PHY 712 Electrodynamics
10-10:50 PM MWF Olin 103

Notes for Lecture 30:

Start reading Chap. 14 –

Radiation by moving charges

1. Motion in a line
2. Motion in a circle
3. Spectral analysis of radiation

24	Mon: 03/13/2023	Chap. 9	Radiation from localized oscillating sources	#17	03/17/2023
25	Wed: 03/15/2023	Chap. 9	Radiation from oscillating sources		
26	Fri: 03/17/2023	Chap. 9 & 10	Radiation and scattering	#18	03/20/2023
27	Mon: 03/20/2023	Chap. 11	Special Theory of Relativity	#19	03/24/2023
28	Wed: 03/22/2023	Chap. 11	Special Theory of Relativity		
29	Fri: 03/24/2023	Chap. 11	Special Theory of Relativity	#20	03/27/2023
30	Mon: 03/27/2023	Chap. 14	Radiation from moving charges	#21	03/29/2023
31	Wed: 03/29/2023	Chap. 14	Radiation from accelerating charged particles		
32	Fri: 03/31/2023	Chap. 14	Synchrotron radiation		

PHY 712 -- Assignment #21

March 27, 2023

Start reading Chap. 14 in **Jackson** .

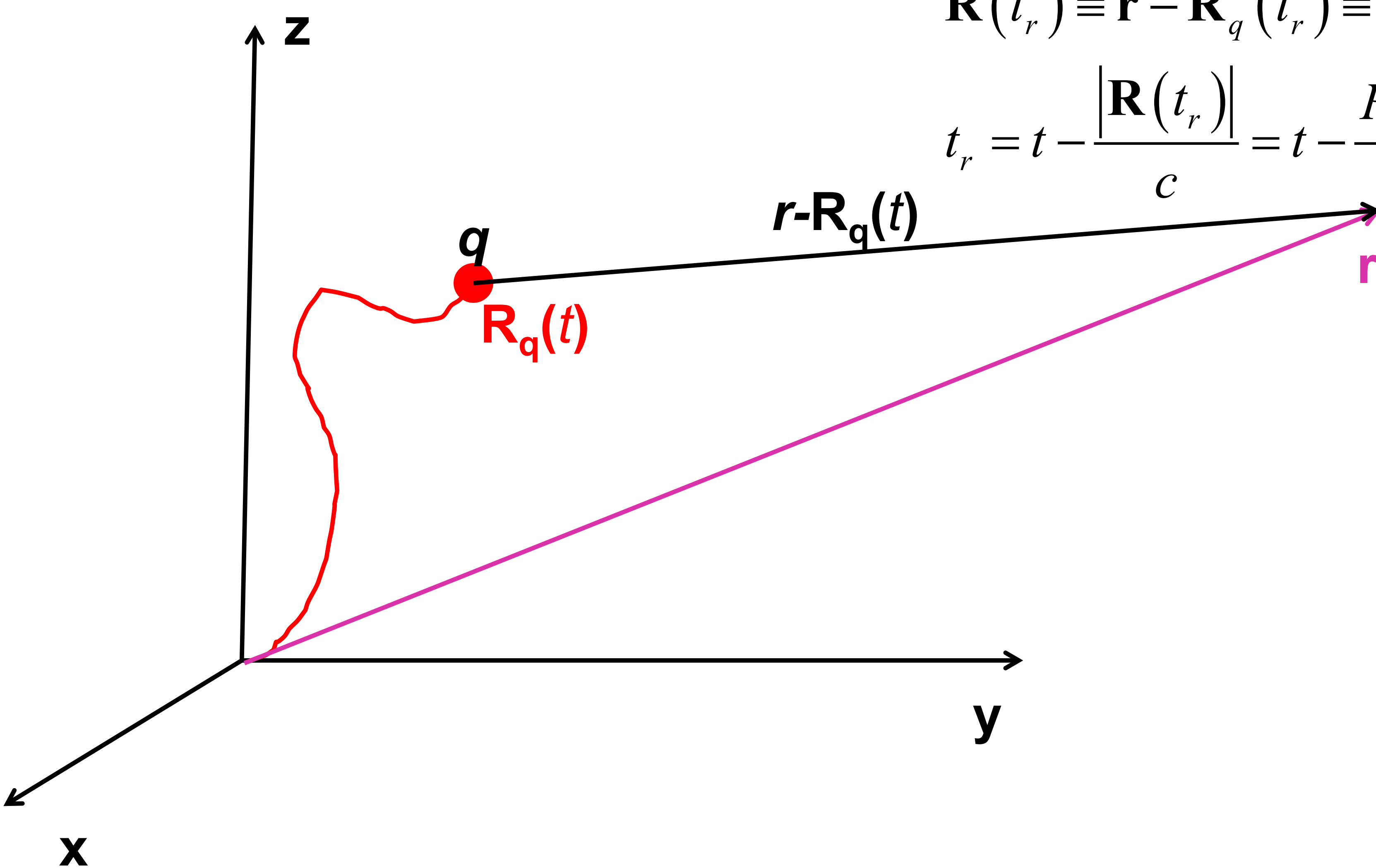
1. Consider an electron moving at constant speed $\beta c \approx c$ in a circular trajectory of radius ρ. Its total energy is $E = \gamma m c^2 = 200$ GeV. Estimate the value of the ratio of the energy lost during one full circle to its total energy. Assume that synchroton radius in this case is $\rho=10^3$ meters. Note if you use the expression for this process analyzed by **Jackson**, please explain the details.

Radiation from a moving charged particle

Variables (notation): $\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$$

$$t_r = t - \frac{|\mathbf{R}(t_r)|}{c} = t - \frac{R(t_r)}{c}$$



Liénard-Wiechert fields (cgs Gaussian units):

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R}\times\left\{\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right]. \tag{19}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right]. \tag{20}$$

In this case, the electric and magnetic fields are related according to

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{R}. \tag{21}$$

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v} \qquad \mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R} \quad \dot{\mathbf{v}} \equiv \frac{d^2\mathbf{R}_q(t_r)}{dt_r{}^2}$$

Comment --

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R}\times\left\{\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right]. \tag{19}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right]. \tag{20}$$

In this case, the electric and magnetic fields are related according to

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{R}. \tag{21}$$

Note that (21) can be demonstrated by evaluating ***R x E(r,t)***

Other helpful identities: ***ax(bxc)=b(a·c)-c(a·b)***

a·(bxc) = b·(cxa) = c·(axb)

Electric field far from source:

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{\left(R-\dfrac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left\{\mathbf{R}\times\left[\left(\mathbf{R}-\frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t)=\frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{R}$$

Note that all of the variables on the right hand side of the equations depend on t_r .

