PHY 712 Electrodynamics
10-10:50 AM MWF in Olin 103

Discussion for Lecture 27:

Start reading Chap. 11

A. Equations in cgs (Gaussian) units

B. Special theory of relativity

C. Lorentz transformation relations

24	Mon: 03/13/2023	Chap. 9	Radiation from localized oscillating sources	#17	03/17/2023
25	Wed: 03/15/2023	Chap. 9	Radiation from oscillating sources		
26	Fri: 03/17/2023	Chap. 9 & 10	Radiation and scattering	#18	03/20/2023
27	Mon: 03/20/2023	Chap. 11	Special Theory of Relativity	#19	03/24/2023
28	Wed: 03/22/2023	Chap. 11	Special Theory of Relativity		
29	Fri: 03/24/2023	Chap. 11	Special Theory of Relativity		

PHY 712 -- Assignment #19

March 20, 2023

Start reading Chapters 11 in **Jackson** .

1. In class, we examined measurements of quantities that could be measured in two different reference frames which were related by the Lorentz transformation matrix. In particular, we are interested in the velocity components such as u_x and and u'_x. For the geometry used in class, work out the algebraic relationships that show that the 4-components of the vector $(\gamma_u c, \gamma_u u_x, \gamma_u u_y, \gamma_u u_z)$ are related by a Lorentz transformation to the corresponding 4-component vector measured in the moving frame -- $(\gamma_{u'} c, \gamma_{u'} u'_x, \gamma_{u'} u'_y, \gamma_{u'} u'_z)$.

Units - SI vs Gaussian

Below is a table comparing SI and Gaussian unit systems. The fundamental units for each system are so labeled and are used to define the derived units.

Variable	SI		Gaussian		SI/Gaussian
	Unit	Relation	Unit	Relation	
length	m	fundamental	cm	fundamental	100
mass	kg	fundamental	gm	fundamental	1000
time	s	fundamental	s	fundamental	1
force	N	$kg \cdot m^2/s$	$dyne$	$gm \cdot cm^2/s$	10^5
current	A	fundamental	$statampere$	$statcoulomb/s$	$\frac{1}{10c}$
charge	C	$A \cdot s$	$statcoulomb$	$\sqrt{dyne \cdot cm^2}$	$\frac{1}{10c}$

Basic equations of electrodynamics

CGS (Gaussian)	SI
$\nabla \cdot \mathbf{D} = 4\pi\rho$	$\nabla \cdot \mathbf{D} = \rho$
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$
$\mathbf{F} = q(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})$	$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$
$u = \frac{1}{8\pi}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$	$u = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$
$\mathbf{S} = \frac{c}{4\pi}(\mathbf{E} \times \mathbf{H})$	$\mathbf{S} = (\mathbf{E} \times \mathbf{H})$

More relationships

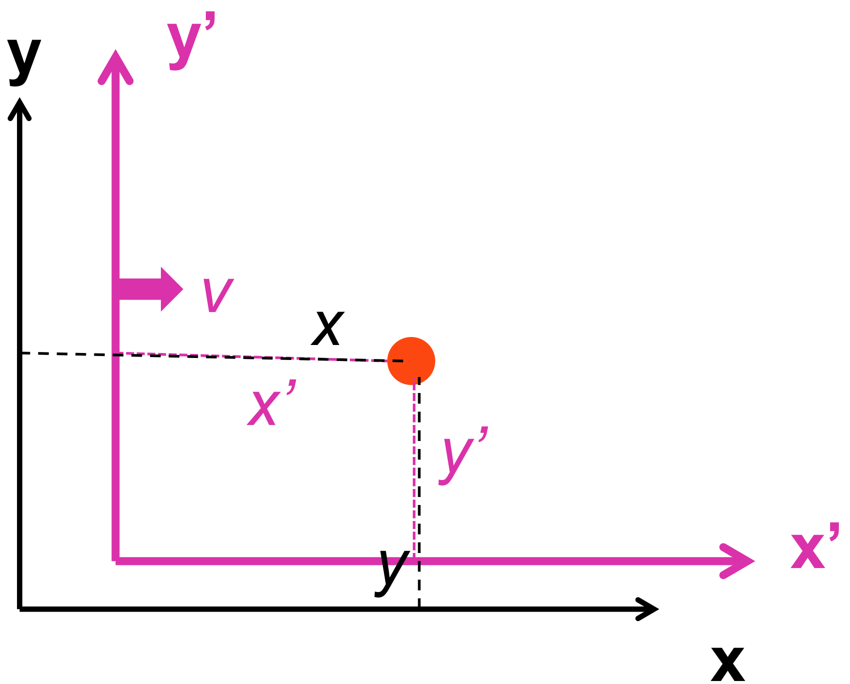
CGS (Gaussian)	MKS (SI)
$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = \epsilon\mathbf{E}$	$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon\mathbf{E}$
$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} = \frac{1}{\mu}\mathbf{B}$	$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} = \frac{1}{\mu}\mathbf{B}$
$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}$	$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}$
$\mathbf{B} = \nabla\times\mathbf{A}$	$\mathbf{B} = \nabla\times\mathbf{A}$

