$

$$=k^4a^6\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2$$

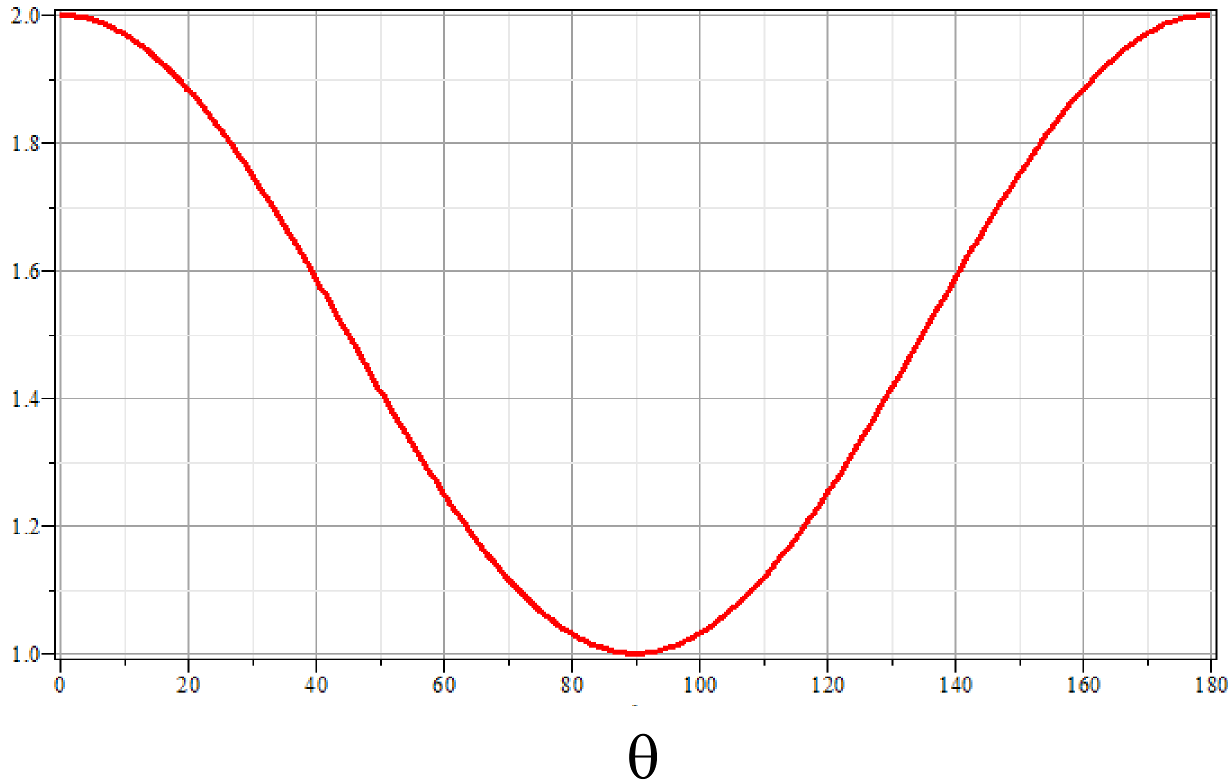
Assuming both incident polarizations are equally likely, average cross section is given by:

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\mathbf{v}};\hat{\mathbf{k}}_0,\hat{\mathbf{v}}_0\right)=\frac{k^4a^6}{2}\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2\left(\cos^2\theta+1\right)$$