Let $\hat{\mathbf{R}}\equiv\dfrac{\mathbf{R}}{R}$ $\quad\boldsymbol{\beta}\equiv\dfrac{\mathbf{v}}{c}$ $\quad\dot{\boldsymbol{\beta}}\equiv\dfrac{\dot{\mathbf{v}}}{c}$

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{cR\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^3}\left\{\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t)=\hat{\mathbf{R}}\times\mathbf{E}(\mathbf{r},t)$$

Poynting vector:

$$\mathbf{S}(\mathbf{r},t)=\frac{c}{4\pi}(\mathbf{E}\times\mathbf{B})$$

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{cR\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^3}\left\{\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t)=\hat{\mathbf{R}}\times\mathbf{E}(\mathbf{r},t) \qquad \mathbf{E}\times\left(\hat{\mathbf{R}}\times\mathbf{E}(\mathbf{r},t)\right)=\hat{\mathbf{R}}\left|\mathbf{E}\right|^2-\mathbf{E}\left(\hat{\mathbf{R}}\cdot\mathbf{E}\right)$$

$$\mathbf{S}(\mathbf{r},t)=\frac{c}{4\pi}\hat{\mathbf{R}}\left|\mathbf{E}(\mathbf{r},t)\right|^2=\frac{q^2}{4\pi cR^2}\hat{\mathbf{R}}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

Note: We have used the fact that

$\hat{\mathbf{R}}\cdot\mathbf{E}(\mathbf{r},t)=0$ which follows from the vector identities.

Power radiated

$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}\hat{\mathbf{R}}\left|\mathbf{E}(\mathbf{r},t)\right|^2 = \frac{q^2}{4\pi c R^2}\hat{\mathbf{R}}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

$$\frac{dP}{d\Omega} = \mathbf{S}\cdot\hat{\mathbf{R}}R^2 = \frac{q^2}{4\pi c}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

In the non-relativistic limit: $\beta << 1$

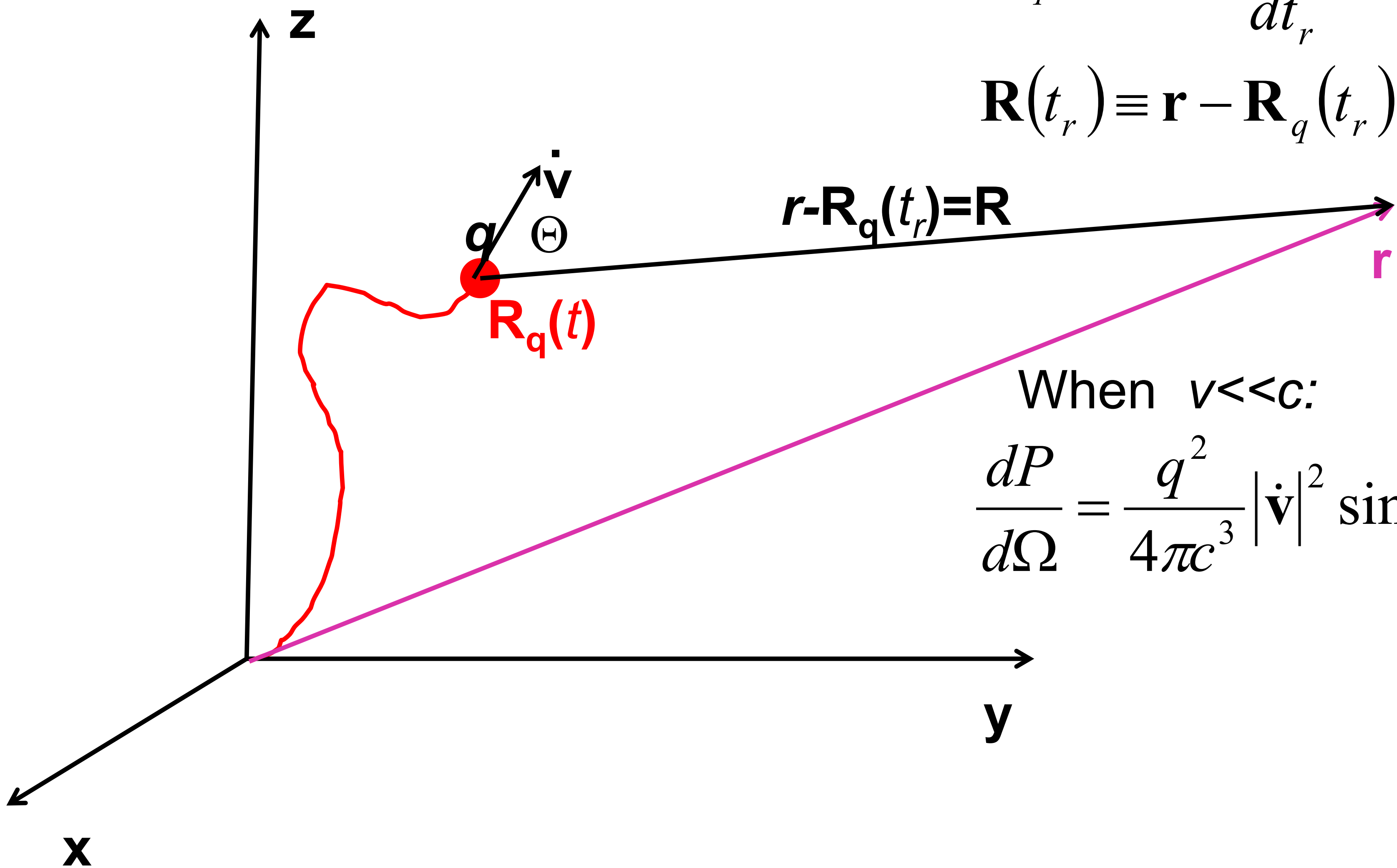
$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c}\left|\hat{\mathbf{R}}\times\left[\hat{\mathbf{R}}\times\dot{\boldsymbol{\beta}}\right]\right|^2 = \frac{q^2}{4\pi c^3}\left|\dot{\mathbf{v}}\right|^2\sin^2\Theta$$

Radiation from a moving charged particle

$$\text{Variables (notation)}:$$

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$$



When $v<<c$:

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}|\dot{\mathbf{v}}|^2 \sin^2\Theta$$