$$\epsilon \quad \Leftrightarrow \quad \epsilon/\epsilon_0$$

$$\mu \quad \Leftrightarrow \quad \mu/\mu_0$$

Notions of special relativity

- The basic laws of physics are the same in all frames of reference (at rest or moving at constant velocity).
- The speed of light in vacuum *c* is the same in all frames of reference.

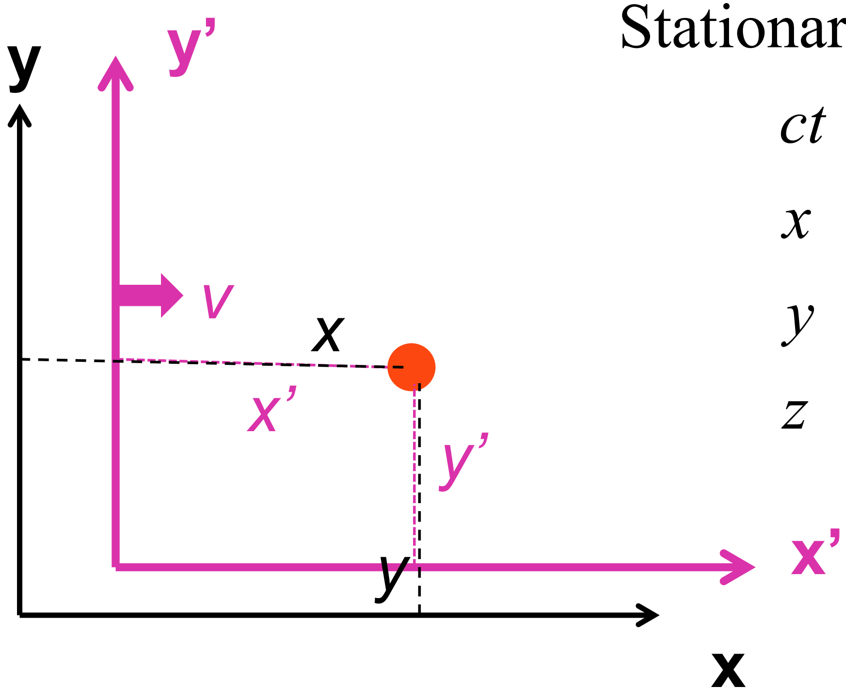


Lorentz transformations

Convenient notation :

$$\beta \equiv \frac{v}{c}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$



Stationary frame		Moving frame
ct	$=$	$\gamma(ct'+\beta x')$
x	$=$	$\gamma(x'+\beta ct')$
y	$=$	y'
z	$=$	z'

Lorentz transformations -- continued

$$\beta \equiv \frac{v}{c} \qquad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\mathcal{L} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathcal{L}^{-1} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \mathcal{L} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} \qquad \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Notice:

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2$$

Examples of other 4-vectors applicable to the Lorentz transformation:

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\beta \equiv \frac{v}{c} \qquad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

$$\mathcal{L} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathcal{L}^{-1} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix} = \mathcal{L} \begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} \qquad \begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix}$$

Note: $\omega t - \mathbf{k}\cdot\mathbf{r} = \omega' t' - \mathbf{k}'\cdot\mathbf{r}'$

In free space:

$$\left(\frac{\omega}{c}\right)^2 - k^2 = \left(\frac{\omega'}{c}\right)^2 - k'^2 = 0$$

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \mathcal{L} \begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} \qquad \begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix}$$

Note: $E^2 - p^2c^2 = E'^2 - p'^2 c^2$

The Doppler Effect

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\mathcal{L} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathcal{L}^{-1} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix} = \mathcal{L} \begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} \qquad \begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix}$$

Note: $\omega t - \mathbf{k}\cdot\mathbf{r} = \omega' t' - \mathbf{k}'\cdot\mathbf{r}'$

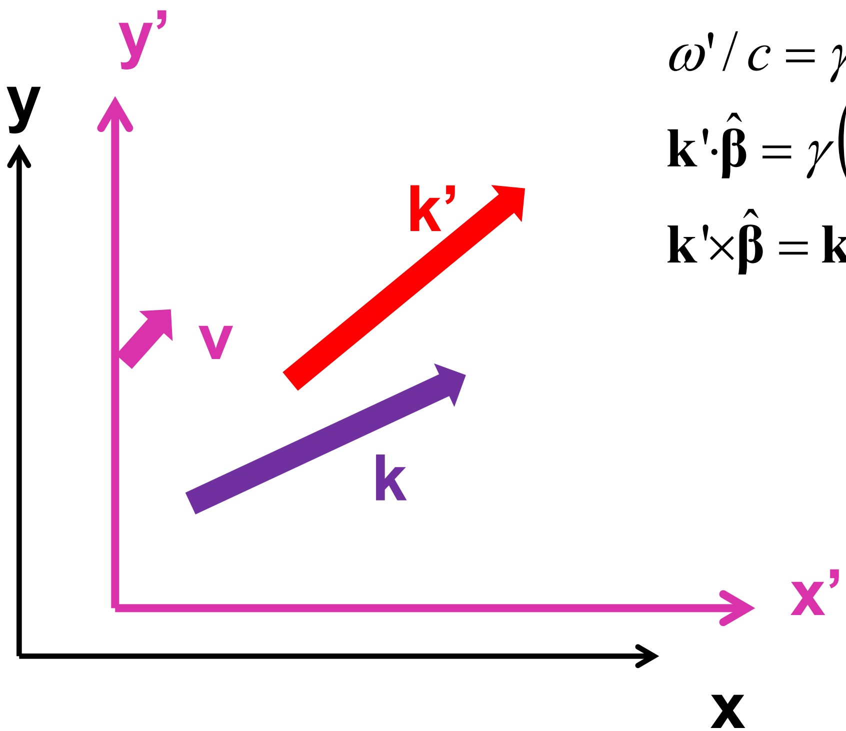
$$\omega'/c = \gamma\left(\omega/c - \beta k_x\right) \qquad k'_x = \gamma\left(k_x - \beta\omega/c\right)$$

$$k'_y = k_y \qquad k'_z = k_z$$

The Doppler Effect -- continued

$$\omega'/c = \gamma(\omega/c - \beta k_x) \qquad k'_x = \gamma(k_x - \beta\omega/c)$$

$$k'_y = k_y \qquad k'_z = k_z$$



More generally:

$$\omega'/c = \gamma(\omega/c - \boldsymbol{\beta}\cdot\mathbf{k}) \equiv \gamma(\omega/c - \beta k\cos\theta)$$

$$\mathbf{k}'\cdot\hat{\boldsymbol{\beta}} = \gamma(\mathbf{k}\cdot\hat{\boldsymbol{\beta}} - \beta\omega/c) \equiv k'\cos\theta' = \gamma(k\cos\theta - \beta\omega/c)$$

$$\mathbf{k}'\times\hat{\boldsymbol{\beta}} = \mathbf{k}\times\hat{\boldsymbol{\beta}}$$