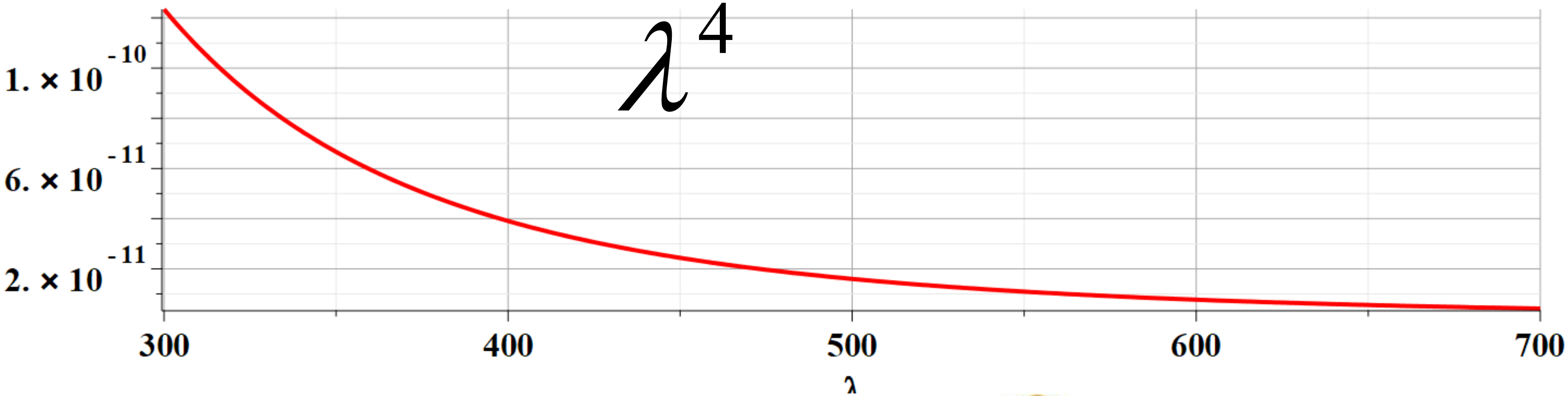
Scattering by dielectric sphere with permittivity ε and radius *a*:

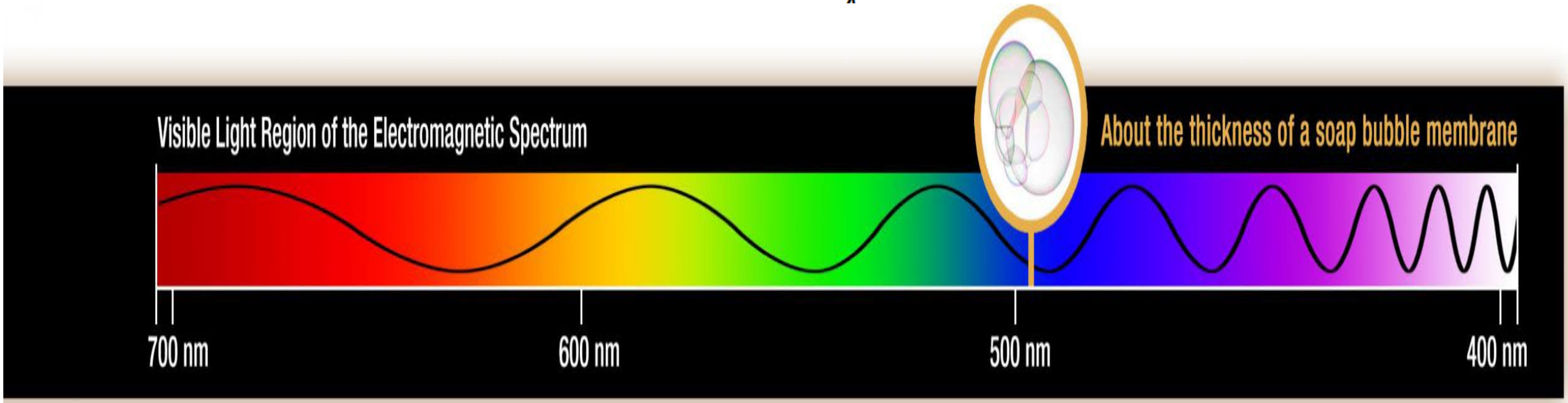


$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\mathbf{v}};\hat{\mathbf{k}}_0,\hat{\mathbf{v}}_0\right)=\frac{k^4a^6}{2}\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2\left(\cos^2\theta+1\right)$$



$$\frac{1}{\lambda^4}$$





Brief introduction to multipole expansion of electromagnetic fields (Chap. 9.7)

Sourceless Maxwell's equations

in terms of **E** and **H** fields with time dependence $e^{-i\omega t}$:

$$\nabla \times \mathbf{E} = ikZ_0\mathbf{H} \qquad \nabla \times \mathbf{H} = -ik\mathbf{E} / Z_0$$