Radiation power in non-relativistic case -- continued

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}|\dot{\mathbf{v}}|^2 \sin^2\Theta$$

$$P = \int d\Omega \frac{dP}{d\Omega} = \frac{2}{3}\frac{q^2}{c^3}|\dot{\mathbf{v}}|^2$$

Radiation distribution in the relativistic case

$$\frac{dP}{d\Omega} = \mathbf{S}\cdot\hat{\mathbf{R}}R^2 = \frac{q^2}{4\pi c}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}\Bigg|_{t_r=t-R/c}$$

This expression gives us the energy per unit field time *t*. We are often interested in the power per unit retarded time $t_r=t-R/c$:

$$\frac{dP_r(t)}{d\Omega} = \frac{dP(t\ )}{d\Omega}\frac{dt}{dt_r} \qquad \frac{dt}{dt_r} = 1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}$$

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^5}\Bigg|_{t_r=t-R/c}$$

Some details –

The power derived from the Poynting vector in terms of the field times is given by:

$$\frac{dP}{d\Omega} = \mathbf{S}\cdot\hat{\mathbf{R}}R^2 = \frac{q^2}{4\pi c}\left.\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}\right|_{t_r=t-R/c}$$

The integrated power would be given by

$$W = \int dt\,\frac{dP(t)}{d\Omega} = \int dt_r \frac{dt}{dt_r}\frac{dP(t)}{d\Omega} \rightarrow \frac{dP_r(t_r)}{d\Omega}$$

More comments

$$t_r = t - \frac{\left|\mathbf{r} - \mathbf{R}_q(t_r)\right|}{c}$$

$$t = t_r + \frac{\left|\mathbf{r} - \mathbf{R}_q(t_r)\right|}{c}$$

$$\frac{dt}{dt_r} = 1 + \left(-\frac{d\mathbf{R}_q(t_r)}{c\,dt_r} \right) \cdot \frac{\mathbf{r} - \mathbf{R}_q(t_r)}{\left|\mathbf{r} - \mathbf{R}_q(t_r)\right|} = 1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}$$

$\Rightarrow$

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \left. \frac{\left| \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^5} \right|_{t_r = t - R/c}$$

Why do you think it useful to measure the power as energy per unit retarded time P_r?

1. Jackson likes to torture us.
2. There should be no difference.
3. ???

Radiation distribution in the relativistic case -- continued

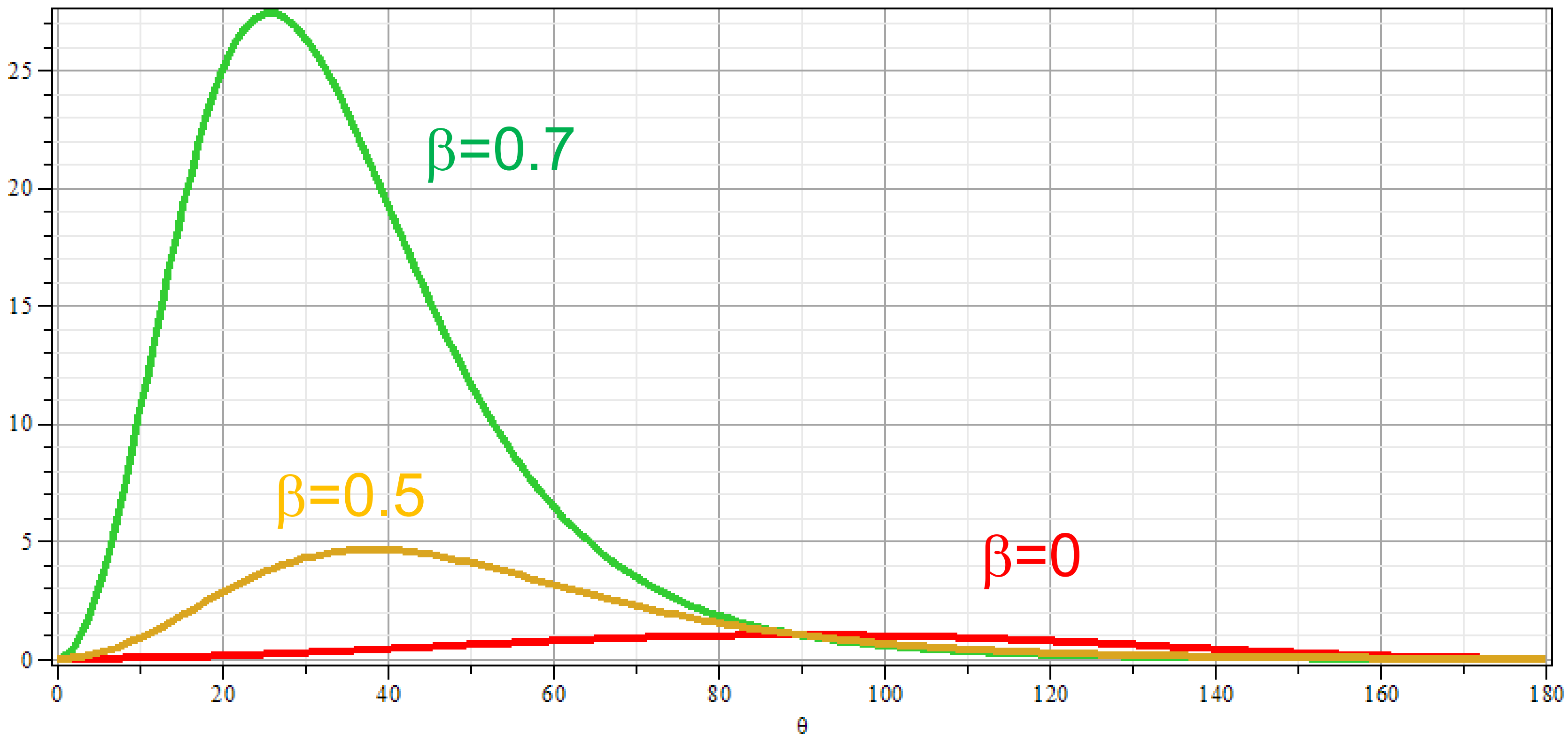
$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \left. \frac{\left| \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^5} \right|_{t_r = t - R/c}$$

For linear acceleration: $\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}} = 0$

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \left. \frac{\left| \hat{\mathbf{R}} \times \left(\hat{\mathbf{R}} \times \dot{\boldsymbol{\beta}} \right) \right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^5} \right|_{t_r = t - R/c} = \frac{q^2}{4\pi c^3} \left| \dot{\mathbf{v}} \right|^2 \frac{\sin^2 \theta}{\left(1 - \beta \cos \theta \right)^5}$$