For $\theta = 0$: $\quad (k = \omega/c)$

$$\omega' = \omega\gamma(1-\beta) \qquad \Rightarrow \omega' = \omega\sqrt{\frac{1-\beta}{1+\beta}}$$

For $\theta \neq 0$: $\quad (k = \omega/c)$

$$\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - \beta)}$$

Electromagnetic Doppler Effect (θ=0)

$$\omega' = \omega\sqrt{\frac{1-\beta}{1+\beta}} \qquad \beta = \frac{v_{\text{source}} - v_{\text{detector}}}{c\left(1 - \frac{v_{\text{source}} v_{\text{detector}}}{c^2}\right)}$$

(details concerning velocities in the following slides.)

Sound Doppler Effect (θ=0)

$$\omega' = \omega\left(\frac{1 \pm v_{\text{detector}}/c_s}{1 \mp v_{\text{source}}/c_s}\right)$$

Lorentz transformation of the velocity

Stationary frame		Moving frame
ct	$=$	$\gamma(ct' + \beta x')$
x	$=$	$\gamma(x' + \beta ct')$
y	$=$	y'
z	$=$	z'

For an infinitesimal increment:

Stationary frame		Moving frame
cdt	$=$	$\gamma(cdt' + \beta dx')$
dx	$=$	$\gamma(dx' + \beta cdt')$
dy	$=$	dy'
dz	$=$	dz'

Lorentz transformation of the velocity -- continued

Stationary frame		Moving frame
cdt	$=$	$\gamma(cdt'+\beta dx')$
dx	$=$	$\gamma(dx'+\beta cdt')$
dy	$=$	dy'
dz	$=$	dz'

Define: $u_x \equiv \dfrac{dx}{dt} \quad u_y \equiv \dfrac{dy}{dt} \quad u_z \equiv \dfrac{dz}{dt}$

$$u'_x \equiv \frac{dx'}{dt'} \quad u'_y \equiv \frac{dy'}{dt'} \quad u'_z \equiv \frac{dz'}{dt'}$$

$$\frac{dx}{dt} = \frac{\gamma(dx'+\beta cdt')}{\gamma(dt'+\beta dx'/c)} = \frac{u'_x + v}{1 + vu'_x/c^2} = u_x$$

$$\frac{dy}{dt} = \frac{dy'}{\gamma(dt'+\beta dx'/c)} = \frac{u'_y}{\gamma(1 + vu'_x/c^2)} = u_y$$