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{H} = 0$$

where $k \equiv \omega / c$ and $Z_0 \equiv \sqrt{\mu_0 / \epsilon_0}$

Decoupled equations:

$$\left(\nabla^2 + k^2\right)\mathbf{E} = 0 \qquad \left(\nabla^2 + k^2\right)\mathbf{H} = 0$$

$$\mathbf{H} = -\frac{i}{kZ_0}\nabla \times \mathbf{E} \qquad \mathbf{E} = \frac{iZ_0}{k}\nabla \times \mathbf{H}$$

Multipole expansion of electromagnetic fields -- continued

Note that:

$$\left(\nabla^2 + k^2\right)\left(\mathbf{r}\cdot\mathbf{E}\right) = 0 \qquad\qquad \left(\nabla^2 + k^2\right)\left(\mathbf{r}\cdot\mathbf{H}\right) = 0$$

Convenient operators for angular momentum analysis

Define: $\mathbf{L} \equiv \frac{1}{i}\left(\mathbf{r}\times\nabla\right)$

Note that $\mathbf{r}\cdot\mathbf{L} = 0$

$$\nabla^2 = \frac{1}{r}\frac{\partial^2 r}{\partial r^2} - \frac{L^2}{r^2}$$

Eigenfunctions:

$$L^2 Y_{lm}(\theta,\phi) = -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]Y_{lm}(\theta,\phi) = l(l+1)Y_{lm}(\theta,\phi)$$

Multipole expansion of electromagnetic fields -- continued

Magnetic multipole field:

$$\mathbf{r} \cdot \mathbf{H}_{lm}^{M} \equiv \frac{l(l+1)}{k} g_l(kr) Y_{lm}(\theta,\phi)$$

$$\mathbf{r} \cdot \mathbf{E}_{lm}^{M} = 0$$

$$\mathbf{L} \cdot \mathbf{E}_{lm}^{M} = l(l+1) Z_0 g_l(kr) Y_{lm}(\theta,\phi)$$

spherical Bessel function

Electric multipole field:

$$\mathbf{r} \cdot \mathbf{E}_{lm}^{E} \equiv -Z_0 \frac{l(l+1)}{k} f_l(kr) Y_{lm}(\theta,\phi)$$

$$\mathbf{r} \cdot \mathbf{H}_{lm}^{E} = 0$$

$$\mathbf{L} \cdot \mathbf{H}_{lm}^{E} = l(l+1) f_l(kr) Y_{lm}(\theta,\phi)$$

spherical Bessel function

Multipole expansion of electromagnetic fields -- continued

Vector spherical harmonics: (for $l > 0$)

$$\mathbf{X}_{lm}(\theta,\phi) = \frac{1}{\sqrt{l(l+1)}}\mathbf{L}Y_{lm}(\theta,\phi)$$

Orthogonality conditions:

$$\int d\Omega\ \mathbf{X}_{l'm'}^{*}(\theta,\phi)\cdot\mathbf{X}_{lm}(\theta,\phi) = \delta_{ll'}\delta_{mm'}$$

$$\int d\Omega\ \mathbf{X}_{l'm'}^{*}(\theta,\phi)\cdot\left(\mathbf{r}\times\mathbf{X}_{lm}(\theta,\phi)\right) = 0$$

General expansion of fields:

$$\mathbf{H} = \sum_{lm}\left[a_{lm}^{E} f_l(kr)\mathbf{X}_{lm}(\theta,\phi) - \frac{i}{k}a_{lm}^{M}\nabla\times\left(g_l(kr)\mathbf{X}_{lm}(\theta,\phi)\right)\right]$$

$$\mathbf{E} = \sum_{lm}\left[\frac{i}{k}a_{lm}^{E}\nabla\times\left(f_l(kr)\mathbf{X}_{lm}(\theta,\phi)\right) + a_{lm}^{M} g_l(kr)\mathbf{X}_{lm}(\theta,\phi)\right]$$

Multipole expansion of electromagnetic fields -- continued

Time averaged power distribution of radiation far from source:

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}\left|\sum_{lm}(-i)^{l+1}\left[a^E_{lm}\mathbf{X}_{lm}(\theta,\phi)\times\hat{\mathbf{r}} + a^M_{lm}\mathbf{X}_{lm}(\theta,\phi)\right]\right|^2$$