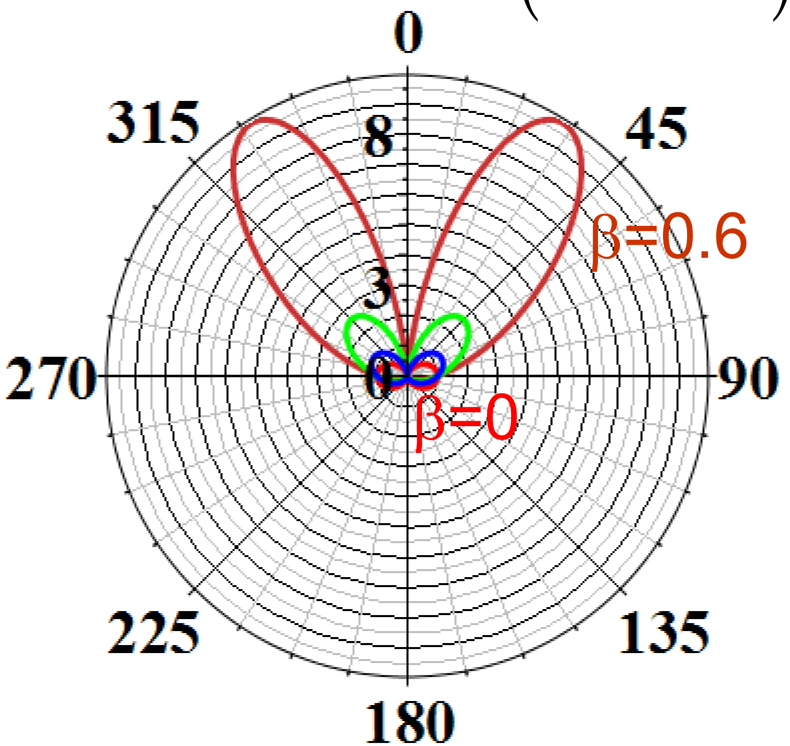
Power from linearly accelerating particle

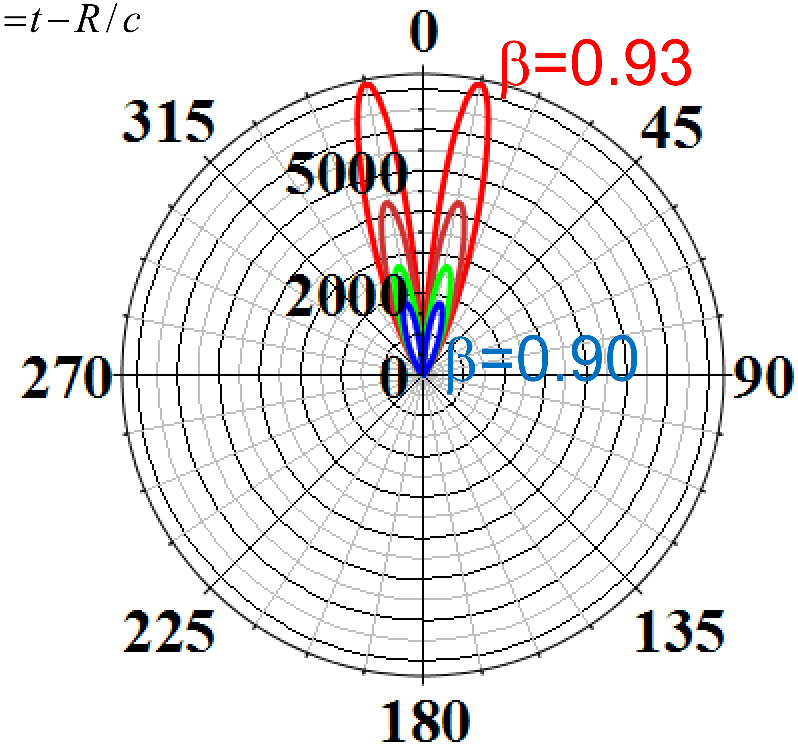
$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{\left|\hat{\mathbf{R}} \times \left(\hat{\mathbf{R}} \times \dot{\boldsymbol{\beta}}\right)\right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^5}\Bigg|_{t_r = t - R/c} = \frac{q^2}{4\pi c^3} \left|\dot{\mathbf{v}}\right|^2 \frac{\sin^2\theta}{\left(1 - \beta\cos\theta\right)^5}$$



Polar plots:

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c}\frac{\left|\hat{\mathbf{R}}\times\left(\hat{\mathbf{R}}\times\dot{\boldsymbol{\beta}}\right)\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^5}\Bigg|_{t_r=t-R/c} = \frac{q^2}{4\pi c^3}\left|\dot{\mathbf{v}}\right|^2\frac{\sin^2\theta}{\left(1-\beta\cos\theta\right)^5}$$





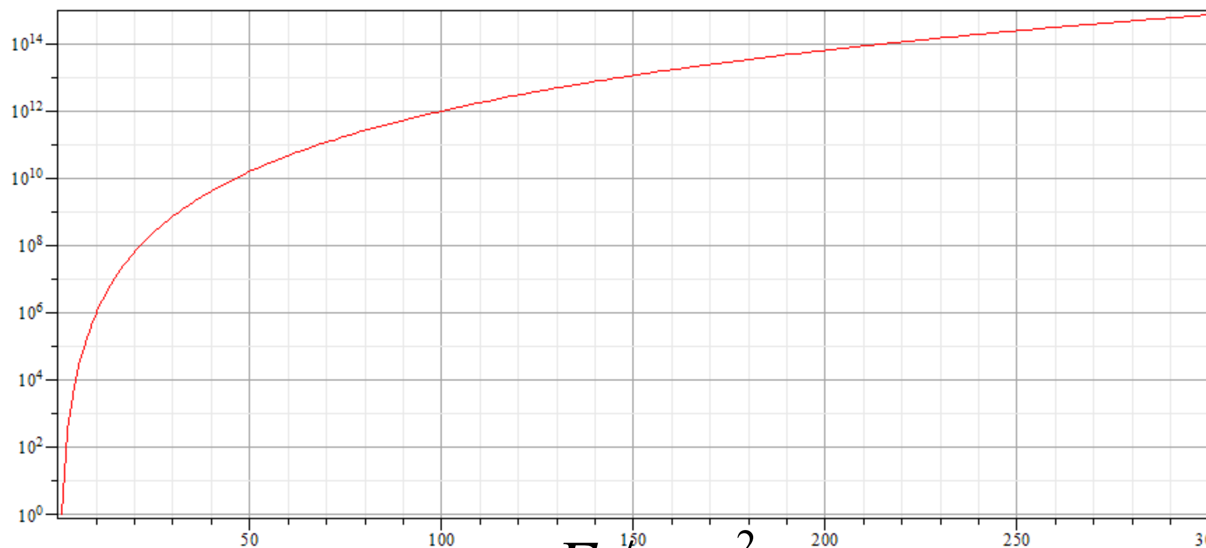
Note – two separate plots are introduced in order to see the drastic change of scale at values of β close to 1.