Summary of velocity relationships

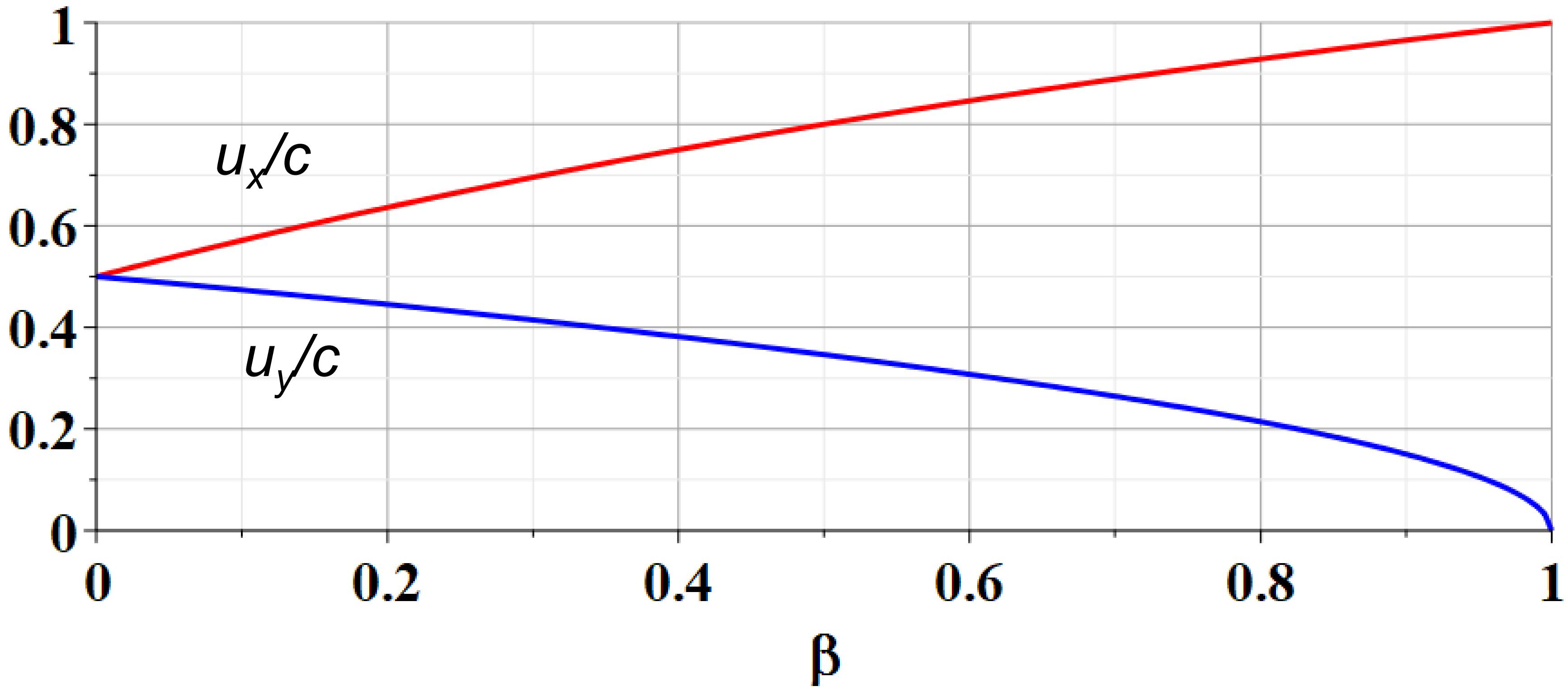
$$u_x = \frac{u'_x + v}{1 + vu'_x / c^2}$$

$$u_y = \frac{u'_y}{\gamma\left(1 + vu'_x / c^2\right)} \equiv \frac{u'_y}{\gamma_v\left(1 + vu'_x / c^2\right)}$$

$$u_z = \frac{u'_z}{\gamma\left(1 + vu'_x / c^2\right)} \equiv \frac{u'_z}{\gamma_v\left(1 + vu'_x / c^2\right)}$$

Example of velocity variation with β:

($u'_x/c = u'_y/c = 0.5$)



Extention to tranformation of acceleration

$$\mathbf{a}_{\|} = \frac{\left(1-\dfrac{v^2}{c^2}\right)^{3/2}}{\left(1+\dfrac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^3}\mathbf{a}'_{\|}$$

$$\mathbf{a}_{\perp} = \frac{\left(1-\dfrac{v^2}{c^2}\right)}{\left(1+\dfrac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^3}\left(\mathbf{a}'_{\perp}+\frac{\mathbf{v}}{c^2}\times\left(\mathbf{a}'\times\mathbf{u}'\right)\right)$$

Comment –
The acceleration equations are obtained by taking the infinitesimal derivative of the velocity relationships and simplifying the expressions. (See Jackson Problem 11.5.)