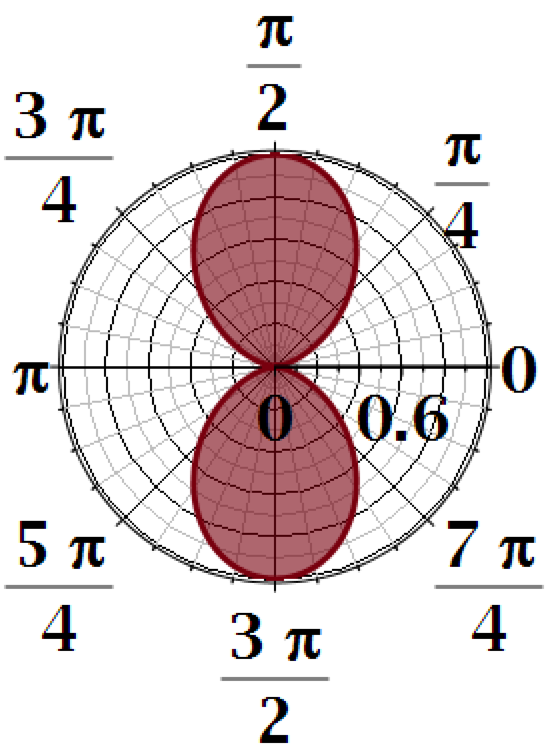
For a pure multipole radiation with either a^E_{lm} or a^M_{lm} :

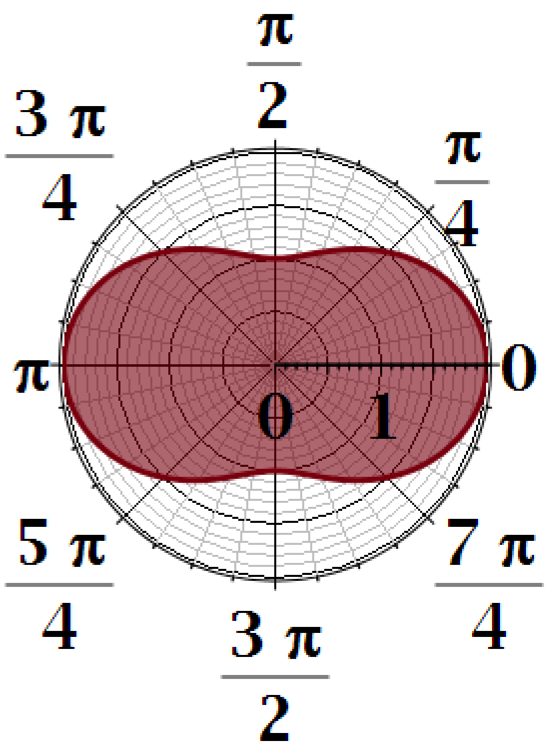
$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}\left|a_{lm}\right|^2\left|\mathbf{X}_{lm}(\theta,\phi)\right|^2$$

$$\left|\mathbf{X}_{lm}(\theta,\phi)\right|^2 = \frac{1}{2l(l+1)}\left(2m^2\left|Y_{lm}\right|^2 + \left(l+m\right)\left(l-m+1\right)\left|Y_{l(m-1)}\right|^2 + \left(l-m\right)\left(l+m+1\right)\left|Y_{l(m+1)}\right|^2\right)$$

For example: $l = 1$

$$\left|\mathbf{X}_{10}(\theta,\phi)\right|^2 = \frac{3}{8\pi}\sin^2\theta \qquad \left|\mathbf{X}_{11}(\theta,\phi)\right|^2 = \left|\mathbf{X}_{1-1}(\theta,\phi)\right|^2 = \frac{3}{16\pi}\left(1+\cos^2\theta\quad\right)$$





For example: $l = 2$

$$|\mathbf{X}_{20}(\theta,\phi)|^2 = \frac{15}{8\pi}\sin^2\theta\,\cos^2\theta \qquad |\mathbf{X}_{21}(\theta,\phi)|^2 = \frac{5}{16\pi}\left(1 - 3\cos^2\theta + 4\cos^4\theta\right) \qquad |\mathbf{X}_{22}(\theta,\phi)|^2 = \frac{5}{16\pi}\left(1 - \cos^4\theta\right)$$