Power from linearly accelerating particle

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \left. \frac{\left|\hat{\mathbf{R}} \times \left(\hat{\mathbf{R}} \times \dot{\boldsymbol{\beta}}\right)\right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^5} \right|_{t_r = t - R/c} = \frac{q^2}{4\pi c^3} \left|\dot{\mathbf{v}}\right|^2 \frac{\sin^2\theta}{\left(1 - \beta\cos\theta\right)^5}$$

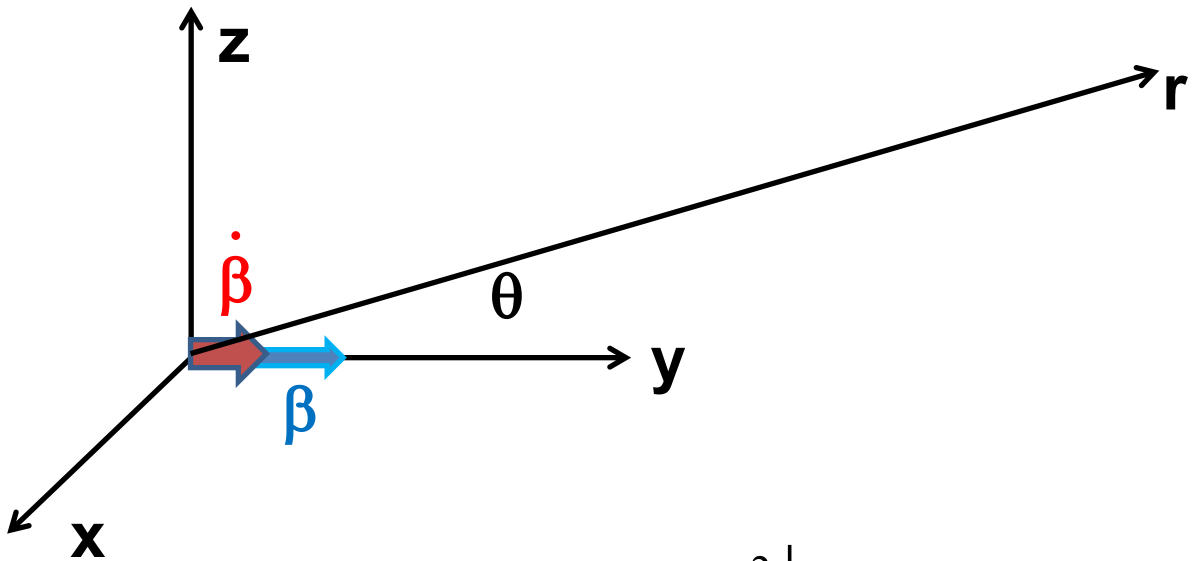
$$P_r(t_r) = \int \frac{dP_r(t_r)}{d\Omega} d\Omega = \frac{2}{3}\frac{q^2}{c^3}\left|\dot{\mathbf{v}}\right|^2 \gamma^6 \quad \text{where} \quad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}} = \frac{E}{mc^2}$$

$$\frac{P_r(\gamma)}{P_r(1)}$$



$$E / mc^2$$

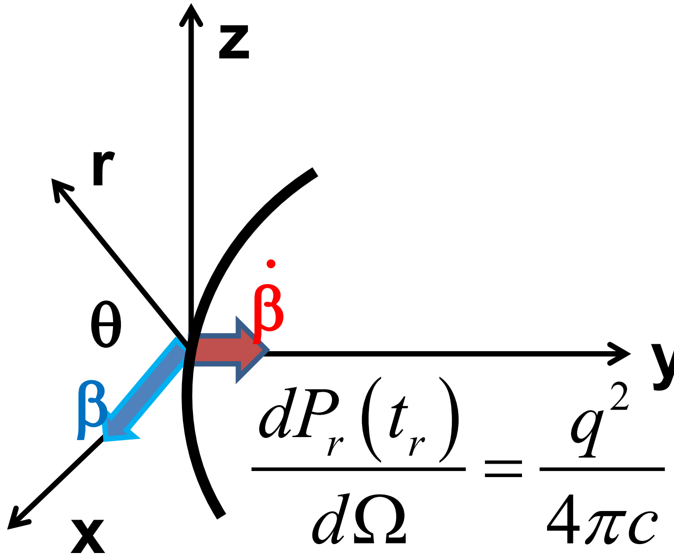
Power distribution for linear acceleration -- continued



$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \left. \frac{\left|\hat{\mathbf{R}} \times \left(\hat{\mathbf{R}} \times \dot{\boldsymbol{\beta}}\right)\right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^5} \right|_{t_r = t - R/c} = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \frac{\sin^2\theta}{\left(1 - \beta\cos\theta\right)^5}$$

$$P_r(t_r) = \int \frac{dP_r(t_r)}{d\Omega} d\Omega = \frac{2}{3}\frac{q^2}{c^3} |\dot{\mathbf{v}}|^2 \gamma^6 \qquad \text{where} \quad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