Velocity transformations continued:

Consider: $u_x = \frac{u'_x + v}{1 + vu'_x/c^2}$ $\quad u_y = \frac{u'_y}{\gamma_v\left(1 + vu'_x/c^2\right)}$ $\quad u_z = \frac{u'_z}{\gamma_v\left(1 + vu'_x/c^2\right)}.$

Note that $\gamma_u \equiv \frac{1}{\sqrt{1-(u/c)^2}} = \frac{1 + vu'_x/c^2}{\sqrt{1-(u'/c)^2}\sqrt{1-(v/c)^2}} = \gamma_v \gamma_{u'}\left(1 + vu'_x/c^2\right)$

$$\Rightarrow \gamma_u c = \gamma_v\left(\gamma_{u'} c + \beta_v \gamma_{u'} u'_x\right)$$

$$\Rightarrow \gamma_u u_x = \gamma_v\left(\gamma_{u'} u'_x + \gamma_{u'} v\right) = \gamma_v\left(\gamma_{u'} u'_x + \beta_v \gamma_{u'} c\right)$$

Note that $\gamma_v \equiv \frac{1}{\sqrt{1-(v/c)^2}}$

$$\Rightarrow \gamma_u u_y = \gamma_{u'} u'_y \qquad \gamma_u u_z = \gamma_{u'} u'_z$$

Velocity 4-vector: $\begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix} = \mathcal{L}_v \begin{pmatrix} \gamma_{u'} c \\ \gamma_{u'} u'_x \\ \gamma_{u'} u'_y \\ \gamma_{u'} u'_z \end{pmatrix}$

Some details:

$$\gamma_u = \gamma_v \gamma_{u'} \left(1 + vu'_x / c^2\right) \quad \Rightarrow \left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{u'^2}{c^2}\right) = \left(1 - \frac{u^2}{c^2}\right)\left(1 + \frac{u_x v}{c^2}\right)^2$$

where $\quad u_x = \dfrac{u'_x + v}{1 + vu'_x / c^2} \qquad u_y = \dfrac{u'_y}{\gamma_v \left(1 + vu'_x / c^2\right)} \qquad u_z = \dfrac{u'_z}{\gamma_v \left(1 + vu'_x / c^2\right)}.$

$$\left(\frac{u_x^2}{c^2} + \frac{u_y^2}{c^2} + \frac{u_z^2}{c^2}\right)\left(1 + \frac{u'_x v}{c^2}\right)^2 = \left(\frac{u'_x}{c} + \frac{v}{c}\right)^2 + \left(\frac{u'^2_y}{c^2} + \frac{u'^2_z}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)$$

$$\frac{u^2}{c^2}\left(1 + \frac{u'_x v}{c^2}\right)^2 = \frac{u'^2}{c^2}\left(1 - \frac{v^2}{c^2}\right) + \left(1 + \frac{u'_x v}{c^2}\right)^2 - \left(1 - \frac{v^2}{c^2}\right)$$

$$\Rightarrow \left(1 - \frac{u^2}{c^2}\right)\left(1 + \frac{u'_x v}{c^2}\right)^2 = \left(1 - \frac{u'^2}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)$$

Significance of 4-velocity vector: $\begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix}$ $\quad \beta_u \equiv \frac{u}{c} \quad \gamma_u \equiv \frac{1}{\sqrt{1-\beta_u^2}}$

Introduce the factor *mc* where *m* is the "rest" mass of the particle characterized by velocity **u**:

$$mc\begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix} = \begin{pmatrix} \gamma_u mc^2 \\ \gamma_u m u_x c \\ \gamma_u m u_y c \\ \gamma_u m u_z c \end{pmatrix} = \begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix}$$

Properties of energy-moment 4-vector:

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \mathcal{L}\begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} \qquad \begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} = \mathcal{L}^{-1}\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix}$$

Note: $E^2 - p^2c^2 = E'^2 - p'^2c^2$

Properties of Energy-momentum 4-vector -- continued

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \begin{pmatrix} \gamma_u mc^2 \\ \gamma_u m u_x c \\ \gamma_u m u_y c \\ \gamma_u m u_z c \end{pmatrix}$$

$$\text{Note: } E^2 - p^2c^2 = \frac{(mc^2)^2}{1-\beta_u^2}\left(1-\left(\frac{u_x}{c}\right)^2-\left(\frac{u_y}{c}\right)^2-\left(\frac{u_z}{c}\right)^2\right) = (mc^2)^2 = E'^2 - p'^2c^2$$