Power distribution for circular acceleration

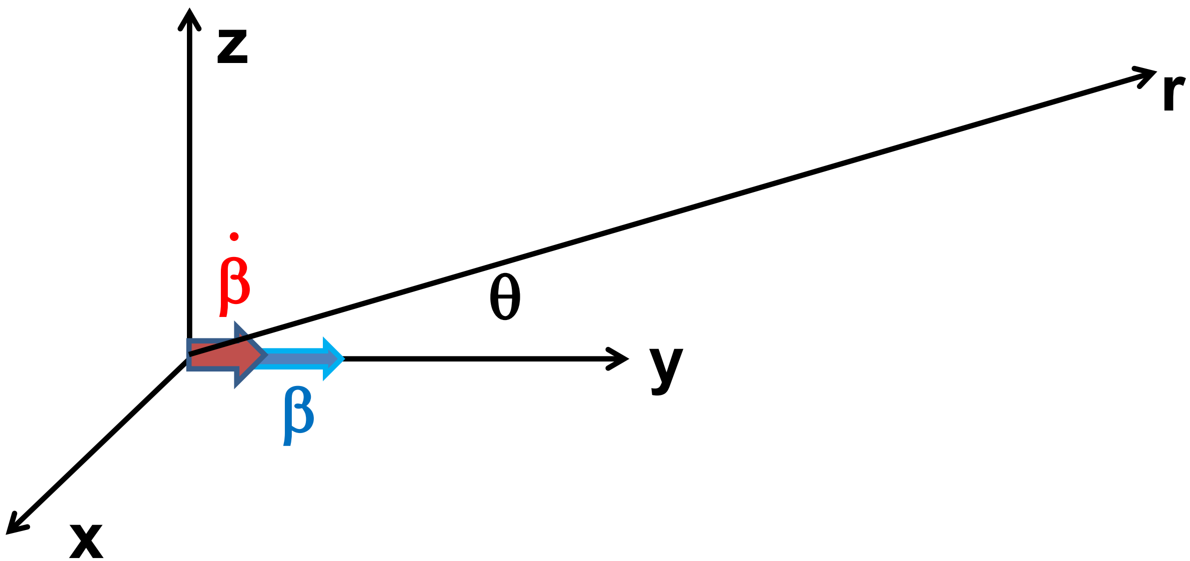


$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \left. \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}} \right]\right|^2}{\left(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^5} \right|_{t_r = t - R/c}$$

$$= \frac{q^2}{4\pi c} \left. \frac{\left|\dot{\boldsymbol{\beta}}\right|^2 \left(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^2 - \left(\hat{\mathbf{R}}\cdot\dot{\boldsymbol{\beta}}\right)^2 \left(1 - \beta^2\right)}{\left(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^5} \right|_{t_r = t - R/c}$$

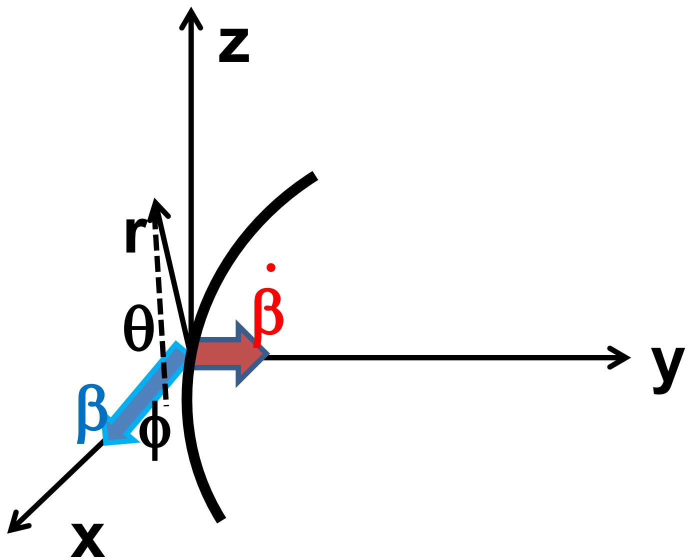
$$P_r(t_r) = \int d\Omega \frac{dP_r(t_r)}{d\Omega} = \frac{2}{3}\frac{q^2}{c^3} \left|\dot{\mathbf{v}}\right|^2 \gamma^4$$

Summary of results --For linear acceleration --



$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{\left|\hat{\mathbf{R}} \times \left(\hat{\mathbf{R}} \times \dot{\boldsymbol{\beta}}\right)\right|^2}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^5}\Bigg|_{t_r = t - R/c} = \frac{q^2}{4\pi c^3} \left|\dot{\mathbf{v}}\right|^2 \frac{\sin^2\theta}{\left(1 - \beta\cos\theta\right)^5}$$

Power distribution for circular acceleration



$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \left. \frac{|\dot{\boldsymbol{\beta}}|^2 \left(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^2 - \left(\hat{\mathbf{R}}\cdot\dot{\boldsymbol{\beta}}\right)^2 \left(1-\beta^2\right)}{\left(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^5} \right|_{t_r = t - R/c}$$

$$= \frac{q^2}{4\pi c^3} \frac{|\dot{\mathbf{v}}|^2}{\left(1 - \beta\cos(\theta)\right)^3} \left(1 - \frac{\cos^2\theta \sin^2\phi}{\gamma^2 \left(1 - \beta\cos(\theta)\right)^2} \right)$$

Angular integrals for the two cases –

Linear acceleration

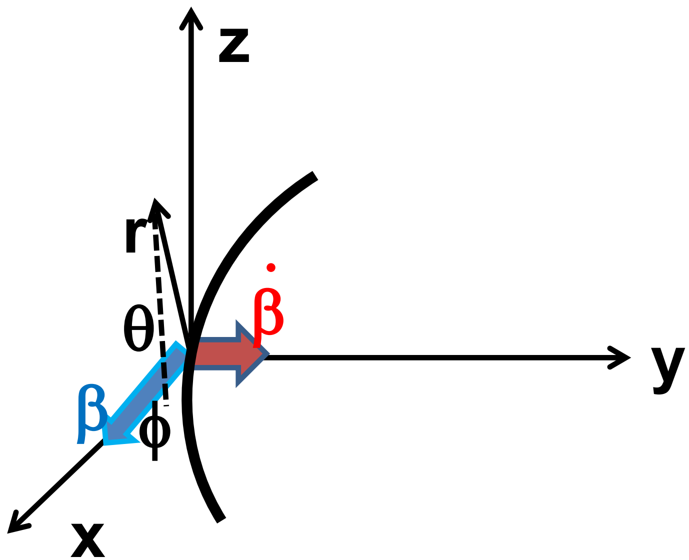
$$P_r\left(t_r\right) = \int \frac{dP_r\left(t_r\right)}{d\Omega} d\Omega = 2\pi \int \frac{q^2}{4\pi c^3} \left|\dot{\mathbf{v}}\right|^2 \frac{\sin^2\theta \; d\sin\theta}{\left(1-\beta\cos\theta\right)^5} = \frac{2}{3}\frac{q^2}{c^3}\left|\dot{\mathbf{v}}\right|^2 \gamma^6$$

Circular acceleration

$$P_r\left(t_r\right) = \int \frac{dP_r\left(t_r\right)}{d\Omega} d\Omega = \int d\phi \; d\sin\theta \frac{q^2}{4\pi c^3} \frac{\left|\dot{\mathbf{v}}\right|^2}{\left(1-\beta\cos(\theta)\right)^3} \left(1 - \frac{\cos^2\theta \sin^2\phi}{\gamma^2\left(1-\beta\cos(\theta)\right)^2}\right)$$

$$= \frac{2}{3}\frac{q^2}{c^3}\left|\dot{\mathbf{v}}\right|^2 \gamma^4$$

Power distribution for circular acceleration



$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{\left|\dot{\boldsymbol{\beta}}\right|^2 \left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^2 - \left(\hat{\mathbf{R}}\cdot\dot{\boldsymbol{\beta}}\right)^2 \left(1-\beta^2\right)}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^5}\Bigg|_{t_r=t-R/c}$$

$$= \frac{q^2}{4\pi c^3} \frac{\left|\dot{\mathbf{v}}\right|^2}{\left(1-\beta\cos(\theta)\right)^3} \left(1 - \frac{\cos^2\theta \sin^2\phi}{\gamma^2\left(1-\beta\cos(\theta)\right)^2}\right)$$

Spectral composition of electromagnetic radiation

Previously we determined the power distribution from a charged particle:

$$\frac{dP(t)}{d\Omega} = \mathbf{S}\cdot\hat{\mathbf{R}}R^2 = \frac{q^2}{4\pi c}\left.\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}\right|_{t_r=t-R/c}$$

Note: Here we are finding power wrt to the field time frame.

$$\equiv \left|\boldsymbol{a}(t)\right|^2$$

where $$\boldsymbol{a}(t) \equiv \sqrt{\frac{q^2}{4\pi c}}\left.\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^3}\right|_{t_r=t-R/c}$$

Time integrated power per solid angle:

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt\,\frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt\left|\boldsymbol{a}(t)\right|^2 = \int_{-\infty}^{\infty} d\omega\left|\tilde{\boldsymbol{a}}(\omega)\right|^2$$

Spectral composition of electromagnetic radiation -- continued

Time integrated power per solid angle :

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt \, \frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt \left|\mathbf{a}(t)\right|^2 = \int_{-\infty}^{\infty} d\omega \left|\tilde{\mathbf{a}}(\omega)\right|^2$$

Fourier amplitude :

$$\tilde{\mathbf{a}}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, \mathbf{a}(t) \, e^{i\omega t} \qquad \mathbf{a}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \, \tilde{\mathbf{a}}(\omega) \, e^{-i\omega t}$$

Parseval's theorem

Marc-Antoine Parseval des Chênes 1755-1836

http://www-history.mcs.st-andrews.ac.uk/Biographies/Parseval.html

Checking:

Fourier amplitude:

$$\tilde{\mathbf{a}}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\ \mathbf{a}(t)\ e^{i\omega t}$$

Amplitude in time:

$$\mathbf{a}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\ \tilde{\mathbf{a}}(\omega)\ e^{-i\omega t}$$

$$\int_{-\infty}^{\infty} dt \left|\mathbf{a}(t)\right|^2 = \int_{-\infty}^{\infty} dt \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\ \tilde{\mathbf{a}}(\omega)\ e^{-i\omega t} \right)\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega'\ \tilde{\mathbf{a}}^*(\omega')\ e^{i\omega' t} \right)$$

$$= \int_{-\infty}^{\infty} d\omega\ \tilde{\mathbf{a}}(\omega) \int_{-\infty}^{\infty} d\omega'\ \tilde{\mathbf{a}}^*(\omega') \left(\frac{1}{2\pi} \right) \int_{-\infty}^{\infty} dt e^{i(\omega'-\omega)t}$$

$$= \int_{-\infty}^{\infty} d\omega\ \tilde{\mathbf{a}}(\omega) \int_{-\infty}^{\infty} d\omega'\ \tilde{\mathbf{a}}^*(\omega') \delta(\omega'-\omega) = \int_{-\infty}^{\infty} d\omega\ \left|\tilde{\mathbf{a}}(\omega)\right|^2$$

Spectral composition of electromagnetic radiation -- continued

Consequences of Parseval's analysis:

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt \frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt \left|\mathbf{a}(t)\right|^2 = \int_{-\infty}^{\infty} d\omega \left|\tilde{\mathbf{a}}(\omega)\right|^2$$

Note that: $\tilde{\mathbf{a}}(\omega) = \tilde{\mathbf{a}}^*(-\omega)$

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} d\omega \left|\tilde{\mathbf{a}}(\omega)\right|^2 = \int_{0}^{\infty} d\omega \left(\left|\tilde{\mathbf{a}}(\omega)\right|^2 + \left|\tilde{\mathbf{a}}(-\omega)\right|^2 \right) \equiv \int_{0}^{\infty} d\omega \frac{\partial^2 I}{\partial\Omega\partial\omega}$$

$$\frac{\partial^2 I}{\partial\Omega\partial\omega} \equiv 2\left|\tilde{\mathbf{a}}(\omega)\right|^2$$

What is the significance of $\frac{\partial^2 I}{\partial\Omega\partial\omega}$?

1. It is purely a mathematical construct
2. It can be measured

Spectral composition of electromagnetic radiation -- continued

For our case: $$\mathcal{a}(t) \equiv \sqrt{\frac{q^2}{4\pi c}} \left. \frac{\left| \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^3} \right|_{t_r = t - R/c}$$

Fourier amplitude:

$$\tilde{\mathcal{a}}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \; e^{i\omega t} \; \mathcal{a}(t)$$

$$= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt \; e^{i\omega t} \left. \frac{\left| \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^3} \right|_{t_r = t - R/c}$$

Spectral composition of electromagnetic radiation -- continued

Fourier amplitude:

$$\tilde{\mathbf{a}}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, \mathbf{a}(t)\, e^{i\omega t}$$

$$= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt \left. \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^3} \right|_{t_r = t - R/c} e^{i\omega t}$$

$$= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \frac{dt}{dt_r} \left. \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^3} \right|_{t_r = t - R/c} e^{i\omega\left(t_r + R(t_r)/c\right)}$$

$$= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \left. \frac{\left|\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right]\right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}\right)^2} \right|_{t_r = t - R/c} e^{i\omega\left(t_r + R(t_r)/c\right)}$$

Spectral composition of electromagnetic radiation -- continued

Exact expression :

$$\tilde{\boldsymbol{a}}(\omega)=\sqrt{\frac{q^2}{8\pi^2 c}}\int_{-\infty}^{\infty} dt_r \left.\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^2}\right|_{t_r=t-R/c} e^{i\omega\left(t_r+R\left(t_r\right)/c\right)}$$