Notion of "rest energy": For $\mathbf{p} \equiv 0$, $E = mc^2$

Define kinetic energy: $E_K \equiv E - mc^2 = \sqrt{p^2c^2 + m^2c^4} - mc^2$

Non-relativistic limit: If $\frac{p}{mc} << 1$, $E_K = mc^2\left(\sqrt{1+\left(\frac{p}{mc}\right)^2} - 1\right)$

$$\approx \frac{p^2}{2m}$$

Summary of relativistic energy relationships

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \begin{pmatrix} \gamma_u mc^2 \\ \gamma_u m u_x c \\ \gamma_u m u_y c \\ \gamma_u m u_z c \end{pmatrix}$$

$$\beta_u \equiv \frac{u}{c} \qquad \gamma_u \equiv \frac{1}{\sqrt{1-\beta_u^2}}$$

$$E = \sqrt{p^2c^2 + m^2c^4} = \gamma_u mc^2$$

$$\text{Check: } \sqrt{p^2c^2 + m^2c^4} = mc^2\sqrt{\gamma_u{}^2\beta_u{}^2 + 1} = \gamma_u mc^2$$

$$\text{Example: } \quad \text{for an electron} \quad mc^2 = 0.5 \text{ MeV}$$

$$\text{for} \qquad E = 200 \text{ GeV}$$

$$\gamma_u = \frac{E}{mc^2} = 4 \times 10^5$$

$$\beta_u = \sqrt{1 - \frac{1}{\gamma_u{}^2}} \approx 1 - \frac{1}{2\gamma_u{}^2} \approx 1 - 3\times 10^{-12}$$

Special theory of relativity and Maxwell's equations

Continuity equation: $$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

Lorenz gauge condition: $$\frac{1}{c}\frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{A} = 0$$

Potential equations: $$\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = 4\pi\rho$$

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \frac{4\pi}{c}\mathbf{J}$$

Field relations: $$\mathbf{E} = -\nabla \Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

More 4-vectors:

$$\text{Time and position :} \quad \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \Rightarrow x^{\alpha}$$

$$\text{Charge and current :} \quad \begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix} \Rightarrow J^{\alpha}$$

$$\text{Vector and scalar potentials :} \quad \begin{pmatrix} \Phi \\ A_x \\ A_y \\ A_z \end{pmatrix} \Rightarrow A^{\alpha}$$

Lorentz transformations

$$\mathcal{L}_v = \begin{pmatrix} \gamma_v & \gamma_v\beta_v & 0 & 0 \\ \gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Time and space: $x^{\alpha} = \mathcal{L}_v x'^{\alpha} \equiv \mathcal{L}_v^{\alpha\beta} x'^{\beta}$

Charge and current: $J^{\alpha} = \mathcal{L}_v J'^{\alpha} \equiv \mathcal{L}_v^{\alpha\beta} J'^{\beta}$

Vector and scalar potential: $A^{\alpha} = \mathcal{L}_v A'^{\alpha} \equiv \mathcal{L}_v^{\alpha\beta} A'^{\beta}$

Summary of results --

$$\text{Time and space:} \qquad x^{\alpha} = \mathcal{L}_v x'^{\alpha} \equiv \mathcal{L}_v{}^{\alpha\beta} x'^{\beta}$$

$$\text{Charge and current:} \qquad J^{\alpha} = \mathcal{L}_v J'^{\alpha} \equiv \mathcal{L}_v{}^{\alpha\beta} J'^{\beta}$$

$$\text{Vector and scalar potential:} \qquad A^{\alpha} = \mathcal{L}_v A'^{\alpha} \equiv \mathcal{L}_v{}^{\alpha\beta} A'^{\beta}$$

Here, the notation varies among the textbooks.

In general, it is convenient to use the matrix multiplication conventions to multiply our 4×4 matrices and 4 vectors

For example: $$\mathcal{L}_v{}^{\alpha\beta} x'^{\beta} = \sum_{\beta=1}^{4} \mathcal{L}_v{}^{\alpha\beta} x'^{\beta} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$