Recall: $\dot{\mathbf{R}}_q\left(t_r\right)\equiv\dfrac{d\mathbf{R}_q\left(t_r\right)}{dt_r}\equiv\mathbf{v}$ $\quad\mathbf{R}\left(t_r\right)\equiv\mathbf{r}-\mathbf{R}_q\left(t_r\right)\equiv\mathbf{R}$

For $r>>R_q\left(t_r\right)$ $\quad R\left(t_r\right)\approx r-\hat{\mathbf{r}}\cdot\mathbf{R}_q\left(t_r\right)$ where $\hat{\mathbf{r}}\equiv\dfrac{\mathbf{r}}{r}$

At the same level of approximation: $\hat{\mathbf{R}}\approx\hat{\mathbf{r}}$

Spectral composition of electromagnetic radiation -- continued

Exact expression:

$$\tilde{a}(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \left. \frac{\left| \hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}} \right)^2} \right|_{t_r = t - R/c} e^{i\omega\left(t_r + R(t_r)/c\right)}$$

Approximate expression:

$$\tilde{a}(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} e^{i\omega(r/c)} \int_{-\infty}^{\infty} dt_r \left. \frac{\left| \hat{\mathbf{r}} \times \left[\left(\hat{\mathbf{r}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}} \right)^2} \right|_{t_r = t - R/c} e^{i\omega\left(t_r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)/c\right)}$$

Resulting spectral intensity expression:

$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{q^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} dt_r \left. \frac{\left| \hat{\mathbf{r}} \times \left[\left(\hat{\mathbf{r}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right] \right|}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}} \right)^2} \right|_{t_r = t - R/c} e^{i\omega\left(t_r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)/c\right)} \right|^2$$

Example – radiation from a collinear acceleration burst

$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2}{4\pi^2 c}\left| \int_{-\infty}^{\infty} dt_r \left.\frac{\left|\hat{\mathbf{r}}\times\left[\left(\hat{\mathbf{r}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)^2}\right|_{t_r=t-R/c} e^{i\omega\left(t_r-\hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)}\right|^2$$

$$\text{Suppose that } \dot{\boldsymbol{\beta}} = \begin{cases} \dfrac{\hat{\boldsymbol{\beta}}\Delta v}{c\tau} & 0 < t_r < \tau \\ 0 & \text{otherwise} \end{cases}$$

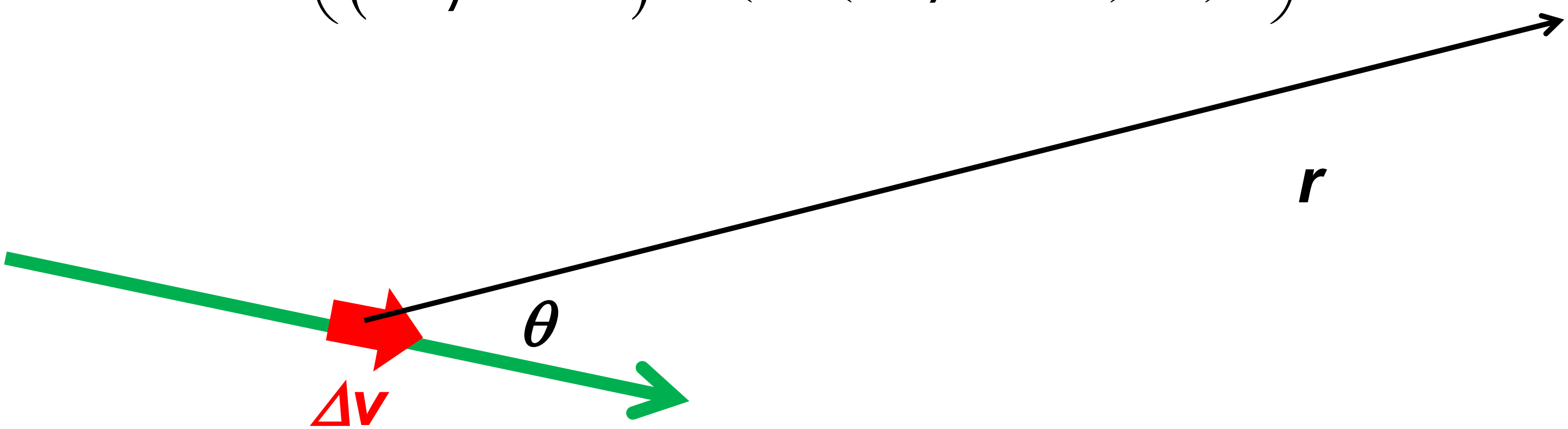
$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2}{4\pi^2 c^3}\left|\frac{\left|\hat{\mathbf{r}}\times\left[\hat{\mathbf{r}}\times\hat{\boldsymbol{\beta}}\right]\right|\Delta v}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}}\right)^2\tau}\right|^2 \left|\int_0^{\tau} dt_r e^{i\omega\left(t_r-\hat{\mathbf{r}}\cdot\boldsymbol{\beta}t_r\right)}\right|^2 \qquad \text{Let } \boldsymbol{\beta}\cdot\hat{\mathbf{r}} = \beta\cos\theta$$

$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2}{4\pi^2 c^3}\left(\frac{\Delta v\sin\theta}{(1-\beta\cos\theta)^2}\frac{\sin(\omega\tau(1-\beta\cos\theta)/2)}{(\omega\tau(1-\beta\cos\theta)/2)}\right)^2$$

Example:

$$\text{Suppose that } \dot{\boldsymbol{\beta}} = \begin{cases} \dfrac{\hat{\boldsymbol{\beta}}\Delta v}{c\tau} & 0 < t_r < \tau \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2}{4\pi^2 c^3}\left(\frac{\Delta v \sin\theta}{\left(1-\beta\cos\theta\right)^2}\frac{\sin(\omega\tau(1-\beta\cos\theta)/2)}{(\omega\tau(1-\beta\cos\theta)/2)}\right)^2$$



Example: "Bremsstrahlung" radiation

Spectral composition of electromagnetic radiation -- continued

Alternative expression --

It can be shown that:

$$\frac{\hat{\mathbf{r}} \times \left[\left(\hat{\mathbf{r}} - \boldsymbol{\beta} \right) \times \dot{\boldsymbol{\beta}} \right]}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}} \right)^2} = \frac{d}{dt_r} \left(\frac{\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \boldsymbol{\beta} \right)}{\left(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}} \right)} \right)$$

Integration by parts and assumptions about the integration limit behaviors shows that the spectral intensity depends on the following integral:

$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{q^2 \omega^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} dt_r \left[\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \boldsymbol{\beta}(t_r) \right) \right] e^{i\omega\left(t_r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)/c\right)} \right|^2$$